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COEFFICIENT INEQUALITY FOR CERTAIN p-VALENT ANALYTIC FUNCTIONS

D. VAMSHEE KRISHNA AND T. RAMREDDY

ABSTRACT. The objective of this paper is to obtain an upper bound to the second Hankel determinant $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for certain p-valent analytic functions, using Toeplitz determinants.

1. Introduction. Let A_p (p is a fixed integer ≥ 1) denote the class of functions f of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots , \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in N = \{1, 2, 3, \ldots\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [**23, 24**] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by several authors in the literature. For example, Noonan and Thomas [**19**] studied about the second Hankel determinant of really mean p-valent functions. Noor [**20**] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [**5**] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [**10**]. One can

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The first author is the corresponding author.

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easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [**2**] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. The Hankel determinant for the function f when $q = 2$ and $n = 2$, known as the second Hankel determinant, is given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus [**9**] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [**13**]. In their work, they have shown that, if $f \in RT$, then $|a_2a_4 - a_3^2| \leq 4/9$. Further, Janteng, Halim and Darus [**8**] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq 1/8$, respectively. Mishra and Gochhayat [**15**] obtained the sharp bound to the non-linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)$ $(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \pi/2)$, defined as $\operatorname{Re}[e^{i\alpha}\Omega_z^\lambda f(z)/z] > \rho \cos \alpha$, using the fractional differential operator denoted by Ω_z^λ, defined by Owa and Srivastava [**21**] and have shown that, if $f \in R_\lambda(\alpha, \rho)$, then $|a_2a_4 - a_3^2| \leq \{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\alpha/9\}$.

Similarly, the same coefficient inequality was calculated for certain subclasses of univalent and multivalent analytic functions by many authors ([**1, 3, 14, 16–18, 25**]).

Motivated by the above-mentioned results obtained by different authors in this direction, in the present paper, we consider the Hankel determinant in the case of $q = 2$ and $n = p + 1$, denoted by $H_2(p + 1)$, given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2. \tag{1.4}$$

Further, we seek an upper bound to the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function f belonging to certain subclasses of p-valent analytic functions, defined as follows.

Definition 1.1. A function $f(z) \in A_p$ is said to be in the class $RT_{b,p}$, where b is a non-zero real number with $p \in N$, if it satisfies the condition

$$\text{Re}\left[1+\frac{1}{b}\left(\frac{1}{p}\frac{f'(z)}{z^{p-1}}-1\right)\right]>0, \quad \text{for all } z \in E. \tag{1.5}$$

It is observed that choosing $b = 1$, we get $RT_{b,p} = RT_{1,p}$, a class consisting of p-valent functions, whose derivative has a positive real part and for the choice of $b = 1$ and $p = 1$, we obtain $RT_{b,p} = RT$.

Definition 1.2. A function $f(z) \in A_p$ is said to be in the class $ST_{b,p}$, where $b \neq 0$ is a real number with $p \in N$, if it satisfies the condition

$$\text{Re}\left[1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f'(z)}-1\right)\right]>0, \quad \text{for all } z \in E. \tag{1.6}$$

It is observed that, choosing $b = 1$, we get $ST_{b,p} = ST_{1,p}$, a class consisting of p-valent starlike functions, defined and studied by Goodman [**6**] and, for the choice of $b = 1$ and $p = 1$, we obtain $ST_{b,p} = ST$.

Definition 1.3. A function $f(z) \in A_p$ is said to be in the class $CV_{b,p}$, where b is a non-zero real number with $p \in N$, if it satisfies the condition

$$\text{Re}\left[1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0, \quad \text{for all } z \in E. \tag{1.7}$$

It is observed that, for the choice of $b = 1$, we get $CV_{b,p} = CV_{1,p}$, a class consisting of p-valent convex functions and for choosing $b = 1$ and $p = 1$, we obtain $CV_{b,p} = CV$.

Some preliminary lemmas required for proving our results are as follows.

