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TOPOLOGICAL FREE ENTROPY DIMENSION FOR APPROXIMATELY DIVISIBLE C*-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a unital separable approximately divisible C*-algebra. We show that $\mathcal{A}$ is generated by two self-adjoint elements and the topological free entropy dimension of any finite generating set of $\mathcal{A}$ is less than or equal to 1.

1. Introduction. The theory of free entropy and free entropy dimension was developed by Voiculescu in the 1990's. It has been a very powerful tool in the recent study of finite von Neumann algebras. In **[10]**, Voiculescu introduced the notion of topological free entropy dimension of elements in a unital C*-algebra as an analogue of free entropy dimension in the context of C*-algebra, and showed that (1) if $x_1, \ldots, x_n$ is a family of free semicircular elements in a unital C*-algebra with a tracial state, then $\delta_{\text{top}}(x_1, \ldots, x_n) = n$, where $\delta_{\text{top}}(x_1, \ldots, x_n)$ is the topological free entropy dimension of $x_1, \ldots, x_n$; (2) if $\{x_1, \ldots, x_n\}$ is the universal n-tuple of self-adjoint contractions, then $\delta_{\text{top}}(x_1, \ldots, x_n) = n$. Recently, Hadwin and Shen **[4]** obtained some interesting results on topological free entropy dimensions of unital C*-algebras, which include the irrational rotation C*-algebras, UHF algebras and minimal tensor products of reduced free group C*-algebras. Thus, it will be interesting to consider the topological free entropy dimensions for a larger class of unital C*-algebras. One goal of this paper is to calculate the topological free entropy dimensions in the unital approximately divisible C*-algebras, which were introduced by Blackadar, Kumjian and Rørdam **[2]**. In that paper, they showed that the class of approximately divisible C*-algebras contains all simple unital AF-algebras and most of the simple unital AH-algebras with real rank 0, as well as every nonrational noncommutative torus.

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Note that Voiculescu's topological free entropy dimension is defined only for the finitely generated C*-algebras. Therefore, it is natural to consider the generator problem for approximately divisible unital C*-algebras before we carry out the calculation of the topological free entropy. In fact, the generator problem for C*-algebras and the one for von Neumann algebras have been studied by many people and many results have been obtained. For example, Olsen and Zame [**7**] showed that if $\mathcal{A}$ is a unital separable C*-algebra and $\mathcal{B}$ is a UHF algebra, then $\mathcal{A}\otimes\mathcal{B}$ is generated by two self-adjoint elements in $\mathcal{A}\otimes\mathcal{B}$. It is clear that such $\mathcal{A}\otimes\mathcal{B}$ is approximately divisible. In this paper, we develop some new techniques and obtain the following result (see Theorem 3.1):

Theorem. *If $\mathcal{A}$ is a unital separable approximately divisible C*-algebra, then $\mathcal{A}$ is generated by two self-adjoint elements in $\mathcal{A}$, i.e., $\mathcal{A}$ is singly generated.*

Then we compute the topological free entropy dimension of any finite family of self-adjoint generators of a unital separable approximately divisible C*-algebra. More specifically, we obtain the following result (see Theorem 4.1):

Theorem. *Let $\mathcal{A}$ be a unital separable approximately divisible C*-algebra. Then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \leq 1,$$

where $x_1, \ldots, x_n$ is any family of self-adjoint generators of $\mathcal{A}$.

The organization of the paper is as follows. In Section 2, we recall the definition of approximately divisible C*-algebra. The generator problem for an approximately divisible C*-algebra is considered in Section 3. The computation of topological free entropy dimension in an approximately divisible C*-algebra is carried out in Section 4.

2. Notation and preliminaries. In this section, we will introduce some notation that will be needed later and recall the definition of an approximately divisible C*-algebra introduced by Blackadar, Kumjian and Rørdam [**2**].

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and $\mathcal{M}_k^{sa}(\mathbb{C})$ the subspace of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}_k$ be the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{sa}(\mathbb{C}))^n$ be the direct sum of n copies of $\mathcal{M}_k^{sa}(\mathbb{C})$.

The following lemma is a well-known fact.

Lemma 2.1. *Suppose $\mathcal{B}$ is a finite-dimensional C*-algebra. Then there exist positive integers r and $k_1, \ldots, k_r$ such that*

$$\mathcal{B} \cong \mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C}).$$

Definition 2.2. Suppose

$$\mathcal{B} \cong \mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C})$$

is a finite-dimensional C*-algebra for some positive integers $r, k_1, \ldots, k_r$. Define the *rank* of $\mathcal{B}$ to be

$$\operatorname{rank}(\mathcal{B}) = k_1 + \cdots + k_r,$$

the *subrank* of $\mathcal{B}$ to be

$$\operatorname{subrank}(\mathcal{B}) = \min\{k_1, \ldots, k_r\}.$$

The following definition is Definition 1.2 in [**2**].

Definition 2.3. A separable unital C*-algebra $\mathcal{A}$ with the unit $I_{\mathcal{A}}$ is approximately divisible if, for every $x_1, \ldots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a finite-dimensional C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that

(1) $I_{\mathcal{A}} \in \mathcal{B}$;
(2) $\operatorname{subrank}(\mathcal{B}) \geq 2$;
(3) $\|x_i y - y x_i\| < \varepsilon$ for $i = 1, \ldots, n$ and all $y \in \mathcal{B}$ with $\|y\| \leq 1$.

The following proposition is taken from Theorem 1.3 and Corollary 2.10 in [**2**].

Proposition 2.4. ([**2**]). *Let $\mathcal{A}$ be a unital separable approximately divisible C*-algebra with the unit $I_{\mathcal{A}}$. Then there exists an increasing sequence $\{\mathcal{A}_m\}_{m=1}^{\infty}$ of subalgebras of $\mathcal{A}$ such that*

(1) $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$,

(2) *for any positive integer* m, $\mathcal{A}'_m \cap \mathcal{A}_{m+1}$ *contains a finite-dimensional C*-subalgebra* $\mathcal{B}$ *with* $I_{\mathcal{A}} \in \mathcal{B}$ *and* $\operatorname{subrank}(\mathcal{B}) \geq 2$,
(3) *for any positive integers* m *and* k, *there is a finite-dimensional C*-subalgebra* $\mathcal{B}$ *of* $\mathcal{A}'_m \cap \mathcal{A}$ *with* $I_{\mathcal{A}} \in \mathcal{B}$ *and* $\operatorname{subrank}(\mathcal{B}) \geq k$.

3. Generator problem for approximately divisible C*algebras. In this section we prove that every unital separable approximately divisible C*-algebra is singly generated, i.e., generated by two self-adjoint elements.

Theorem 3.1. *If* $\mathcal{A}$ *is a unital, separable approximately divisible C*-algebra, then* $\mathcal{A}$ *is singly generated.*

Proof. Since $\mathcal{A}$ is separable, there exists a sequence of self-adjoint elements $\{x_i\}_{i=1}^{\infty} \subset \mathcal{A}$ that generate $\mathcal{A}$ as a C*-algebra.

Claim 3.2. *There exists a sequence of finite-dimensional subalgebras* $\{\mathcal{B}_n\}_{n=1}^{\infty}$ *of* $\mathcal{A}$ *so that the following hold*:

(1) *for all* $n \in \mathbb{N}$, $I_{\mathcal{A}} \in \mathcal{B}_n$, *where* $I_{\mathcal{A}}$ *is the unit of* $\mathcal{A}$;
(2) $\operatorname{subrank}(\mathcal{B}_1) \geq 4$, *and for any* $n \geq 2$,

$$\operatorname{subrank}(\mathcal{B}_n) \geq n \cdot (\operatorname{rank}(\mathcal{B}_1))^2 \cdots (\operatorname{rank}(\mathcal{B}_{n-1}))^2 + 3;$$

(3) *if* $n \neq m$, *then* $\mathcal{B}_n$ *commutes with* $\mathcal{B}_m$;
(4) *for any* $n \in \mathbb{N}$,

$$\operatorname{dist}(x_p, \mathcal{A}'_n \cap \mathcal{A}) < 2^{-n}, \quad \textit{for all } 1 \leq p \leq n,$$

where $\operatorname{dist}(x_p, \mathcal{B}'_n \cap \mathcal{A}) = \inf\{\|x_p - y\| : y \in \mathcal{B}'_n \cap \mathcal{A}\}$.

Proof of the claim. It follows from Proposition 2.4 that there exists an increasing sequence $\{\mathcal{A}_m\}_{m=1}^{\infty}$ of subalgebras of $\mathcal{A}$ such that

(a) $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$,
(b) for any positive integer m, $\mathcal{A}'_m \cap \mathcal{A}_{m+1}$ contains a finite-dimensional C*-subalgebra $\mathcal{B}$ with $I_{\mathcal{A}} \in \mathcal{B}$ and $\operatorname{subrank}(\mathcal{B}) \geq 2$,
(c) for any positive integers m and k, there is a finite-dimensional C*-subalgebra $\mathcal{B}$ of $\mathcal{A}'_m \cap \mathcal{A}$ with $I_{\mathcal{A}} \in \mathcal{B}$ and $\operatorname{subrank}(\mathcal{B}) \geq k$.

