ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 6, 2014

SEMIGROUP COMPACTIFICATIONS OF ZAPPA PRODUCTS

H.D. JUNGHENN AND P. MILNES

ABSTRACT. A group G with subgroups S and T satisfying $G = ST$ and $S \cap T = \{e\}$ gives rise to functions $[t, s] \in S$ and $\langle t, s\rangle \in T$ such that $(st)(s't') = (s[t, s'])(\langle t, s'\rangle t')$. This notion may be extended to arbitrary semigroups S, T with identities, producing the Zappa product of S and T, a generalization of direct and semidirect product. Necessary and sufficient conditions are given for a semigroup compactification of a Zappa product G of topological semigroups S and T to be canonically isomorphic to a Zappa product of compactifications of S and T. The result is applied to various types of compactifications of G, including the weakly almost periodic and almost periodic compactifications.

1. Introduction. Let G be a semigroup with identity e, and let S and T be subsemigroups containing e such that every member of G is uniquely expressible as a product st with $s \in S$ and $t \in T$. It follows that there exist functions $[\cdot, \cdot] : T \times S \to S$ and $\langle \cdot, \cdot\rangle : T \times S \to T$ such that

$$ts = [t, s]\langle t, s\rangle, \quad s \in S,\ t \in T.$$

Identifying G with $S \times T$, we see that multiplication in G may be expressed as

$$(s, t)(s', t') = (s[t, s'], \langle t, s'\rangle t'). \tag{1}$$

Associativity and the identity property imply the following relations:

$$[t, e] = e, \qquad [e, s] = s, \qquad \langle e, s\rangle = e, \qquad \langle t, e\rangle = t, \tag{2}$$

$$[tt', s] = \big[t, [t', s]\big], \qquad [t, ss'] = [t, s]\big[\langle t, s\rangle, s'\big] \tag{3}$$

$$\langle t, ss'\rangle = \langle\langle t, s\rangle, s'\rangle\rangle, \quad \text{and} \quad \langle tt', s\rangle = \langle t, [t', s]\rangle\langle t', s\rangle. \tag{4}$$

Conversely, if S and T are semigroups with identities e and $[\cdot, \cdot]$ and $\langle \cdot, \cdot\rangle$ are mappings that satisfy (2)–(4), then $G := S \times T$ is a semigroup with identity (e, e) under multiplication given by (1). G is then called

Received by the editors on June 27, 2012, and in revised form on October 18, 2012.
DOI:10.1216/RMJ-2014-44-6-1903

a *Zappa product* of S and T, which we denote by $G = S \times_Z T$. If S and T are groups, then G is a group and

$$(s,t)^{-1} = (e, t^{-1})(s^{-1}, e) = ([t^{-1}, s^{-1}], \langle t^{-1}, s^{-1} \rangle).$$

$S \times_Z T$ is a *semidirect product* if either of the Zappa product mappings $[\cdot,\cdot]$ or $\langle \cdot,\cdot \rangle$ is trivial (i.e., $[t,\cdot]$ or $\langle \cdot, s\rangle$ is the identity mapping), and a direct product if both are trivial. Zappa products of groups were first studied in **[10]** and subsequently in, for example, **[8, 9]**. The semigroup case was considered in **[6]** in the context of finite automata.

Our goal in this paper is to give necessary and sufficient conditions for a compactification of a topological Zappa product to be a Zappa product of compactifications. Our results complement and extend those of **[3, 4, 5, 7]**, which consider the direct and semidirect product cases. (See also **[1]** for a summary.)

2. Semigroup compactifications. In this section, we give a brief overview of those aspects of the theory of semigroup compactifications that will be needed in the sequel. For details, the reader is referred to **[1]**.

Let G be a semitopological semigroup with identity, $C(G)$ the space of bounded, continuous, complex-valued functions on G, and $L(s)$, respectively, $R(s)$, the left, respectively, right, translation operator on $C(G)$. A (*right topological*) *compactification* of G is a pair (ψ, G'), where G' is a compact, Hausdorff, right topological semigroup and $\psi : G \mapsto G'$, the *compactification map*, is a continuous homomorphism with dense range such that the mappings $x \mapsto \psi(s)x$ are continuous. It follows that $F := \psi^*(C(G'))$ is a translation invariant C^*subalgebra of $C(G)$ with the property that, for $x \in G^F$, the spectrum of F, $x_\ell(F) \subseteq F$, where $x_\ell(f)(s) := x(L(s)f)$. We shall call such an algebra *m-admissible*. Note that $\{x_\ell(f) \mid x \in G^F\} = \overline{R(G)f}^p$ is the closure of $R(G)f$ in the topology of pointwise convergence on $C(G)$. Conversely, if $F \subseteq C(G)$ is m-admissible, then G^F is a compactification of G, where the compactification map $\varepsilon_F : G \mapsto G^F$ is evaluation. The pair (ε_F, G^F) is called the (*canonical*) *F-compactification* of G. (See **[1,** pages 108–109].)