2. Preliminary results.

Let $\mathscr{P}$ denote the class of functions

$$p(z) = (1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots) = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \tag{2.1}$$

for all $z \in E$,

which are regular in E and satisfy $\operatorname{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called as Carathéodory function [**4**].

Lemma 2.1. [**22, 26**] *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $(1+z)/(1-z)$.*

Lemma 2.2. [**7**] *The power series for p given in (2.1) converges in the unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3 \ldots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case, $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [**7**] is due to Caratheodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$, respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2\operatorname{Re}[c_1^2 c_2] - 2|c_2|^2 - 4|c_1|^2] \geq 0,$$

it is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \quad \text{for some } x, |x| \leq 1. \tag{2.2}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} \tag{2.4}$$
for some z, with $|z| \leq 1$.

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [**11, 12**], used by several authors in the literature.

3. Main results.

Theorem 3.1. *If $f(z) \in RT_{b,p}$ ($b \neq 0$ is a real number) with $p \in N$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2bp}{(p+2)}\right]^2$$

and the inequality is sharp.

Proof. Since $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in RT_{b,p}$, from Definition 1.1, there exists an analytic function $p \in \mathscr{P}$ in the unit disc E with $p(0) = 1$ and $\operatorname{Re}[p(z)] > 0$ such that

$$\left[1 + \frac{1}{b}\left(\frac{1}{p}\frac{f'(z)}{z^{p-1}} - 1\right)\right] = p(z) \tag{3.1}$$
$$\Longleftrightarrow \left\{(b-1)pz^{p-1} + f'(z)\right\} = bp \times \left\{z^{p-1}p(z)\right\}.$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$\left[(b-1)pz^{p-1} + \left\{pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right\}\right]$$
$$= bp \times \left[z^{p-1}\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right].$$

Upon simplification, we obtain

$$[(p+1)a_{p+1}z^p + (p+2)a_{p+2}z^{p+1} + (p+3)a_{p+3}z^{p+2} + \cdots] \tag{3.2}$$
$$= bp[c_1 z^p + c_2 z^{p+1} + c_3 z^{p+2} + \cdots].$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2} respectively in (3.2), we get

$$a_{p+1} = \frac{bpc_1}{(p+1)}; \qquad a_{p+2} = \frac{bpc_2}{(p+2)}; \qquad a_{p+3} = \frac{bpc_3}{(p+3)}. \tag{3.3}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.3) in the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in RT_{b,p}$, after simplifying, we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{b^2p^2}{(p+1)(p+2)^2(p+3)} \times \left|(p+2)^2c_1c_3 - (p+1)(p+3)c_2^2\right|. \tag{3.4}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4), respectively, from Lemma 2.2 on the right-hand side of (3.4), we have

$$\begin{aligned}\left|(p+2)^2c_1c_3 - (p+1)(p+3)c_2^2\right| &= \Big|(p+2)^2c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\ &\quad + 2(4-c_1^2)(1-|x|^2)z\} - (p+1)(p+3) \\ &\quad \times \frac{1}{4}\left\{c_1^2 + x(4-c_1^2)\right\}^2\Big|.\end{aligned}$$

Using the facts $|z| < 1$ and $|xa + yb| \leq |x||a| + |y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$\begin{aligned}4\left|(p+2)^2c_1c_3 - (p+1)(p+3)c_2^2\right| &\leq |c_1^4 + 2(p+2)^2c_1(4-c_1^2) \\ &\quad + 2c_1^2(4-c_1^2)|x| \\ &\quad - \left\{c_1^2 + 2(p+2)^2c_1 + 4(p+1)(p+3)\right\}(4-c_1^2)|x|^2|.\end{aligned} \tag{3.5}$$