Instead of proving Claim 3.1 directly, we will prove a stronger result by replacing statement (3) in Claim 3.1 with the following one:

(3′) there exist two increasing sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ of positive integers such that, for any $n \in \mathbb{N}$, $s_n \le t_n \le s_{n+1}$ and $\mathcal{B}_n \subseteq \mathcal{A}'_{s_n} \cap \mathcal{A}_{t_n}$.

We prove this stronger claim by induction on n.

Base step. Note that $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$. For $x_1 \in \mathcal{A}$, there are a positive integer s_1 and a self-adjoint element $y_1^{(1)} \in \mathcal{A}_{s_1}$ such that $\|x_1 - y_1^{(1)}\| < 1/2$. By restriction (b) on the subalgebras $\{\mathcal{A}_m\}_{m=1}^{\infty}$, we know that there exist two finite-dimensional subalgebras $\mathcal{C}_{s_1+1}, \mathcal{C}_{s_1+2}$ of $\mathcal{A}$ such that, $I_{\mathcal{A}} \in \mathcal{C}_{s_1+1}$ and $I_{\mathcal{A}} \in \mathcal{C}_{s_1+2}$; $\mathcal{C}_{s_1+1} \subseteq \mathcal{A}'_{s_1} \cap \mathcal{A}_{s_1+1}$ and $\mathcal{C}_{s_1+2} \subseteq \mathcal{A}'_{s_1+1} \cap \mathcal{A}_{s_1+2}$; subrank $(\mathcal{C}_{s_1+1})$ and subrank $(\mathcal{A}_{s_1+2})$ are at least 2.

Let $t_1 = s_1 + 2$, $\mathcal{B}_1 = C^*(\mathcal{C}_{s_1+1}, \mathcal{C}_{t_1})$ (the *-subalgebra generated by $\mathcal{C}_{s_1+1}$ and $\mathcal{C}_{t_1}$ in $\mathcal{A}$). Then subrank $(\mathcal{B}_1) \ge 4$ because $\mathcal{C}_{s_1+1}$ and $\mathcal{C}_{t_1}$ commute, and $\mathcal{B}_1 \subseteq \mathcal{A}'_{s_1} \cap \mathcal{A}_{t_1}$.

Inductive step. Now suppose the stronger claim is true when $n \le k-1$, i.e., there exists a family of finite-dimensional C*-algebras $\{\mathcal{B}_n\}_{n=1}^{k-1}$ of $\mathcal{A}$, and two increasing sequences of positive integers $\{s_n\}_{n=1}^{k-1}$ and $\{t_n\}_{n=1}^{k-1}$ that satisfy (1), (2), (3′) and (4).

For $x_1, \ldots, x_k$ in $\mathcal{A}$, from restriction (a) on $\{\mathcal{A}_m\}_{m=1}^{\infty} \subseteq \mathcal{A}$, we know that there are a positive integers s_k with $s_k \ge t_{k-1}$ and self-adjoint elements $y_1^{(k)}, \ldots, y_k^{(k)}$ in $\mathcal{A}_{s_k}$ such that $\|x_i - y_i^{(k)}\| < 2^{-k}$ for $1 \le i \le k$. From restriction (b) on $\{\mathcal{A}_m\}_{m=1}^{\infty} \subseteq \mathcal{A}$, there exists a family $\{\mathcal{C}_{s_k+1}, \mathcal{C}_{s_k+2}, \ldots\}$ of finite-dimensional subalgebras in $\mathcal{A}$ such that:

(i) $I_{\mathcal{A}} \in \mathcal{C}_{s_k+i}$, for all $i \ge 1$;
(ii) $\mathcal{C}_{s_k+i} \subseteq \mathcal{A}'_{s_k+i-1} \cap \mathcal{A}_{s_k+i}$, for all $i \ge 1$;
(iii) subrank $(\mathcal{C}_{s_k+i}) \ge 2$, for all $i \ge 1$.

By (ii), we know that $\{\mathcal{C}_{s_k+1}, \mathcal{C}_{s_k+2}, \ldots\}$ is a commuting sequence of subalgebras of $\mathcal{A}$. Combining with (iii), we get that there is a positive integer t_k such that

$$\text{subrank } (C^*(\mathcal{C}_{s_k+1}, \ldots, \mathcal{C}_{t_k})) \ge k \cdot (\text{rank}\,(\mathcal{B}_1))^2 \cdots (\text{rank}\,(\mathcal{B}_{k-1}))^2 + 3,$$

where $C^*(\mathcal{C}_{s_k+1}, \ldots, \mathcal{C}_{t_k})$ is the C*-subalgebra generated by $\mathcal{C}_{s_k+1}, \ldots,$

$\mathcal{C}_{t_k}$ in $\mathcal{A}$. Moreover, $C^*(\mathcal{C}_{s_k+1}, \ldots, \mathcal{C}_{t_k})$ contains $I_{\mathcal{A}}$, and it is a finite-dimensional C*-subalgebra in $\mathcal{A}'_{s_k} \cap \mathcal{A}_{t_k}$. Let

$$\mathcal{B}_k = C^*(\mathcal{C}_{s_k+1}, \ldots, \mathcal{C}_{t_k}),$$

and it is not hard to check that $\mathcal{B}_1, \ldots, \mathcal{B}_k$ satisfy conditions (1), (2), (3′) and (4) in the stronger claim. This completes the proof of the claim.

Let $\{\mathcal{B}_n\}_{n=1}^{\infty}$ be as in Claim 3.1. For any positive integer n, since $\mathcal{B}_n$ is a finite-dimensional C*-algebra, there exist positive integers r_n and $k_1^{(n)}, \ldots, k_{r_n}^{(n)}$ such that

$$\mathcal{B}_n \cong \mathcal{M}_{k_1^{(n)}}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_{r_n}^{(n)}}(\mathbb{C}).$$

Let $\{e_{ij}^{(n,s)} : 1 \le i, j \le k_s^{(n)}\}$ be the canonical system of matrix units for $\mathcal{M}_{k_s^{(n)}}$. We can further assume that $\{e_{ij}^{(n,s)} : 1 \le i, j \le k_s^{(n)}, 1 \le s \le r_n\}$ consists of a system of matrix units of $\mathcal{B}_n$. Note that $\mathcal{B}_n$ contains the unit $I_{\mathcal{A}}$ of $\mathcal{A}$, so

$$\sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)}} e_{ii}^{(n,s)} = I_{\mathcal{A}}.$$

Define

$$p_n = \sum_{s=1}^{r_n} e_{k_s^{(n)}, k_s^{(n)}}^{(n,s)} \quad \text{for } n \ge 1. \tag{1}$$

Then p_n is a projection in $\mathcal{B}_n$. Since subrank$(\mathcal{B}_n) \ge 2$, it is clear that

$$p_n e_{11}^{(n,s)} = 0 \quad \text{for } 1 \le s \le r_n. \tag{2}$$

Claim 3.3. *Let $\{x_n\}_{n=1}^{\infty}$, $\{\mathcal{B}_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be defined as above. For any positive integer n, there exists $z_n = z_n^* \in \mathcal{A}$ with $\|z_n\| = 2^{-(r_1+\cdots+r_n+1)}$ so that*

(1) $(I_{\mathcal{A}} - p_n)p_{n-1} \cdots p_1 \cdot z_n \cdot p_1 \cdots p_{n-1}(I_{\mathcal{A}} - p_n) = z_n$,
(2) dist$(x_j, C^*(\mathcal{B}_1, \ldots, \mathcal{B}_n, z_n)) < 2^{-n}$ *for* $1 \le j \le n$, *where* $C^*(\mathcal{B}_1, \ldots, \mathcal{B}_n, z_n)$ *is the* C*-*subalgebra generated by* $\mathcal{B}_1, \ldots, \mathcal{B}_n$, z_n *in* $\mathcal{A}$.