A compactification (ψ, G') of G is an *extension* of a compactification (θ, G'') and (θ, G'') is a *factor* of (ψ, G'), written $(\theta, G'') \lesssim (\psi, G')$, if there exists a continuous map $\pi : G' \to G''$, the *canonical homomorphism*, such that $\pi \circ \psi = \theta$. If π is one-to-one, then π is an *isomorphism* and (ψ, G') and (θ, G'') are said to be (*canonically*)*isomorphic*, written $(\psi, G') \cong (\theta, G'')$ or simply $G' \cong G''$. Every compactification (ψ, G') of G is isomorphic to the F-compactification (ε_F, G^F), where $F = \psi^*(C(G'))$. The relation $\lesssim$ partially orders the collection of (equivalence classes of) compactifications of G rendering it a complete lattice with upper bound G^{LMC} (see below) and lower bound the trivial compactification $\{\varepsilon(e)\}$.

We shall frequently make use of the following fact: If $\varphi : G_1 \mapsto G$ is a continuous function from a semitopological semigroup G_1 into G and if $F \subseteq C(G)$ and $F_1 \subseteq C(G_1)$ are m-admissible with $\varphi^*(F) \subseteq F_1$, then φ has an extension $\overline{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} G_1^{F_1} & \xrightarrow{\overline{\varphi}} & G^F \\ {\scriptstyle \varepsilon_{F_1}}\uparrow & & {\scriptstyle \varepsilon_F}\uparrow \\ G_1 & \xrightarrow{\varphi} & G \end{array}$$

The map $\overline{\varphi}(x)$ is defined by $\overline{\varphi}(x)(f) = x(\varphi^*(f))$ and is a homomorphism if and only if φ is.

A compactification of G possessing a given property $\mathcal{P}$ is called the *universal $\mathcal{P}$-compactification* of G if it is an extension of every compactification with property $\mathcal{P}$ [**1,** page 115]. Here are some examples of F-compactifications and their universal properties.

- *Left multiplicatively continuous functions.* $F = LMC(G)$ is the algebra of all $f \in C(G)$ such that $s \mapsto x(L(s)f)$ is continuous for each multiplicative mean x. (Equivalently, $R(S)f$ is relatively compact in the topology of pointwise convergence on S.) G^{LMC} is the universal right topological semigroup compactification of G.
- *Left continuous functions.* $F = LC(G)$ is the algebra of all $f \in C(G)$ such that $s \mapsto L(s)f$ is norm continuous. G^{LC} is the right topological semigroup compactification of G that is universal with respect to the property that the mapping $(s, x) \mapsto \varepsilon_F(s)x$ from $G \times G^F$ to G^F is continuous. This implies that G^F has the *restricted joint continuity property*, namely, that multiplication on $\varepsilon_F(K) \times G^{LC}$ is continuous for

each compact set $K \subset G$.

- *Left continuous functions on compacta.* $F = K(G)$ is the algebra of all $f \in LMC(G)$ such that $R(S)f$ is relatively compact in the topology of uniform convergence on compacta (equivalently: the restriction of $s \mapsto L(s)f$ to compact sets is norm continuous). G^K is the right topological semigroup compactification of G that is universal with respect to the restricted joint continuity property described in the preceding example.
- *Distal functions.* $F = D(G)$ is the algebra of all $f \in LMC(G)$ such that $(uvw)(f) = (uw)(f)$ for all $u, v^2 = v, w \in G^{LMC}$. G^D is the universal right topological group compactification of G.
- *Weakly almost periodic functions.* $F = WAP(G)$ is the algebra of all functions $f \in C(G)$ such that $R(G)f$ (equivalently, $L(G)f$) is relatively weakly compact. G^{WAP} is the universal semitopological semigroup compactification of G.
- *Almost periodic functions.* $F = AP(G)$ is the algebra of all $f \in C(G)$ such that $R(G)f$ (equivalently, $L(G)f$) is relatively norm compact. G^{AP} is the universal topological semigroup compactification of G.
- *Strongly almost periodic functions.* $F = SAP(G)$ is the algebra generated by the finite dimensional unitary subspaces of $C(G)$. G^{SAP} is the universal topological group compactification of G. By Ellis's theorem on separate and joint continuity of group actions, $SAP(G) = WAP(G) \cap D(G)$.