Consider

$$\begin{aligned}&\left\{c_1^2+2(p+2)^2c_1+4(p+1)(p+3)\right\}\\ &\quad=\left[\left\{c_1+(p+2)^2\right\}^2-(p+2)^4+4(p+1)(p+3)\right]\\ &\quad=\left[\left\{c_1+(p+2)^2\right\}^2-\left(\sqrt{p^4+8p^3+20p^2+16p+4}\right)^2\right]\\ &\quad=\left[c_1+\left\{(p+2)^2+\left(\sqrt{p^4+8p^3+20p^2+16p+4}\right)\right\}\right]\times\\ &\qquad\left[c_1+\left\{(p+2)^2-\left(\sqrt{p^4+8p^3+20p^2+16p+4}\right)\right\}\right].\end{aligned}$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b\geq 0$, on the right-hand side of the above expression, on simplifying, we get

$$\begin{aligned}(3.6)\quad &-\left\{c_1^2+2(p+2)^2c_1+4(p+1)(p+3)\right\}\\ &\qquad\leq-\left\{c_1^2-2(p+2)^2c_1+4(p+1)(p+3)\right\}\end{aligned}$$

From expressions (3.5) and (3.6), we obtain

$$\begin{aligned}4\left|(p+2)^2c_1c_3-(p+1)(p+3)c_2^2\right|&\leq|c_1^4+2(p+2)^2c_1(4-c_1^2)\\ &\quad+2c_1^2(4-c_1^2)|x|-\left\{c_1^2-2(p+2)^2c_1\right.\\ &\quad\left.+4(p+1)(p+3)\right\}(4-c_1^2)|x|^2|.\end{aligned}$$

Choosing $c_1=c\in[0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we get

$$\begin{aligned}(3.7)\quad &4\left|(p+2)^2c_1c_3-(p+1)(p+3)c_2^2\right|\leq\left[c^4+2(p+2)^2c(4-c^2)+\right.\\ &\quad\left.2c^2(4-c^2)\mu+\left\{c^2-2(p+2)^2c+4(p+1)(p+3)\right\}(4-c^2)\mu^2\right].\\ &\quad=F(c,\mu),\quad 0\leq\mu=|x|\leq 1\text{ and }0\leq c\leq 2,\end{aligned}$$

where

$$\begin{aligned}(3.8)\quad F(c,\mu)=&\left[c^4+2(p+2)^2c(4-c^2)+2c^2(4-c^2)\mu+\right.\\ &\left.\left\{c^2-2(p+2)^2c+4(p+1)(p+3)\right\}(4-c^2)\mu^2\right].\end{aligned}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ given in (3.8) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = \left[2c^2 + 2\left\{c^2 - 2(p+2)^2c + 4(p+1)(p+3)\right\}\mu\right] \times (4-c^2). \tag{3.9}$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.9), we observe that $\partial F/\partial \mu > 0$. Therefore, $F(c, \mu)$ becomes an increasing function of μ, and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Moreover, for a fixed $c \in [0, 2]$, we have

$$\max_{0 \le \mu \le 1} F(c, \mu) = F(c, 1) = G(c). \tag{3.10}$$

Therefore, replacing μ by 1 in $F(c, \mu)$, upon simplification, we obtain

$$G(c) = -2c^4 - 4p(p+4)c^2 + 16(p+1)(p+3). \tag{3.11}$$

$$G'(c) = -8c\left\{c^2 + p(p+4)\right\}. \tag{3.12}$$

From (3.12), we observe that $G'(c) \le 0$, for every $c \in [0, 2]$ with $p \in N$. Therefore, $G(c)$ is a decreasing function of c in the interval $[0, 2]$, whose maximum value occurs at $c = 0$. From (3.11), we obtain G-maximum at $c = 0$, given by

$$G_{\max} = G(0) = 16(p+1)(p+3). \tag{3.13}$$

From relations (3.7) and (3.13), after simplifying, we get

$$\left|(p+2)^2 c_1 c_3 - (p+1)(p+3)c_2^2\right| \le 4(p+1)(p+3). \tag{3.14}$$

Simplifying the relations (3.4) and (3.14), we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \left[\frac{2bp}{(p+2)}\right]^2. \tag{3.15}$$