Proof of the claim. By Claim 3.2, for any positive integer n,

$$\operatorname{dist}(x_j, \mathcal{B}_n' \cap \mathcal{A}) < 2^{-n} \quad \text{for } 1 \le j \le n.$$

Thus there exist self-adjoint elements $y_1^{(n)}, \ldots, y_n^{(n)}$ in $\mathcal{A}$ that commute with $\mathcal{B}_n$ and

$$\|x_j - y_j^{(n)}\| < 2^{-n} \quad \text{for } 1 \le j \le n.$$

Step 1. Let

$$z_1 = \frac{1}{2^{1+r_1}} \cdot \frac{\sum_{s=1}^{r_1} e_{22}^{(1,s)} y_1^{(1)}}{\| \sum_{s=1}^{r_1} e_{22}^{(1,s)} y_1^{(1)} \|}. \tag{3}$$

With subrank $(\mathcal{B}_1) \ge 4$, we have

$$(I_{\mathcal{A}} - p_1) \cdot z_1 \cdot (I_{\mathcal{A}} - p_1) = z_1.$$

To prove $\operatorname{dist}(x_1, C^*(\mathcal{B}_1, , z_1)) < 2^{-1}$, it is sufficient to show that $y_1^{(1)} \in C^*(\mathcal{B}_1, , z_1)$. By equation (3.3) and the fact that $y_1^{(1)}$ commutes with $\mathcal{B}_1$, we know

$$y_1^{(1)} = \left(2^{1+r_1} \cdot \| \sum_{s=1}^{r_1} e_{22}^{(1,s)} y_1^{(1)} \| \right) \cdot \left(\sum_{s=1}^{r_1} \sum_{i=1}^{k_s^{(1)}} e_{i,2}^{(1,s)} \cdot z_1 \cdot e_{2,i}^{(1,s)} \right).$$

This implies that $y_1^{(1)}$ is in the C*-algebra generated by $\mathcal{B}_1$ and z_1, whence

$$\operatorname{dist}(x_1, C^*(\mathcal{B}_1, z_1)) \le \operatorname{dist}(x_1, y_1^{(1)}) < 2^{-1}.$$

Step 2. Now let us construct z_n for any positive integer $n \ge 2$. Let

$$\begin{aligned} \Delta_{n-1} = \{ & (i_1, s_1) \times (j_1, t_1) \times (i_2, s_2) \times (j_2, t_2) \times \cdots \\ & \times (i_{n-1}, s_{n-1}) \times (j_{n-1}, t_{n-1}) : \\ & \quad 1 \le i_1 \le k_{s_1}^{(1)}, 1 \le j_1 \le k_{t_1}^{(1)}, 1 \le s_1, t_1 \le r_1, \\ & \quad \cdots, 1 \le i_{n-1} \le k_{s_{n-1}}^{(n-1)}, 1 \le j_{n-1} \le k_{t_{n-1}}^{(n-1)}, \\ & \quad 1 \le s_{n-1}, t_{n-1} \le r_{n-1} \}. \end{aligned}$$

It is not hard to check that the cardinality of the set Δ_{n-1} satisfies

$$\operatorname{card}(\Delta_{n-1}) = \prod_{i=1}^{n-1} (\operatorname{rank}(\mathcal{B}_i))^2.$$

Hence, for any $1 \leq j \leq n$, there is a one-to-one mapping $f_j^{(n)}$ from the index set Δ_{n-1} onto the set

$$\{i \in \mathbb{N} \,|(j-1) \cdot \operatorname{card}(\Delta_{n-1}) + 2 \leq i \leq j \cdot \operatorname{card}(\Delta_{n-1}) + 1\}.$$

For any index

$$\alpha = (i_1, s_1) \times (j_1, t_1) \times \cdots \times (i_{n-1}, s_{n-1}) \times (j_{n-1}, t_{n-1}) \in \Delta_{n-1}$$

and any $1 \leq j \leq n$, we define

$$\alpha(y_j^{(n)}) = e^{(n-1, s_{n-1})}_{k^{(n-1)}_{s_{n-1}}, i_{n-1}} \cdots e^{(1,s_1)}_{k^{(1)}_{s_1}, i_1} \cdot y_j^{(n)} \cdot e^{(1,t_1)}_{j_1, k^{(1)}_{t_1}} \cdots e^{(n-1, t_{n-1})}_{j_{n-1}, k^{(n-1)}_{t_{n-1}}} \in \mathcal{A}. \tag{4}$$

By Claim 3.1, we know that $\operatorname{subrank}(\mathcal{B}_n) \geq n \cdot \operatorname{card}(\Delta_{n-1}) + 3$. It follows that

$$\begin{aligned} z_n = c_n \cdot \sum_{s=1}^{r_n} \sum_{j=1}^{n} \sum_{\alpha \in \Delta_{n-1}} \Bigg(& e^{(n,s)}_{f_j^{(n)}(\alpha), f_j^{(n)}(\alpha)+1} \cdot \alpha(y_j^{(n)}) \\ & + \left(e^{(n,s)}_{f_j^{(n)}(\alpha), f_j^{(n)}(\alpha)+1} \cdot \alpha(y_j^{(n)}) \right)^* \Bigg) \end{aligned} \tag{5}$$

is well defined and belongs to $\mathcal{A}$, where c_n is a constant such that

$$\|z_n\| = 2^{-(r_1 + \cdots + r_n + 1)}. \tag{6}$$

From the construction of z_n, it follows as f is injective that $z_n = z_n^*$ and

$$z_n = (I_{\mathcal{A}} - p_n) \cdot p_{n-1} \cdots p_1 \cdot z_n \cdot p_1 \cdots p_{n-1} \cdot (I_{\mathcal{A}} - p_n).$$

To prove $\operatorname{dist}(x_j, C^*(\mathcal{B}_1, \ldots, \mathcal{B}_n, z_n)) < 2^{-n}$ for $1 \leq j \leq n$, it is sufficient to prove that $\{y_1^{(n)}, \ldots, y_n^{(n)}\} \subseteq C^*(\mathcal{B}_1, \ldots, \mathcal{B}_n, z_n)$. Because $\mathcal{B}_n$ commutes with $y_1^{(n)}, \ldots, y_n^{(n)}$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-1}$, and $f_j^{(n)}$ is one-to-one, it follows from equation (3.5) that, for any $\alpha \in \Delta_{n-1}$ and $1 \leq j \leq n$,

$$\alpha(y_j^{(n)}) = \sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)}} e^{(n,s)}_{i, f_j^{(n)}(\alpha)} \cdot \left(\frac{1}{c_n} z_n \right) \cdot e^{(n,s)}_{f_j^{(n)}(\alpha)+1, i}.$$

This implies that $\alpha(y_j^{(n)}) \in C^*(\mathcal{B}_1, \ldots, \mathcal{B}_n, z_n)$.

Suppose $\alpha = (i_1, s_1) \times (j_1, t_1) \times \cdots \times (i_{n-1}, s_{n-1}) \times (j_{n-1}, t_{n-1}) \in \Delta_{n-1}$. Since $\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}$ are commuting, it follows from equation (3.4) that, for any $1 \le j \le n$,

$$e^{(n-1,s_{n-1})}_{i_{n-1},i_{n-1}} \cdots e^{(1,s_1)}_{i_1,i_1} \cdot y^{(n)}_j \cdot e^{(1,t_1)}_{j_1,j_1} \cdots e^{(n-1,t_{n-1})}_{j_{n-1},j_{n-1}}$$
$$= e^{(n-1,s_{n-1})}_{i_{n-1},k^{(n-1)}_{s_{n-1}}} \cdots e^{(1,s_1)}_{i_1,k^{(1)}_{s_1}} \cdot \alpha(y^{(n)}_j) \cdot e^{(1,t_1)}_{k^{(1)}_{t_1},j_1} \cdots e^{(n-1,t_{n-1})}_{k^{(n-1)}_{t_{n-1}},j_{n-1}}$$

and

$$y^{(n)}_j = \sum_{s_1,t_1=1}^{r_1} \sum_{i_1=1}^{k^{(1)}_{s_1}} \sum_{j_1=1}^{k^{(1)}_{t_1}} \cdots \sum_{s_{n-1},t_{n-1}=1}^{r_{n-1}} \sum_{i_{n-1}=1}^{k^{(n-1)}_{s_{n-1}}} \sum_{j_{n-1}=1}^{k^{(n-1)}_{t_{n-1}}} e^{(n-1,s_{n-1})}_{i_{n-1},i_{n-1}} \cdots e^{(1,s_1)}_{i_1,i_1} \cdot y^{(n)}_j \cdot e^{(1,t_1)}_{j_1,j_1} \cdots e^{(n-1,t_{n-1})}_{j_{n-1},j_{n-1}}.$$

Thus, $y^{(n)}_j \in C^*(\mathcal{B}_1, \ldots, \mathcal{B}_n, z_n)$ for any $1 \le j \le n$. This completes the proof of the claim.