We also define

$$LCWAP(G) := LC(G) \cap WAP(G)$$

and

$$KWAP(G) := K(G) \cap WAP(G).$$

It is clear from the inclusion relations among the function spaces that $G^{SAP} \lesssim G^{AP} \lesssim G^{WAP}$ and $G^{LC} \lesssim G^K$. Moreover, if G contains a dense group, then $G^{SAP} \cong G^{AP}$; if G is a K-space or is metrizable, then $G^{LC} \cong G^K$; and if G is a topological group, then $G^{WAP} \lesssim G^{LC}$.

For more details about these spaces the reader is referred to [**1,** Chapter 4].

3. A general compactification theorem. For the remainder of the paper we assume that S and T are topological semigroups and that the Zappa product mappings of $G = S \times_Z T$ are jointly continuous, so that G is a topological semigroup.

Given an m-admissible algebra $F \subseteq C(G)$, we seek conditions under which G^F is (canonically isomorphic to) a Zappa product of compactifications (θ, S') of S and (ψ, T') of T. This means that the compactification map of $S' \times_Z T'$ is the product map $\theta \times \psi$. In this case, we write $G^F \cong S' \times_Z T'$. Note that we then have

$$\begin{aligned}\big([\psi(t), \theta(s)], \langle\psi(t), \theta(s)\rangle\big) &= \big(\theta(e), \psi(t)\big)\big(\theta(s), \psi(e)\big)\\ &= (\theta \times \psi)\big((e,t)(s,e)\big)\\ &= \big(\theta([t,s]), \psi(\langle t,s\rangle)\big),\end{aligned}$$

i.e., the Zappa product maps on $S' \times T'$ are extensions of the corresponding maps on $S \times T$. This implies that if G is a semidirect (direct) product then $S' \times_Z T'$ is a semidirect (direct) product.

Theorem 3.3 below is the basis for the compactification results in this paper. The corresponding theorem for the special case of semidirect products was proved in [**5**] using results on tensor products. We give a direct, self-contained proof based on Lemmas 3.1 and 3.2. The following notation will be convenient:

$$\begin{aligned}&q_S : S \longmapsto S \times T, \quad s \mapsto (s,e);\\ &q_T : T \longmapsto S \times T, \quad t \mapsto (e,t);\\ &p_S : S \times T \longmapsto S, \quad (s,t) \mapsto s;\\ &p_T : S \times T \longmapsto T, \quad (s,t) \mapsto t;\\ &r_S = q_S \circ p_S; \qquad r_T = q_T \circ p_T.\end{aligned}$$

Lemma 3.1. *Let $G = S \times T$ have a multiplication relative to which it is a semigroup with identity (e,e). Then G is a Zappa product if and only if the mappings q_S and q_T are homomorphisms and $r_S \cdot r_T = \mathrm{id}_G$. If G is a Zappa product, then it is a semidirect product if and only if* *either p_S and p_T is a homomorphism, and a direct product if and only if both maps are homomorphisms.*

Proof. The conditions on the maps q and r reduce to the identities

$$(s,e)(s',e) = (ss',e), \quad (e,t)(e,t') = (e,tt'), \quad \text{and} \quad (s,e)(e,t) = (s,t).$$

For the sufficiency of the first assertion, define

$$[t,s] = p_S\big((e,t)(s,e)\big) \quad \text{and} \quad \langle t,s\rangle = p_T\big((e,t)(s,e)\big)$$

so that $(e,t)(s,e) = ([t,s],\langle t,s\rangle)$ It is straightforward to check that identities (2)–(4) in the introduction are satisfied. For example, $(e,t) = (e,t)(e,e) = ([t,e],\langle t,e\rangle)$ implies that $[t,e] = e$ and $\langle t,e\rangle = t$, and the calculations

$$\begin{aligned}([tt',s],\langle tt',s\rangle) &= (e,tt')(s,e) = (e,t)(e,t')(s,e)\\ &= (e,t)([t',s],e)(e,\langle t',s\rangle)\\ &= ([t,[t',s]],\langle t,[t',s]\rangle)(e,\langle t',s\rangle)\\ &= ([t,[t',s]],e)(e,\langle t,[t',s]\rangle)(e,\langle t',s\rangle)\\ &= ([t,[t',s]],e)(e,\langle t,[t',s]\rangle\langle t',s\rangle)\\ &= ([t,[t',s]],\langle t,[t',s]\rangle\langle t',s\rangle)\end{aligned}$$

show that $[tt',s] = [t,[t',s]]$ and $\langle tt',s\rangle = \langle t,[t',s]\rangle\langle t',s\rangle$. The necessity and the remaining assertions of the lemma are also straightforward. □