By setting $c_1 = c = 0$ and selecting $x = -1$ in expressions (2.2) and (2.4), we find that $c_2 = -2$ and $c_3 = 0$, respectively. Using these values in (3.14), we observe that equality is attained, which shows that our result is sharp. This completes the proof of Theorem 3.1. □

Remark 3.2. Choosing $b = 1$, we get $RT_{b,p} = RT_{1,p}$. From (3.15), we have

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \left[\frac{2p}{(p+2)}\right]^2.$$

Remark 3.3. For the choice of $b = 1$ and $p = 1$, we get $RT_{b,p} = RT$. From (3.15), we obtain $|a_2 a_4 - a_3^2| \leq 4/9$. This inequality is sharp and the result coincides with that of Janteng, Halim and Darus [**9**].

Theorem 3.4. *If $f(z) \in ST_{b,p}(b \geq 1/(2p))$ with $p \in N$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq [bp]^2\,,$$

and the inequality is sharp.

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be in the class $ST_{b,p}$, and, from Definition 1.2, there exists an analytic function $p \in \mathscr{P}$ in the unit disc E with $p(0) = 1$ and $\operatorname{Re}[p(z)] > 0$ such that

$$(3.16) \quad \left[1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right] = p(z)$$
$$\implies \{(b-1)pf(z) + zf'(z)\} = bp \times \{f(z) \times p(z)\}\,.$$

Replacing $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.16), we have

$$\left[(b-1)p\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\} + z\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}\right\}\right]$$
$$= bp \times \left[\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\} \times \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right].$$

After simplifying, we get

$$(3.17) \quad [a_{p+1}z^p + 2a_{p+2}z^{p+1} + 3a_{p+3}z^{p+2} + \cdots]$$
$$= bp \times [c_1 z^p + (c_2 + c_1 a_{p+1})\, z^{p+1} + (c_3 + c_2 a_{p+1} + c_1 a_{p+2})\, z^{p+2} + \cdots].$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2}, respectively, on both sides of (3.17), upon simplification, we obtain

$$(3.18) \qquad a_{p+1} = bpc_1; \qquad a_{p+2} = \frac{bp}{2}\left\{c_2 + bpc_1^2\right\};$$
$$a_{p+3} = \frac{bp}{6}\left\{2c_3 + 3bpc_1c_2 + b^2p^2c_1^3\right\}.$$

Considering the second Hankel functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in ST_{b,p}$ and substituting the values of a_{p+1}, a_{p+2} and a_{p+3}

from the relation (3.18), after simplifying, we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{b^2p^2}{12}\left|4c_1c_3 - 3c_2^2 - p^2b^2c_1^4\right|. \tag{3.19}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4), respectively, from Lemma 2.2 on the right-hand side of (3.19), applying the same procedure as described in Theorem 3.1, upon simplification, we obtain

$$\begin{aligned} 4\left|4c_1c_3 - 3c_2^2 - p^2b^2c_1^4\right| \le |\left(1-4b^2p^2\right)c_1^4 + 8c_1(4-c_1^2) + 2c_1^2(4-c_1^2)|x| \\ -(c_1+2)(c_1+6)(4-c_1^2)|x|^2|. \end{aligned} \tag{3.20}$$

Choosing $c_1 = c \in [0,2]$, applying the same procedure as described in Theorem 3.1 and replacing $|x|$ by μ on the right-hand side of (3.20), we obtain

$$\begin{aligned} 4\left|4c_1c_3 - 3c_2^2 - p^2b^2c_1^4\right| \le [\left(4b^2p^2-1\right)c^4 + 8c(4-c^2) + 2c^2(4-c^2)\mu \\ +(c-2)(c-6)(4-c^2)\mu^2] = F(c,\mu), \quad \text{for } 0 \le \mu = |x| \le 1, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} F(c,\mu) = [\left(4b^2p^2-1\right)c^4 + 8c(4-c^2) + 2c^2(4-c^2)\mu \\ +(c-2)(c-6)(4-c^2)\mu^2]. \end{aligned} \tag{3.22}$$