Let $\{x_n\}_{n=1}^\infty$, $\{\mathcal{B}_n\}_{n=1}^\infty$, $\{r_n\}_{n=1}^\infty$, $\{p_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be as above. From equation (3.2), part (1) of Claim 3.2 and the construction of z_n, we can get some basic facts about z_n. Let us list them below:

$$p_n z_n = z_n p_n = 0. \tag{7}$$

$$z_n \cdot e^{(m,s)}_{11} = e^{(m,s)}_{11} \cdot z_n = 0 \quad \text{for} \quad m \le n \text{ and } 1 \le s \le r_m. \tag{8}$$

$$z_n \cdot z_m = 0 \quad \text{for any } n \ne m. \tag{9}$$

Let $p_0 = I_{\mathcal{A}}$ and $r_0 = 0$. For any $n \ge 1$, let

$$a_n = p_1 \cdots p_{n-1} \sum_{s=1}^{r_n} 2^{-r_1-\cdots-r_{n-1}-s} \cdot e^{(n,s)}_{11} + z_n, \tag{10}$$

$$b_n = 2^{-2n} p_1 \cdots p_{n-1} \sum_{s=1}^{r_n} \sum_{i=1}^{k^{(n)}_s - 1} (e^{(n,s)}_{i,i+1} + e^{(n,s)}_{i+1,i}). \tag{11}$$

From equations (3.7), (3.8) and (3.9), we have

$$a_n \cdot a_m = 0 \quad \text{for} \quad n \ne m. \tag{12}$$

Combining equations (3.6), (3.8) and the fact that $e_{11}^{(n,s)} \cdot e_{11}^{(n,s_1)} = e_{11}^{(n,s_1)} \cdot e_{11}^{(n,s)} = 0$ ($s \neq s_1$), it is clear that

$$\begin{aligned} \|a_n\| &= \max\{\{\|p_1 \cdots p_{n-1} \cdot 2^{-r_1-\cdots-r_{n-1}-s} \cdot e_{11}^{(n,s)}\|\}_{s=1}^{r_n}, \|z_n\|\} \\ &= 2^{-r_1-\cdots-r_n-1} \leq 2^{-n}. \end{aligned} \tag{13}$$

It follows from equation (3.11) that

$$\|b_n\| \leq 2 \cdot 2^{-2n} \cdot \|\sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)}-1} e_{i,i+1}^{n,s}\| \leq 2^{-2n+1} \leq 2^{-n}. \tag{14}$$

It induces that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are all convergent series in $\mathcal{A}$. Let

$$a = \sum_{n=1}^{\infty} a_n, \qquad b = \sum_{n=1}^{\infty} b_n. \tag{15}$$

It is clear that $a = a^* \in \mathcal{A}$ and $b = b^* \in \mathcal{A}$.

Claim 3.4. *Let $\{\mathcal{B}_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ and a, b be defined as above. Then*

$$\{\mathcal{B}_1, z_1, \mathcal{B}_2, z_2, \ldots\} \subseteq C^*(a,b),$$

where $C^*(a,b)$ is the C^*-subalgebra generated by a and b in $\mathcal{A}$.

Proof of the claim. It is sufficient to prove that, for any $n \geq 1$,

$$\{\mathcal{B}_1, \ldots, \mathcal{B}_n, z_1, \ldots, z_n\} \subseteq C^*(a,b).$$

We will prove it by using an induction on n.

Step 1. We shall prove $\{\mathcal{B}_1, z_1\} \subseteq C^*(a,b)$. It follows from equations (3.10), (3.12), (3.15) and part (1) of Claim 3.2 that for all $k \in \mathbb{N}$,

$$\begin{aligned} (2a)^k &= (2a_1)^k + \sum_{n=2}^{\infty} (2a_n)^k \\ &= e_{11}^{(1,1)} + \sum_{s_1=2}^{r_1} (2^{-s_1+1})^k e_{11}^{(1,s_1)} + (2z_1)^k + \sum_{n=2}^{\infty} (2a_n)^k. \end{aligned}$$

Therefore,

$$\|(2a)^k - e_{11}^{(1,1)}\| = \left\| \sum_{s_1=2}^{r_1} (2^{-s_1+1})^k e_{11}^{(1,s_1)} + (2z_1)^k + \sum_{n=2}^{\infty} (2a_n)^k \right\|.$$

Combining equations (3.6), (3.12) and inequality (3.13), we have

$$\left\| \sum_{s_1=2}^{r_1} (2^{-s_1+1})^k e_{11}^{(1,s_1)} + (2z_1)^k + \sum_{n=2}^{\infty} (2a_n)^k \right\| \longrightarrow 0, \quad \text{as } k \to \infty.$$

Hence, $\|(2a)^k - e_{11}^{(1,1)}\| \to 0$ as k goes to ∞, which implies $e_{11}^{(1,1)} \in C^*(a,b)$.

By the construction of the element b, it is not hard to check that $\{e_{ij}^{(1,1)} : 1 \le i,j \le k_1^{(1)}\}$ are contained in the C*-subalgebra generated by $e_{11}^{(1,1)}$ and b in $\mathcal{A}$. Therefore, $\{e_{ij}^{(1,1)} : 1 \le i,j \le k_1^{(1)}\} \subseteq C^*(a,b)$.

It follows from the construction of a that

$$(I_{\mathcal{A}} - e_{11}^{(1,1)}) \cdot a \cdot (I_{\mathcal{A}} - e_{11}^{(1,1)}) = \sum_{s=2}^{r_1} 2^{-s} e_{11}^{(1,s)} + z_1 + \sum_{n=2}^{\infty} a_n.$$

By equations (3.7), (3.8), (3.9) and (3.12), we have

$$\Big(4(I_{\mathcal{A}} - e_{11}^{(1,1)}) \cdot a \cdot (I_{\mathcal{A}} - e_{11}^{(1,1)})\Big)^k = e_{11}^{(1,2)} + \sum_{s=3}^{r_1} 2^{k(-s+2)} e_{11}^{(1,s)} + (4z_1)^k + \sum_{n=2}^{\infty} (4a_n)^k.$$

By equation (3.6) and inequality (3.13), we have

$$\left\| \Big(4(I_{\mathcal{A}} - e_{11}^{(1,1)}) \cdot a \cdot (I_{\mathcal{A}} - e_{11}^{(1,1)})\Big)^k - e_{11}^{(1,2)} \right\| = \left\| \sum_{s=3}^{r_1} 2^{k(-s+2)} e_{11}^{(1,s)} + (4z_1)^k + \sum_{n=2}^{\infty} (4a_n)^k \right\| \longrightarrow 0, \quad \text{as } k \to \infty.$$

This implies $e_{11}^{(1,2)}$ is in $C^*(a,b)$, whence $\{e_{ij}^{(1,2)} : 1 \le i,j \le k_2^{(1)}\}$ are in $C^*(a,b)$. Repeating the preceding process, we get that $\{e_{ij}^{(1,s)} : 1 \le i,j \le k_s^{(1)}, 1 \le s \le r_1\}$ is contained in $C^*(a,b)$; therefore, $\mathcal{B}_1$ is contained in $C^*(a,b)$. Whence p_1 is contained in $C^*(a,b)$. By part (1) of Claim 3.2, we know that

$$(I_{\mathcal{A}} - p_1)z_1 = z_1;$$

and, from equation (3.10) and part (1) of Claim 3.2, it is clear that

$$(I_{\mathcal{A}} - p_1)\sum_{n=2}^{\infty} a_n = 0.$$

Whence, by the construction of a,

$$(I_{\mathcal{A}} - p_1)\cdot a = \sum_{s=1}^{r_1} 2^{-s} e_{11}^{(1,s)} + z_1.$$

This indicates that z_1 is contained in $C^*(a,b)$ since p_1 and $e_{ij}^{(1,s)}$ are in $C^*(a,b)$. Now we conclude that both $\mathcal{B}_1$ and z_1 are contained in $C^*(a,b)$.

Step 2. Assume that $\{\mathcal{B}_1,\dots,\mathcal{B}_{n-1},z_1,\dots,z_{n-1}\}\subseteq C^*(a,b)$. We need to prove that $\{\mathcal{B}_n,z_n\}\subseteq C^*(a,b)$. By equation (3.2) and the construction of the elements a, b (see equations (3.10), (3.11) and (3.15)), we know that

$$\begin{aligned}(p_1\cdots p_{n-1})\,a = \sum_{i=n}^{\infty} a_i = p_1\cdots p_{n-1}\sum_{s=1}^{r_n} 2^{-r_1-\cdots-r_{n-1}-s}\cdot e_{11}^{(n,s)}\\ + z_n + \sum_{i=n+1}^{\infty} a_i,\end{aligned}$$

and

$$\begin{aligned}(p_1\cdots p_{n-1})\,b\,(p_1\cdots p_{n-1}) &= \sum_{i=n}^{\infty} b_i\\ &= \left(2^{-2n}p_1\cdots p_{n-1}\sum_{s=1}^{r_n}\sum_{i=1}^{k_s^{(n)}-1}(e_{i,i+1}^{(n,s)} + e_{i+1,i}^{(n,s)})\right)\\ &\quad + \sum_{i=n+1}^{\infty} b_i.\end{aligned}$$

From equations (3.10) and (3.12) and part (1) of Claim 3.2,

$$\left\| \left(2^{r_1+\cdots+r_{n-1}+1}\right)\left(\sum_{i=n}^{\infty} a_i\right)^k - (p_1\cdots p_{n-1})\, e_{11}^{(n,1)} \right\| \longrightarrow 0, \qquad \text{as } k\to\infty.$$