Lemma 3.2. *Let X and Y be topological spaces with X' dense in X. Suppose that f is a bounded, complex-valued function on $X \times Y$ such that $f(X',\cdot)$ is relatively compact in $C(Y)$ and $f(\cdot,Y) \subseteq C(X)$. Then $f(X,\cdot)$ is relatively compact in $C(Y)$, f is jointly continuous, and $x \mapsto f(x,\cdot)$ is continuous.*

Proof. Set $K = \overline{f(X',\cdot)} \subseteq C(Y)$, and let $x \in X$ and $x'_\alpha \in X'$ with $x'_\alpha \to x$. Taking a subnet if necessary, we may assume that $f(x'_\alpha,\cdot) \to g \in K$. Given $\epsilon > 0$ choose α_0 such that

$$|f(x'_\alpha,y) - g(y)| < \epsilon \quad \text{for all } y \in Y \text{ and } \alpha \geq \alpha_0.$$

Since $f(\cdot,y)$ is continuous, taking limits we see that $g(y) = f(x,y)$. Therefore, $f(x,\cdot) \in K$; hence, $f(X,\cdot)$ is relatively compact in $C(Y)$.

Now let $(x_\alpha, y_\alpha) \to (x_0, y_0)$ in $X \times Y$. By the result of the first paragraph, we may assume that $f(x_\alpha, \cdot) \to h \in C(Y)$. As above, given $\epsilon > 0$, there exists α_0 such that

$$|f(x_\alpha, y) - h(y)| < \epsilon \quad \text{for all } \alpha \geq \alpha_0 \text{ and } y \in Y. \tag{5}$$

It follows that $h(y) = f(x_0, y)$ and hence $f(x_\alpha, \cdot) \to f(x_0, \cdot)$. Since h is continuous and $y_\alpha \to y_0$, it follows easily that $\{f(x_\alpha, y_\alpha)\}$ is a Cauchy net and hence converges to some $L \in \mathbb{C}$. From (5) we see that $L = h(y_0)$. Therefore, $L = f(x_0, y_0)$ and $f(x_\alpha, y_\alpha) \to f(x_0, y_0)$. □

Theorem 3.3. *Let $F \subseteq C(G)$ be m-admissible. Then $G^F \cong S' \times_Z T'$ for some compactifications of S and T if and only if* (a) $r_S^*(F) \cup r_T^*(F) \subseteq F$ *and* (b) *for each $f \in F$, either $f(S, \cdot)$ is relatively compact in $C(T)$ or $f(\cdot, T)$ is relatively compact in $C(S)$. If* (a) *and* (b) *hold, then $S' \cong S^A$ and $T' \cong T^B$, where $A := q_S^*(F)$ and $B := q_T^*(F)$.*

Proof. For the necessity, suppose that $G^F \cong S' \times_Z T'$ for some compactifications (θ, S') of S and (ψ, T') of T, and let $\varphi : G^F \mapsto S' \times_Z T'$ denote the compactification isomorphism. Given $f \in F$, choose $g \in C(S' \times_Z T')$ such that $f = \varepsilon_F^* \circ \varphi^*(g)$. Since the mapping $x \mapsto g(x, \cdot) : S' \mapsto C(T')$ is norm continuous, $g(S', \cdot)$ is norm compact in $C(T')$. Since $f(s,t) = g\big(\theta(s), \psi(t)\big)$, $f(S, \cdot)$ is norm compact in $C(T)$. Similarly, $f(\cdot, T)$ is norm compact in $C(S)$. Setting $h := r_{S'}^*(g)$, we have

$$\varepsilon_F^* \circ \varphi^*(h)(s,t) = g\big(\theta(s), \psi(e)\big) = f(s, e),$$

so $r_S^*(f) = \varepsilon_F^* \circ \varphi^*(h) \in F$. Similarly, $r_T^*(f) \in F$. Thus, conditions (a) and (b) hold.