Applying the same procedure as described in Theorem 3.1, differentiating $F(c,\mu)$ in (3.22) partially with respect to μ, for $0 < \mu < 1$ and for fixed c with $0 < c < 2$, we observe that

$$\frac{\partial F}{\partial \mu} = \left\{2c^2 + 2(c-2)(c-6)\mu\right\} \times (4-c^2) > 0. \tag{3.23}$$

Further, for fixed $c \in [0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c). \tag{3.24}$$

From equations (3.22) and (3.24), after simplifying, we get

$$G(c) = 4\left(b^2p^2-1\right)c^4 + 48, \tag{3.25}$$

$$G'(c) = 16\left(b^2p^2-1\right)c^3. \tag{3.26}$$

From expression (3.26), we observe that $G'(c) \leq 0$, for every $c \in [0,2]$ and, for fixed b, where $(b \geq 1/(2p))$ with $p \in N$, which shows that $G(c)$ is a monotonically decreasing function of c in the interval $[0,2]$ and hence its maximum value occurs at $c = 0$ only. From expression (3.25), we obtain

$$\max_{0\leq c\leq 2} G(c) = G(0) = 48. \tag{3.27}$$

From (3.21) and (3.27), after simplifying, we get

$$\left|4c_1c_3 - 3c_2^2 - p^2b^2c_1^4\right| \leq 12. \tag{3.28}$$

Upon simplifying expressions (3.19) and (3.28), we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq [bp]^2\,. \tag{3.29}$$

By setting $c_1 = c = 0$ and selecting $x = 1$ in expressions (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$, respectively. Using these values in (3.28), we observe that equality is attained, which shows that our result is sharp. This completes the proof of Theorem 3.4. □

Remark 3.5. Choosing $b = 1$, we get $ST_{b,p} = ST_{1,p}$, for which, from (3.29), we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq p^2.$$

Remark 3.6. For the choice of $b = 1$ and $p = 1$, we get $ST_{b,p} = ST$. From (3.29), we obtain $|a_2a_4 - a_3^2| \leq 1$. This inequality is sharp, and the result coincides with that of Janteng, Halim and Darus [**8**].

Theorem 3.7. *If $f(z) \in CV_{b,p}$ $(b \geq 1/(2p))$ with $p \in N$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{b^2p^4\left[6(bp+1)^2+(p+1)(p+3)\left\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\right\}\right]}{(p+1)(p+2)^2(p+3)\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\}}\right].$$

Proof. Let $f(z) \in z^p + \sum_{n=p+1}^{\infty} a_n z^n \in CV_{b,p}$. From Definition 1.3, there exists an analytic function $p \in \mathscr{P}$ in the unit disc E with $p(0) = 1$

and $\operatorname{Re}\{p(z)\} > 0$ such that

$$\begin{aligned}(3.30)\quad \operatorname{Re}\left[1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] &= p(z)\\ \Longrightarrow [\{(b-1)p+1\}f'(z)+zf''(z)] &= bp\times[f'(z)\times p(z)].\end{aligned}$$

Substituting the equivalent expressions for $f'(z)$, $f''(z)$ and $p(z)$ in series in the expression (3.30), we have

$$\begin{aligned}&\left[\{(b-1)p+1\}\left\{pz^{p-1}+\sum_{n=p+1}^{\infty}na_nz^{n-1}\right\}\right.\\ &\qquad\left.+z\left\{p(p-1)z^{p-2}+\sum_{n=p+1}^{\infty}n(n-1)a_nz^{n-2}\right\}\right]\\ &\qquad=\left[bp\left\{pz^{p-1}+\sum_{n=p+1}^{\infty}na_nz^{n-1}\right\}\times\left\{1+\sum_{n=1}^{\infty}c_nz^n\right\}\right].\end{aligned}$$