Using a similar argument as in the case $n = 1$, we can show that $\{p_1\cdots p_{n-1}e_{ij}^{(n,s)} : 1\le i,j\le k_s^{(n)}, 1\le s\le r_n\}$ are contained in the C*-subalgebra generated by $\sum_{i=n}^{\infty} a_i$ and $\sum_{i=n}^{\infty} b_i$ in $\mathcal{A}$. From the fact that $\mathcal{B}_1,\ldots,\mathcal{B}_n$ are mutually commuting subalgebras, it follows that

$$e_{ij}^{(n,s)} = \sum_{s_1=1}^{r_1}\sum_{i_1=1}^{k_{s_1}^1}\cdots\sum_{s_{n-1}=1}^{r_{n-1}}\sum_{i_{n-1}=1}^{k_{s_{n-1}}^{(n-1)}} e_{i_{n-1},k_{s_{n-1}}^{(n-1)}}^{(n-1,s_{n-1})} \cdots e_{i_1,k_{s_1}^{(1)}}^{(1,s_1)}\cdot\left(p_1\cdots p_{n-1}e_{ij}^{(n,s)}\right)\cdot e_{k_{s_1}^{(1)},i_1}^{(1,s_1)}\cdots e_{k_{s_{n-1}}^{(n-1)},i_{n-1}}^{(n-1,s_{n-1})}$$

is in $C^*(a,b)$, which implies that $\{e_{ij}^{(n,s)} : 1\le i,j\le k_s^{(n)}, 1\le s\le r_n\}$; therefore, $\mathcal{B}_n$, p_n and z_n, are contained in $C^*(a,b)$. This completes the proof of the claim.

By Claims 3.3 and 3.4, $\mathcal{A}$ is generated by two self-adjoint elements a and b. Therefore, $\mathcal{A}$ is singly generated. □

It is mentioned in **[2]** that if $\mathcal{A}$ and $\mathcal{B}$ are unital separable C*-algebras and $\mathcal{B}$ is approximately divisible, then so is $\mathcal{A}\otimes\mathcal{B}$. Combined with the theorem above, we have the following corollary.

Corollary 3.5. *If $\mathcal{A}$ is a unital separable C*-algebra and $\mathcal{B}$ is a unital separable approximately divisible C*-algebra, and α is a C*-cross norm, then $\mathcal{A}\otimes_\alpha\mathcal{B}$ is singly generated.*

Note that a UHF algebra is approximately divisible and nuclear (only one tensor product). Therefore, Theorem 9 in **[7]** is a corollary of our theorem.

Corollary 3.6. *If $\mathcal{A}$ is a unital separable C*-algebra and $\mathcal{B}$ is a* UHF *algebra, then $\mathcal{A}\otimes\mathcal{B}$ is singly generated.*

4. Topological free entropy dimension. In this section we show that the topological free entropy dimension of any finite generating set of a unital separable approximately divisible C*-algebra is less than or equal to 1.

4.1. Preliminaries. We are going to recall Voiculescu's definition of topological free entropy dimension of an n-tuple of self-adjoint elements in a unital C*-algebra.

For any element $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$, define the operator norm on $\mathcal{M}_k(\mathbb{C})^n$ by

$$\|(A_1, \ldots, A_n)\| = \max\{\|A_1\|, \ldots, \|A_n\|\}.$$

For every $\omega > 0$, we define the ω-$\|\cdot\|$-ball $\mathrm{Ball}\,(B_1, \ldots, B_n; \omega, \|\cdot\|)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{B}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\| < \omega.$$

Suppose $\mathcal{F}$ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the *covering number* $\nu_\infty(\mathcal{F}, \omega)$ to be the minimal number of ω-$\|\cdot\|$-balls whose union covers $\mathcal{F}$ in $\mathcal{M}_k(\mathbb{C})^n$.

Define $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ to be the unital noncommutative polynomials in the indeterminates $X_1, \ldots, X_n$. Let $\{p_m\}_{m=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ with rational complex coefficients. (Here "rational complex coefficients" means that the real and imaginary parts of all coefficients of p_m are rational numbers).

Suppose $\mathcal{A}$ is a unital C*-algebra, $x_1, \ldots, x_n, y_1, \ldots, y_t$ are self-adjoint elements of $\mathcal{A}$. For any $\omega, \varepsilon > 0$, positive integers k and m, define

$$\begin{aligned}\Gamma_{\mathrm{top}}(x_1, \ldots, x_n; k, \varepsilon, m) = \{&(A_1, \ldots, A_n) \in (\mathcal{M}_k^{sa}(\mathbb{C}))^n : \\ &|\|p_j(A_1, \ldots, A_n)\| - \|p_j(x_1, \ldots, x_n)\|| < \varepsilon, \quad \forall\ 1 \le j \le m\},\end{aligned}$$

and define

$$\nu_\infty(\Gamma_{\mathrm{top}}(x_1, \ldots, x_n; k, \varepsilon, m), \omega)$$

to be the covering number of the set $\Gamma_{\mathrm{top}}(x_1, \ldots, x_n; k, \varepsilon, m)$ by ω-$\|\cdot\|$-balls in the metric space $(\mathcal{M}_k^{sa}(\mathbb{C}))^n$ equipped with operator norm.

Define

$$\delta_{\text{top}}(x_1,\ldots,x_n;\omega) = \inf_{\substack{\varepsilon>0\\ m\in\mathbb{N}}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_{\text{top}}(x_1,\ldots,x_n;k,\varepsilon,m),\omega))}{-k^2\log\omega},$$

and

$$\delta_{\text{top}}(x_1,\ldots,x_n) = \limsup_{\omega\to 0^+} \delta_{\text{top}}(x_1,\ldots,x_n;\omega).$$

Define $\Gamma_{\text{top}}(x_1,\ldots,x_n : y_1,\ldots,y_t;k,\varepsilon,m)$ to be the set of $(A_1,\ldots,A_n) \in (\mathcal{M}_k^{sa}(\mathbb{C}))^n$ such that there is $(B_1,\ldots,B_t) \in (\mathcal{M}_k^{sa}(\mathbb{C}))^t$ satisfying

$$(A_1,\ldots,A_n,B_1,\ldots,B_t) \in \Gamma_{\text{top}}(x_1,\ldots,x_n,y_1,\ldots,y_t;k,\varepsilon,m).$$

Then, similarly, we can define

$$\begin{aligned}&\delta_{\text{top}}(x_1,\ldots,x_n : y_1,\ldots,y_t;\omega)\\ &= \inf_{\varepsilon>0,m\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_{\text{top}}(x_1,\ldots,x_n : y_1,\ldots,y_t;k,\varepsilon,m),\omega))}{-k^2\log\omega};\end{aligned}$$

and

$$\delta_{\text{top}}(x_1,\ldots,x_n : y_1,\ldots,y_t) = \limsup_{\omega\to 0^+} \delta_{\text{top}}(x_1,\ldots,x_n : y_1,\ldots,y_t;\omega).$$

Lemma 4.1. *Suppose $\mathcal{A}$ is a unital C^*-algebra, $x_1,\ldots,x_n$, $y_1,\ldots,y_t$ are self-adjoint elements in $\mathcal{A}$ and $x_1,\ldots,x_n$ generate $\mathcal{A}$. Suppose $p \in \{p_m\}_{m=1}^\infty$ and $\omega > 0$. Then the following are true*:

(1) $\delta_{\text{top}}(x_1,\ldots,x_n;\omega) = \delta_{\text{top}}(x_1,\ldots,x_n : y_1,\ldots,y_t;\omega)$,
(2) $\delta_{\text{top}}(p(x_1,\ldots,x_n) : x_1,\ldots,x_n;\omega) = \delta_{\text{top}}(p(x_1,\ldots,x_n) : x_1,\ldots,x_n,y_1,\ldots,y_t;\omega)$,
(3) $\delta_{\text{top}}(x_1,\ldots,x_n) \geq \delta_{\text{top}}(p(x_1,\ldots,x_n) : x_1,\ldots,x_n)$.

Proof. The proof of (1) and (2) are straightforward adaptations of the proof of Proposition 1.6 in [**9**]. Lemma 4.1 (3) is proved by Hadwin and Shen in [**4**]. □

The following lemma is Lemma 2.3 in [**2**], and it will be used in the proofs of Theorem 4.5 and Theorem 4.8.