Upon simplification, we obtain

$$\begin{aligned}(3.31)\quad &[(p+1)a_{p+1}z^{p-1}+2(p+2)a_{p+2}z^{p}+3(p+3)a_{p+3}z^{p+1}+\cdots]\\ &\quad= bp\times[pc_1z^{p-1}+\{pc_2+(p+1)c_1a_{p+1}\}z^{p}\\ &\qquad+\{pc_3+(p+1)c_2a_{p+1}+(p+2)c_1a_{p+2}\}z^{p+1}+\cdots].\end{aligned}$$

Equating the coefficients of like powers of z^{p-1}, z^p and z^{p+1}, respectively, on both sides of (3.31), after simplifying, we get

$$(3.32)\quad \begin{aligned}a_{p+1} &= \frac{bp^2}{(p+1)}c_1;\\ a_{p+2} &= \frac{bp^2}{2(p+2)}\left\{c_2+bpc_1^2\right\};\\ a_{p+3} &= \frac{bp^2}{6(p+3)}\left\{2c_3+3bpc_1c_2+b^2p^2c_1^3\right\}.\end{aligned}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.32) in the functional $|a_{p+1}a_{p+3}-a_{p+2}^2|$ for the function $f\in CV_{b,p}$, upon simplification, we obtain

$$|a_{p+1}a_{p+3}-a_{p+2}^2| = \frac{b^2p^4}{12(p+1)(p+2)^2(p+3)}\left|4(p+2)^2c_1c_3+6bpc_1^2c_2 -3(p+1)(p+3)c_2^2-(p^2+4p+1)b^2p^2c_1^4\right|.$$

The above expression is equivalent to

$$|a_{p+1}a_{p+3}-a_{p+2}^2| = \frac{b^2p^4}{12(p+1)(p+2)^2(p+3)}\times \left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|, \tag{3.33}$$

where

$$d_1=4(p+2)^2;\qquad d_2=6bp;\qquad d_3=-3(p+1)(p+3); \qquad d_4=-(p^2+4p+1)b^2p^2. \tag{3.34}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4), respectively, from Lemma 2.2 on the right-hand side of (3.33), we have

$$|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| = |d_1c_1\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2c_1^2\times\frac{1}{2}\{c_1^2+x(4-c_1^2)\} + d_3\times\frac{1}{4}\{c_1^2+x(4-c_1^2)\}^2+d_4c_1^4|.$$

Using the facts $|z|<1$ and $|xa+yb|\le|x||a|+|y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \le |(d_1+2d_2+d_3+4d_4)c_1^4+2d_1c_1(4-c_1^2) + 2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}(4-c_1^2)|x|^2|. \tag{3.35}$$

Using the values of d_1, d_2, d_3 and d_4 from (3.34), upon simplification, we obtain

$$(d_1+2d_2+d_3+4d_4) = \left\{-4b^2p^2(p^2+4p+1)+12bp+(p^2+4p+7)\right\}; \qquad d_1=4(p+2)^2;\qquad (d_1+d_2+d_3)=\left\{p^2+(6b+4)p+7\right\}. \tag{3.36}$$

$$\begin{aligned}(3.37)\quad &\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}\\ &\qquad=\left\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\}.\end{aligned}$$

Considering the expression on the right-hand side of (3.37), we have

$$\begin{aligned}&\left\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\}\\ &\quad=(p^2+4p+7)\times\left[c_1^2+\frac{8(p+2)^2}{(p^2+4p+7)}c_1+\frac{12(p+1)(p+3)}{(p^2+4p+7)}\right].\end{aligned}$$

$$\begin{aligned}&=(p^2+4p+7)\times\\ &\left[\left\{c_1+\frac{4(p+2)^2}{(p^2+4p+7)}\right\}^2-\frac{16(p+2)^4}{(p^2+4p+7)^2}+\frac{12(p+1)(p+3)}{(p^2+4p+7)}\right].\end{aligned}$$