Lemma 4.2. *Let $\mathcal{B}$ be a finite-dimensional C*-algebra, which is isomorphic to $\mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C})$. For any $\varepsilon > 0$, there is a $\delta > 0$ such that, whenever $\mathcal{A}$ is a unital separable C*-algebra with the unit $I_{\mathcal{A}}$ and $\{a_{ij}^{(s)} : 1 \le i, j \le k_s, 1 \le s \le r\}$ in $\mathcal{A}$ satisfying*

(1) $\|(a_{ij}^{(s)})^* - a_{ji}^{(s)}\| \le \delta$ *for all* i, j, s,
(2) $\|\sum_{s=1}^{r} \sum_{i=1}^{k_s} a_{ii}^{(s)} - I_{\mathcal{A}}\| \le \delta$,
(3) $\|a_{ij}^{(s)} a_{jj_1}^{(s)} - a_{ij_1}^{(s)}\| \le \delta$ *for all* i, j, j_1, s, $\|a_{ij}^{(s)} a_{i_1 j_1}^{(s_1)}\| \le \delta$ *if* $s \ne s_1$ *or* $j \ne i_1$,

*then there is a set $\{e_{ij}^{(s)} : 1 \le i, j \le k_s, 1 \le s \le r\}$ of matrix units for a copy of $\mathcal{B}$ (i.e., a faithful unital *-homomorphic image of $\mathcal{B}$) in $\mathcal{A}$ satisfying $\|a_{ij}^{(s)} - e_{ij}^{(s)}\| < \varepsilon$ for all i, j, s.*

4.2. Upper bound of topological free entropy dimension in an approximately divisible C*-algebra. The following lemma is Lemma 6 in [**3**].

Lemma 4.3. *The following statements are true*:

(1) *Let $\mathcal{U}_k$ be the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$, $\omega > 0$. Then*
$$\left(\frac{1}{\omega}\right)^{k^2} \le \nu_\infty(\mathcal{U}_k, \omega) \le \left(\frac{9\pi e}{\omega}\right)^{k^2}.$$
(2) *If d is the metric from a norm $\|\cdot\|$ on $\mathbb{R}^m$ and $\mathbb{B}$ is the unit ball of $\mathbb{R}^m$, then for $\omega > 0$,*
$$\left(\frac{1}{\omega}\right)^{m} \le \nu_d(\mathbb{B}, \omega) \le \left(\frac{3}{\omega}\right)^{m}.$$

Let $\mathcal{B}$ be a finite-dimensional C*-algebra which is isomorphic to $\mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C})$ for some positive integers $k_1, \ldots, k_r$. To simplify the notation, we will use $\{e_{st}^{(\iota)}\}_{s,t,\iota}$ to denote a set $\{e_{st}^{(\iota)} : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$ of matrix units for $\mathcal{B}$, let $\{\mathrm{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota}$ denote the set $\{\, e_{st}^{(\iota)} + (e_{st}^{(\iota)})^*/2 : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$, and let $\{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota}$ denote the set $\{\, e_{st}^{(\iota)} - (e_{st}^{(\iota)})^*/2\sqrt{-1} : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$.

Lemma 4.4. *Let $\mathcal{A}$ be a unital separable approximately divisible C^*-algebra with unit $I_{\mathcal{A}}$, and $\{x_1, \ldots, x_n\}$ be a family of self-adjoint generators of $\mathcal{A}$. Then, for any $\omega > 0$ and positive integer N, there exists a finite-dimensional C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ with a set of matrix units $\{e_{st}^{(\iota)}\}_{s,t,\iota} = \{e_{st}^{(\iota)} : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$, a positive integer m_0 and $1 > \varepsilon_0 > 0$, such that*

(1) $I_{\mathcal{A}} \in \mathcal{B}$,
(2) $\operatorname{subrank}(\mathcal{B}) \ge N$,
(3) *for any $m \ge m_0$, $\varepsilon \le \varepsilon_0$, and any $k \ge 1$, if*

$$\begin{aligned}&(A_1, \ldots, A_n, \{B_{st}^{(\iota)}\}_{s,t,\iota}, \{C_{st}^{(\iota)}\}_{s,t,\iota})\\ &\quad \in \Gamma_{\text{top}}(x_1, \ldots, x_n, \{\operatorname{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\operatorname{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m),\end{aligned}$$

then there exists a set $\{P_{st}^{(\iota)} : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$ of matrix units for a copy of $\mathcal{B}$ in $\mathcal{M}_k(\mathbb{C})$ so that

$$\left\| A_j - \sum_{1 \le \iota \le r} \sum_{1 \le s \le k_\iota} P_{ss}^{(\iota)} A_j P_{ss}^{(\iota)} \right\| \le 2\omega.$$

Proof. Suppose $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$, where $\mathcal{A}_m$ is as in Proposition 2.4. For any $\omega > 0$, any positive integer N and self-adjoint elements $x_1, \ldots, x_n$, there are self-adjoint elements $y_1, \ldots, y_n$ in some $\mathcal{A}_m$ such that $\|x_j - y_j\| < \omega/2$ for all $1 \le j \le n$. From part (3) of Proposition 2.4, there exists a finite-dimensional subalgebra $\mathcal{B}$ of $\mathcal{A}_m' \cap \mathcal{A}$ such that $I_{\mathcal{A}} \in \mathcal{B}$ and $\operatorname{subrank}(\mathcal{B}) \ge N$. Let $\{e_{st}^{(\iota)} : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$ be a set of matrix units for $\mathcal{B}$. Then, for $1 \le j \le n$,

$$\begin{aligned}&\left\| x_j - \sum_{1 \le \iota \le r} \sum_{1 \le s \le k_\iota} e_{ss}^{(\iota)} x_j e_{ss}^{(\iota)} \right\|\\ &\quad = \left\| (x_j - y_j) + y_j - \sum_{1 \le \iota \le r} \sum_{1 \le s \le k_\iota} e_{ss}^{(\iota)} (x_j - y_j) e_{ss}^{(\iota)} \right.\\ &\qquad \left. - \sum_{1 \le \iota \le r} \sum_{1 \le s \le k_\iota} e_{ss}^{(\iota)} y_j e_{ss}^{(\iota)} \right\|\\ &\quad = \left\| (x_j - y_j) - \sum_{1 \le \iota \le r} \sum_{1 \le s \le k_\iota} e_{ss}^{(\iota)} (x_j - y_j) e_{ss}^{(\iota)} \right\|\end{aligned}$$

$$\leq \|(x_j - y_j)\| + \left\| \sum_{1\leq\iota\leq r} \sum_{1\leq s\leq k_\iota} e_{ss}^{(\iota)}(x_j - y_j)e_{ss}^{(\iota)} \right\|$$
$$\leq \|(x_j - y_j)\| + \max\{\|e_{ss}^{(\iota)}(x_j - y_j)e_{ss}^{(\iota)}\|\}$$
$$< \frac{\omega}{2} + \frac{\omega}{2} = \omega.$$

Let $R = \max\{\|x_1\|, \ldots, \|x_n\|, 1\}$.

By Lemma 4.2, there are $0 < \varepsilon_0 < \min\{1, \omega/2\}$ and positive integer m_0, such that, for any $m \geq m_0$, $\varepsilon \leq \varepsilon_0$ and $k \geq 1$, if

$$(A_1, \ldots, A_n, \{B_{st}^{(\iota)}\}_{s,t,\iota}, \{C_{st}^{(\iota)}\}_{s,t,\iota}) \in \Gamma_{\text{top}}(x_1, \ldots, x_n, \{\mathrm{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m), \tag{16}$$

then there exists a set $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\} \subset \mathcal{M}_k^{sa}(\mathbb{C})$ such that

(a) $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$ is exactly a set of matrix units for a copy of $\mathcal{B}$ in $\mathcal{M}_k(\mathbb{C})$,
(b) For any $1 \leq \iota \leq r, 1 \leq s, t \leq k_\iota$,
$$\|P_{st}^{(\iota)} - (B_{st}^{(\iota)} + \sqrt{-1} \cdot C_{st}^{(\iota)})\| < \frac{\omega}{24R \cdot \mathrm{rank}\,(\mathcal{B})}.$$

Let $D_{st}^{(\iota)} = B_{st}^{(\iota)} + \sqrt{-1} \cdot C_{st}^{(\iota)}$. We have

$$\left\| A_j - \sum_{1\leq\iota\leq r} \sum_{1\leq s\leq k_\iota} P_{ss}^{(\iota)} A_j P_{ss}^{(\iota)} \right\|$$
$$\leq \left\| A_j - \sum_{1\leq\iota\leq r} \sum_{1\leq s\leq k_\iota} D_{ss}^{(\iota)} A_j D_{ss}^{(\iota)} \right\|$$
$$+ \left\| \sum_{1\leq\iota\leq r} \sum_{1\leq s\leq k_\iota} (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) A_j D_{ss}^{(\iota)} \right\|$$
$$+ \left\| \sum_{1\leq\iota\leq r} \sum_{1\leq s\leq k_\iota} D_{ss}^{(\iota)} A_j (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) \right\|$$
$$+ \left\| \sum_{1\leq\iota\leq r} \sum_{1\leq s\leq k_\iota} (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) A_j (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) \right\|$$
$$\leq \omega + \varepsilon + \frac{\omega}{6} + \frac{\omega}{6} + \frac{\omega}{6} \leq 2\omega.$$

□

Theorem 4.5. *Suppose $\mathcal{A}$ is a unital separable approximately divisible C^*-algebra generated by self-adjoint elements $x_1, \ldots, x_n$. Then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \leq 1.$$

Proof. For any positive integer N, $1 > \omega > 0$, from Lemma 4.4, there exists a finite-dimensional C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ with a set of matrix units $\{e_{st}^{(\iota)}\}_{s,t,\iota} = \{e_{st}^{(\iota)} : 1 \leq s,t \leq k_\iota, 1 \leq \iota \leq r\}$, a positive integer m_0 and $1 > \varepsilon_0 > 0$, such that

(a) $I_{\mathcal{A}} \in \mathcal{B}$, where $I_{\mathcal{A}}$ is the unit of $\mathcal{A}$,
(b) subrank $(\mathcal{B}) \geq N$,
(c) for $m \geq m_0$ and $\varepsilon \leq \varepsilon_0$, and for any $k \geq 1$, if

$$(A_1, \ldots, A_n, \{B_{st}^{(\iota)}\}_{s,t,\iota}, \{C_{st}^{(\iota)}\}_{s,t,\iota}) \in \Gamma_{\text{top}}(x_1, \ldots, x_n, \{\text{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\text{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m), \tag{17}$$

then there exists a set $\{P_{st}^{(\iota)} : 1 \leq s,t \leq k_\iota, 1 \leq \iota \leq r\}$ of matrix units for a copy of $\mathcal{B}$ in $\mathcal{M}_k(\mathbb{C})$ so that

$$\left\| A_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} P_{ss}^{(\iota)} A_j P_{ss}^{(\iota)} \right\| \leq 2\omega.$$

Note that $\{P_{ss}^{(\iota)} : 1 \leq \iota \leq r,\ 1 \leq s \leq k_\iota\}$ is a family of mutually orthogonal projections with the sum I_k in $\mathcal{M}_k(\mathbb{C})$. There is some unitary matrix $U \in \mathcal{U}_k$ such that $U^* P_{ss}^{(\iota)} U (= Q_{ss}^{(\iota)})$ is diagonal for any $1 \leq \iota \leq r$ and $1 \leq s \leq k_\iota$. Then, for any $1 \leq j \leq n$,

$$\left\| A_j - U \left(\sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} (U^* A_j U) Q_{ss}^{(\iota)} \right) U^* \right\| \leq 2\omega. \tag{18}$$

Thus, for $1 \leq j \leq n$, $R = \max\{\|x_1\|, \ldots, \|x_n\|, 1\}$,

$$\left\| \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} (U^* A_j U) Q_{ss}^{(\iota)} \right\| \leq \|A_j\| + 2\omega \leq 4R.$$

Therefore,

$$\Big(\sum_{1\le\iota\le r} \sum_{1\le s\le k_\iota} Q_{ss}^{(\iota)}(U^*A_1U)Q_{ss}^{(\iota)}, \dots, \sum_{1\le\iota\le r} \sum_{1\le s\le k_\iota} Q_{ss}^{(\iota)}(U^*A_nU)Q_{ss}^{(\iota)} \Big) \in \mathrm{Ball}\,(0,\dots,0;4R,\|\cdot\|), \tag{19}$$

i.e., it is contained in the ball centered at $(0,\dots,0)$ with radius $4R$ in $(\mathcal{M}_k(\mathbb{C}))^n$.

Since $\{P_{st}^{(\iota)} : 1 \le s,t \le k_\iota, 1 \le \iota \le r\}$ is a system of matrix units for a copy of $\mathcal{B}$ in $\mathcal{M}_k(\mathbb{C})$ such that

$$\sum_{1\le\iota\le r 1\le s\le k_\iota} P_{ss}^{(\iota)} = I_k,$$

we know that there is a unital embedding from $\mathcal{B}$ into $\mathcal{M}_k(\mathbb{C})$. It follows that there are positive integers $c_1,\dots,c_r$ satisfying

(i) $\mathrm{rank}\, P_{11}^{(\iota)} = \cdots = \mathrm{rank}\, P_{k_\iota,k_\iota}^{(\iota)} = c_\iota$ for all $1 \le \iota \le r$, where $\mathrm{rank}\, T$ is the rank of the matrix T for any T in $\mathcal{M}_k(\mathbb{C})$; and
(ii) $c_1k_1 + \cdots + c_rk_r = k$.

By the restriction on the C*-algebra $\mathcal{B}$ (see condition (b) as above), we know that subrank $(B) \ge N$, i.e.,

$$\min\{k_1,\dots,k_r\} \ge N.$$

By (ii), we obtain that

$$\min\{c_1,\dots,c_r\} \le \frac{k}{N}. \tag{20}$$

By (i) and the fact that $Q_{ss}^{(\iota)} = U^*P_{ss}^{(\iota)}U$, we know that

$$\begin{aligned}\mathrm{rank}\, Q_{11}^{(\iota)} &= \cdots = \mathrm{rank}\, Q_{k_\iota,k_\iota}^{(\iota)} = \mathrm{rank}\, P_{11}^{(\iota)} \\ &= \cdots = \mathrm{rank}\, P_{k_\iota,k_\iota}^{(\iota)} = c_\iota, \quad \text{for } 1 \le \iota \le r.\end{aligned}$$

Thus, the real-dimension of the linear space $\sum_{1\le\iota\le r}\sum_{1\le j\le k_\iota} Q_{jj}^{(\iota)}$

$\mathcal{M}_k(\mathbb{C})^{sa}Q_{jj}^{(\iota)}$ is

$$\dim_{\mathbb{R}}\left(\sum_{1\le\iota\le r}\sum_{1\le j\le k_\iota}Q_{jj}^{(\iota)}\mathcal{M}_k(\mathbb{C})^{sa}Q_{jj}^{(\iota)}\right)=c_1^2k_1+\cdots+c_r^2k_r. \tag{21}$$

By inequality (4.4), we get

$$c_1^2k_1+\cdots+c_r^2k_r\le\frac{k}{N}(c_1k_1+\cdots+c_rk_r)=\frac{k^2}{N}. \tag{22}$$

For any such family of positive integers $c_1,\ldots,c_r$ with $c_1k_1+\cdots+c_rk_r=k$, and the family of mutually orthogonal diagonal projections $\{Q_{ss}^{(\iota)}\}_{1\le s\le k_\iota,1\le\iota\le r}$ with

$$\operatorname{rank}(Q_{ss}^{(\iota)})=c_\iota,\quad \text{for all } 1\le\iota\le r,$$

we define

$$\Omega(\{Q_{ss}^{(\iota)}\}_{s,\iota})=\left\{\left(\sum_{1\le\iota\le r}\sum_{1\le s\le k_\iota}Q_{ss}^{(\iota)}T_1Q_{ss}^{(\iota)},\ldots,\sum_{1\le\iota\le r}\sum_{1\le s\le k_\iota}Q_{ss}^{(\iota)}T_nQ_{ss}^{(\iota)}\right):\right.$$
$$\left. T_i=T_i^*\in\mathcal{M}_k(\mathbb{C}),\quad \text{for all } 1\le i\le n\right\},$$

which is a subset of $(\mathcal{M}_k^{sa}(\mathbb{C}))^n$.

Therefore, combining part (2) of Lemma 4.3, equation (4.5) and inequality (4.6), for any $\omega>0$, we have

$$\nu_\infty(\Omega(\{Q_{ss}^{(\iota)}\}_{s,\iota})\cap\operatorname{Ball}(0,\ldots,0;4R,\|\cdot\|);\omega)\le\left(\frac{12R}{\omega}\right)^{nk^2/N}. \tag{23}$$

Let

$$\Lambda=\{(c_1,\ldots,c_r):\exists k_1,\ldots,k_r\in\mathbb{N}\text{ such that } c_1k_1+\cdots+c_rk_r=k\}.$$

By inequality (4.4), we know that the cardinality of the set Λ satisfies

$$\operatorname{card}(\Lambda)\le\left(\frac{k}{N}\right)^r. \tag{24}$$

Let

$$\Omega = \cup_{(c_1,\ldots,c_r)\in\Lambda} \Big\{ \Omega(\{Q_{ss}^{(\iota)}\}_{s,\iota}) \mid \{Q_{ss}^{(\iota)}\}_{1\le s\le k_\iota, 1\le \iota\le r} \text{ is a family of mutually orthogonal diagonal projections with rank}\,(Q_{ss}^{(\iota)}) = c_\iota, \quad \forall\, 1 \le \iota \le r \Big\}.$$

By inequalities (4.7) and (4.8), we know that

$$\nu_\infty(\Omega \cap \mathrm{Ball}\,(0,\ldots,0;4R,\|\ \|);\omega) \le \left(\frac{12R}{\omega}\right)^{nk^2/N} \cdot \left(\frac{k}{N}\right)^r. \tag{25}$$

Based on (4.1), (4.2), (4.3), (4.9), and part (2) of Lemma 4.3, now it is a standard argument to show that

$$\begin{aligned}
\delta_{\mathrm{top}}(x_1,\ldots,x_n : \{\mathrm{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota}; 4\omega)& \\
\le \limsup_{k\to\infty} \frac{\log\Big((k/N)^r(12R/\omega)^{nk^2/N}(9\pi e/\omega)^{k^2}\Big)}{-k^2\log(4\omega)}& \\
= \frac{-(1+n/N)\log\omega + n/N\log(12R) + \log(9\pi e)}{-\log 4\omega}&.
\end{aligned}$$

By part (1) of Lemma 4.1,

$$\delta_{\mathrm{top}}(x_1,\ldots,x_n;4\omega) \le \frac{-(1+n/N)\log\omega + n/N\log(12R) + \log(9\pi e)}{-\log 4\omega}.$$

Therefore,

$$\delta_{\mathrm{top}}(x_1,\ldots,x_n) = \limsup_{\omega\to 0^+} \delta_{\mathrm{top}}(x_1,\ldots,x_n;4\omega) \le 1 + \frac{n}{N}.$$

Since N is arbitrarily large, $\delta_{\mathrm{top}}(x_1,\ldots,x_n) \le 1$. □

4.3. Lower bound of topological free entropy dimension of an approximately divisible C*-algebra. Using the idea in the proof of Lemma 5.3.5 in [**4**], we can prove the following lemma.