Upon simplification, the above expression can also be expressed as

$$\begin{aligned}&\left\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\}=(p^2+4p+7)\\ &\times\left[\left\{c_1+\frac{4(p+2)^2}{(p^2+4p+7)}\right\}^2-\left\{\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}^2\right].\end{aligned}$$

$$\begin{aligned}(3.38)\quad &\left\{(p^2+4p+1)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\}\\ &\qquad=(p^2+4p+7)\times\\ &\left[c_1+\left\{\frac{4(p+2)^2}{(p^2+4p+7)}+\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}\right]\\ &\times\left[c_1+\left\{\frac{4(p+2)^2}{(p^2+4p+7)}-\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}\right].\end{aligned}$$

Applying the same procedure as described in Theorem 3.1, from expressions (3.37) and (3.38), we obtain

$$\begin{aligned}(3.39)\quad &-\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}\\ &\qquad\leq-\left\{(p^2+4p+1)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\right\}.\end{aligned}$$

Substituting the calculated values from (3.36) and (3.39) on the right-hand side of the relation (3.35), we have

$$\begin{aligned}(3.40)\quad & 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\ & \le |\left\{-4b^2p^2(p^2+4p+1) + 12bp + (p^2+4p+7)\right\} c_1^4 \\ & + 8(p+2)^2c_1(4-c_1^2) + 2\left\{p^2 + (6b+4)p + 7\right\} c_1^2(4-c_1^2)|x| \\ & - \left\{(p^2+4p+1)c_1^2 - 8(p+2)^2c_1 + 12(p+1)(p+3)\right\} \\ & (4-c_1^2)|x|^2|.\end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of (3.40), we obtain

$$\begin{aligned}(3.41)\quad & 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\ & \le [\left\{-4b^2p^2(p^2+4p+1) + 12bp + (p^2+4p+7)\right\} c^4 \\ & + 8(p+2)^2c(4-c^2) + 2\left\{p^2 + (6b+4)p + 7\right\} c^2(4-c^2)\mu \\ & + \left\{(p^2+4p+1)c^2 - 8(p+2)^2c + 12(p+1)(p+3)\right\} (4-c^2)\mu^2], \\ & = F(c,\mu), \quad \text{for } 0 \le \mu = |x| \le 1,\end{aligned}$$

where

$$\begin{aligned}(3.42)\quad F(c,\mu) = & \left[\left\{-4b^2p^2(p^2+4p+1) + 12bp + (p^2+4p+7)\right\} c^4\right. \\ & + 8(p+2)^2c(4-c^2) + 2\left\{p^2 + (6b+4)p + 7\right\} c^2(4-c^2)\mu \\ & \left. + \left\{(p^2+4p+1)c^2 - 8(p+2)^2c + 12(p+1)(p+3)\right\} (4-c^2)\mu^2\right].\end{aligned}$$

The function $F(c,\mu)$ is maximized on the closed region $[0,1] \times [0,2]$. Differentiating $F(c,\mu)$ in (3.42) partially with respect to μ, we obtain

$$\begin{aligned}(3.43)\quad \frac{\partial F}{\partial \mu} = & [2\left\{p^2 + (6b+4)p + 7\right\} c^2 \\ & + 2\left\{(p^2+4p+1)c^2 - 8(p+2)^2c + 12(p+1)(p+3)\right\} \mu] \times (4-c^2).\end{aligned}$$

For every $c \in [0,2]$ and for fixed b, where $(b \ge 1/(2p))$ with $p \in N$, from (3.43), we observe that $\partial F/\partial \mu > 0$. Consequently, $F(c,\mu)$ is an increasing function of μ, and hence, it cannot have a maximum value at any point in the interior of the closed region $[0,1] \times [0,2]$. Moreover, for fixed $c \in [0,2]$, we have

$$(3.44)\qquad \max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in (3.42), upon simplification, we obtain

$$G(c) = 2[-\left\{2b^2p^4 + 8b^2p^3 + (1+2b^2)p^2 + 4p + 7)\right\} c^4 + 24(bp+1)c^2 + 24(p+1)(p+3)], \tag{3.45}$$

$$G'(c) = 2[-4\left\{2b^2p^4 + 8b^2p^3 + (1+2b^2)p^2 + 4p + 7)\right\} c^3 + 48(bp+1)c], \tag{3.46}$$

$$G''(c) = 2[-12\left\{2b^2p^4 + 8b^2p^3 + (1+2b^2)p^2 + 4p + 7)\right\} c^2 + 48(bp+1)]. \tag{3.47}$$

To obtain the optimum value of $G(c)$, consider $G'(c) = 0$. From (3.46), we get

$$-8c[\left\{2b^2p^4 + 8b^2p^3 + (1+2b^2)p^2 + 4p + 7\right\} c^2 - 12(bp+1)] = 0. \tag{3.48}$$

We now discuss the following cases.

Case 1. If $c = 0$, then, from the expression (3.47), we obtain

$$G''(c) = 96(bp+1) > 0, \quad \text{for } \left(b \geq \frac{1}{2p}\right), \text{ where } p \in N.$$

Therefore, by the second derivative test, $G(c)$ has minimum value at $c = 0$, which is ruled out.

Case 2. If $c \neq 0$, then, from (3.48), we obtain

$$c^2 = \left\{\frac{12(bp+1)}{2b^2p^4 + 8b^2p^3 + (1+2b^2)p^2 + 4p + 7}\right\} > 0, \quad \text{for } \left(b \geq \frac{1}{2p}\right), \text{ with } p \in N. \tag{3.49}$$

Using the value of c^2 given in (3.49) in (3.47), after simplifying, we get

$$G''(c) = -192(bp+1) < 0, \quad \text{for } \left(b \geq \frac{1}{2p}\right), \text{ where } p \in N.$$

From the second derivative test, $G(c)$ has maximum value at c^2. Substituting c^2 value in (3.45), the maximum value of $G(c)$, given by (3.50)

$$G_{\max} = 48 \times \left[\frac{6(bp+1)^2+(p+1)(p+3)\left\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\right\}}{\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\}} \right].$$

We consider only the maximum value of $G(c)$ at c, where c^2 is given by (3.49). From the expressions (3.41) and (3.50), after simplifying, we get

$$\begin{aligned}(3.51)\quad |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leq 12 \\ &\times \left[\frac{6(bp+1)^2+(p+1)(p+3)\left\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\right\}}{\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\}} \right].\end{aligned}$$

From expressions (3.33) and (3.51), upon simplification, we obtain

$$\begin{aligned}(3.52)\quad |a_{p+1}a_{p+3} - a_{p+2}^2| &\leq \\ &\left[\frac{b^2p^4\left[6(bp+1)^2+(p+1)(p+3)\left\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\right\}\right]}{(p+1)(p+2)^2(p+3)\{2b^2p^4+8b^2p^3+(1+2b^2)p^2+4p+7\}} \right].\end{aligned}$$

This completes the proof of Theorem 3.7. □

Remark 3.8. For $b = 1$, we get $CV_{b,p} = CV_{1,p}$ and, from (3.52), we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{p^4\left\{6(p+1)+(p+3)\left(2p^4+8p^3+3p^2+4p+7\right)\right\}}{(p+2)^2(p+3)(2p^4+8p^3+3p^2+4p+7)} \right].$$

Remark 3.9. For the choice of $b = 1$ and $p = 1$, we get $CV_{1,1} = CV$, for which, from (3.52), we obtain $|a_2a_4 - a_3^2| \leq 1/8$. This inequality is sharp, and the result coincides with that of Janteng, Halim and Darus [**8**].

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Department of Mathematics, GIT, GITAM University, Visakhapatnam-530 045, A.P., India
Email address: **vamsheekrishna1972@gmail.com**

Department of Mathematics, Kakatiya University, Warangal-506 009, A.P., India
Email address: **reddytr2@gmail.com**