Lemma 4.6. *Suppose* $m_1, m_2, \ldots, m_r$ *is a family of positive integers with summation* m *and* $m_1,\ldots,m_r \ge N$ *for some positive integer* N*. Suppose* $k_1,\ldots,k_m$ *is a family of positive integers with summation* k *and, for every* $1 \le s \le r$, $k_{m_1+\cdots+m_{s-1}+1} = \cdots = k_{m_1+\cdots+m_s}$ $(m_0 = 0)$.

If $A = A^* \in \mathcal{M}_k(\mathbb{C})$, and for some $U \in \mathcal{U}_k$,

$$\left\| A - U \begin{pmatrix} 1 \cdot I_{k_1} & 0 & \cdots & 0 \\ 0 & 2 \cdot I_{k_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & m \cdot I_{k_m} \end{pmatrix} U^* \right\| \leq \frac{2}{N^3},$$

then, for any $\omega > 0$, we have

$$\nu_\infty(\Omega(A), \omega) \geq (8C_1\omega)^{-k^2} \left(\frac{2C}{\omega} \right)^{-56k^2/N},$$

for some constants $C_1, C > 1$ independent of k, ω, where

$$\Omega(A) = \{W^*AW : W \in \mathcal{U}_k\}.$$

Suppose $\mathcal{A}$ is a unital C*-algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements of $\mathcal{A}$ that generates $\mathcal{A}$ as a C*-algebra. In the definition of topological free entropy dimension, it requires that the "Γ-set" $\Gamma_{\text{top}}(x_1, \ldots, x_n)$ is "eventually" nonempty, more specifically, for any $m \in \mathbb{N}$, $\varepsilon > 0$, there is a sequence of positive integers $k_1 < k_2 < \cdots$ such that, for $s \geq 1$,

$$\Gamma_{\text{top}}(x_1, \ldots, x_n : y_1, \ldots, y_t; k_s, \varepsilon, m) \neq \emptyset.$$

Actually, this requirement is equivalent to saying that $\mathcal{A}$ is an MF-algebra (see [**5**]). The notion of MF algebra was introduced by Blackadar and Kirchberg [**1**]. A separable C*-algebra is an MF-algebra if and only if it can be embedded into $\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $n_k, k = 1, 2, \ldots$.

Now we are ready to prove the main theorem in this subsection.

Theorem 4.7. *Let $\mathcal{A}$ be a unital separable approximately divisible C*-algebra generated by self-adjoint elements $x_1, \ldots, x_n$. If $\mathcal{A}$ is an MF-algebra, then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geq 1.$$

Proof. For any positive integer N, by part (3) of Proposition 2.4, there is a finite-dimensional C*-subalgebra $\mathcal{B}$ containing the unit of $\mathcal{A}$ with $\operatorname{subrank}(\mathcal{B}) \geq N$. Therefore, there are positive integers

$r, k_1, \dots, k_r$ such that

$$\mathcal{B} \simeq \mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C}).$$

Let $\{e_{st}^{(\iota)} : 1 \le \iota \le r, 1 \le s, t \le k_\iota\}$ be a system of matrix units for $\mathcal{B}$. Let

$$z_N = \sum_{\iota=1}^{r} \sum_{s=1}^{k_\iota} \left(s + \sum_{j=1}^{\iota-1} k_j \right) \cdot e_{ss}^{(\iota)}.$$

Let $\{p_m\}_{m=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n\rangle$ with rational complex coefficients. Note that $\{p_m(x_1, \dots, x_n)\}_{m=1}^{\infty}$ is a norm-dense set in $\mathcal{A}$. Thus, there exists a polynomial $p_{m_N} \in \{p_m\}_{m=1}^{\infty}$ such that $p_{m_N}(x_1, \dots, x_n)$ is self-adjoint and $\|p_{m_N}(x_1, \dots, x_n) - z_N\| \le 1/N^3$.

For sufficiently small $\varepsilon > 0$ and sufficiently large positive integers m and k, if

$$\begin{aligned}(B, A_1, \dots, A_n, \{C_{st}^{(\iota)}\}_{s,t,\iota}, \{D_{st}^{(\iota)}\}_{s,t,\iota}) \\ \in \Gamma_{\text{top}}(p_{m_N}(x_1, \dots, x_n), x_1, \dots, x_n, \{\mathrm{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota}, \\ \{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m),\end{aligned}$$

then, by Lemma 4.2, there exists a set $\{P_{st}^{(\iota)} : 1 \le s, t \le k_\iota, 1 \le \iota \le r\}$ of matrix units for a copy of $\mathcal{B}$ in $\mathcal{M}_k(\mathbb{C})$, such that

$$\left\| B - \sum_{\iota=1}^{r} \sum_{s=1}^{k_\iota} \left(s + \sum_{j=1}^{\iota-1} k_j \right) \cdot P_{ss}^{(\iota)} \right\| \le \frac{2}{N^3}.$$

Let U be a unitary matrix in $\mathcal{M}_k(\mathbb{C})$ such that, for any $1 \le s \le k_\iota$ and $1 \le \iota \le r$, $U^* P_{ss}^{(\iota)} U (= Q_{ss}^{(\iota)})$ is diagonal. Then, from the preceding inequality,

$$\left\| B - U \left(\sum_{\iota=1}^{r} \sum_{s=1}^{k_\iota} \left(s + \sum_{j=1}^{\iota-1} k_j \right) \cdot Q_{ss}^{(\iota)} \right) U^* \right\| \le \frac{2}{N^3}.$$

From Lemma 4.6, for any $\omega > 0$, when m is large enough and ε is small enough, there are some constants $C, C_1 > 1$ independent of k and

ω, such that

$$\nu_\infty(\Gamma_{\text{top}}(p_{m_N}(x_1,\ldots,x_n):x_1,\ldots,x_n, \{\mathrm{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota},\{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota};k,\varepsilon,m),\omega) \geq (8C_1\omega)^{-k^2}\left(\frac{2C}{\omega}\right)^{-56k^2/N}.$$

Therefore,

$$\delta_{\text{top}}(p_{m_N}(x_1,\ldots,x_n):x_1,\ldots,x_n,\{\mathrm{Re}(e_{st}^{(\iota)})\}_{s,t,\iota},\{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota})\geq 1-\frac{56}{N}.$$

By Lemma 4.1,

$$\begin{aligned}\delta_{\text{top}}(p_{m_N}(x_1,\ldots,x_n)&:x_1,\ldots,x_n,\{\mathrm{Re}\,(e_{st}^{(\iota)})\}_{s,t,\iota},\{\mathrm{Im}\,(e_{st}^{(\iota)})\}_{s,t,\iota})\\ &=\delta_{\text{top}}(p_{m_N}(x_1,\ldots,x_n):x_1,\ldots,x_n)\\ &\leq\delta_{\text{top}}(x_1,\ldots,x_n),\end{aligned}$$

whence $\delta_{\text{top}}(x_1,\ldots,x_n)\geq 1-56/N$. Since N is an arbitrary positive integer, we obtain

$$\delta_{\text{top}}(x_1,\ldots,x_n)\geq 1. \qquad \square$$

Combining Theorem 3.1, Theorem 4.1 and Theorem 4.2, we have the following result.

Theorem 4.8. *Let $\mathcal{A}$ be a unital separable approximately divisible C^*-algebra. If $\mathcal{A}$ is an MF-algebra, then*

$$\delta_{\text{top}}(x_1,\ldots,x_n)=1,$$

where $x_1,\ldots,x_n$ is any family of self-adjoint generators of $\mathcal{A}$.

Note. The original version of this paper was included in the first author's dissertation and **[6]**.

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