We now have the commutative diagram

$$\begin{array}{ccccccc}
S^A & \xrightarrow{\overline{q}_S} & G^F & \xrightarrow{\varphi} & S' \times_Z T' & \xrightarrow{p_{S'}} & S' \\
\varepsilon_A \uparrow & & \varepsilon_F \uparrow & & \theta \times \psi \uparrow & & \theta \uparrow \\
S & \xrightarrow{q_S} & G & \xrightarrow{\text{id}} & G & \xrightarrow{p_S} & S
\end{array}$$

with a similar diagram for T. Set $\gamma = p_{S'} \circ \varphi \circ \overline{q}_S : S^A \mapsto S'$. Since $\gamma \circ \varepsilon_A = \theta \circ p_S \circ q_S = \theta$, γ is a compactification homomorphism. To see that γ is one-to-one and hence an isomorphism, note first that

$$\big(\gamma(\varepsilon_A(s)), \psi(e)\big) = (\theta \times \psi) \circ q_S(s) = \varphi \circ \overline{q}_S\big(\varepsilon_A(s)\big);$$

hence, $(\gamma(x), \psi(e)) = \varphi \circ \overline{q}_{_S}(x)$ for all $x \in S^A$. But φ and $\overline{q}_{_S}$ are one-to-one, the latter because $\overline{q}_{_S}^* : C(G^F) \mapsto C(S^A)$ is surjective. Therefore, γ is a compactification isomorphism and $S^A \cong S'$. Similarly, $T^B \cong T'$.

Conversely, suppose conditions (a) and (b) hold. Set $\pi = \varepsilon_{_A} \times \varepsilon_{_B} : G \mapsto S^A \times T^B$. As a first step, we show that $\pi^* C(S^A \times T^B) \subseteq F$. Let $g_1,\, h_1 \in F$, $g := q_{_S}^*(g_1)$, $h := q_{_T}^*(h_1)$, $\varepsilon_{_A}^*(\widehat{g}) = g$, and $\varepsilon_{_B}^*(\widehat{h}) = h$. Then

$$\pi^*(\widehat{g} \otimes \widehat{h}) = g \otimes h = r_{_S}^*(g_1) \cdot r_{_T}^*(h_1) \in F.$$

By the Stone-Weierstrass theorem, the functions $\widehat{g} \otimes \widehat{h}$ span a dense subspace of $C(S^A \times T^B)$, verifying the assertion.

It follows that π has a continuous extension $\overline{\pi} : G^F \mapsto S^A \times T^B$ such that $\overline{\pi} \circ \varepsilon_{_F} = \pi$. We claim that $\overline{\pi}^* C(S^A \times T^B) \mapsto C(G^F)$ is surjective. This will imply that $\overline{\pi}$ is one-to-one and hence a homeomorphism. Let $\widehat{f} \in C(G^F)$ and $f = \varepsilon_{_F}^*(\widehat{f})$. Assume without loss of generality that $K := \overline{f(S, \cdot)}$ is compact in $C(T)$. Since

$$f(s,t) = q_{_S}^*\big(R(e,t)f\big)(s) = q_{_T}^*\big(L(s,e)f\big)(t),$$

$f(\cdot, t) \in A$ and $K \subseteq B$. Define $\widetilde{f}$ on $S^A \times T$ by $\widetilde{f}(x,t) = x(f(\cdot,t))$ and note that $\widetilde{f}(\cdot,t)$ is continuous for each t and $\widetilde{f}(\varepsilon_{_A}(s), \cdot) = f(s, \cdot) \in K$. By Lemma 3.2, $\widetilde{f}$ is jointly continuous, $x \mapsto \widetilde{f}(x, \cdot)$ is continuous and $\widetilde{f}(S^A, \cdot) \subseteq K$. For $(x,y) \in S^A \times T^B$, define $\breve{f}(x,y) = y(\widetilde{f}(x,\cdot))$. Then $\breve{f}$ is separately continuous and $x \mapsto \breve{f}(x, \cdot)$ is norm continuous so that, by Lemma 3.2, the function $\breve{f}$ is jointly continuous. Moreover, $\breve{f}(\varepsilon_{_A}(s), \varepsilon_{_B}(t)) = \widetilde{f}(\varepsilon_{_A}(s), t) = f(s,t)$, hence $\overline{\pi}^* \breve{f} = \widehat{f}$, verifying the claim.

Now give $S^A \times T^B$ the unique multiplication that makes $\overline{\pi}$ a semigroup isomorphism. Since $\overline{\pi} \circ \varepsilon_{_F} = \pi$, $\overline{\pi}$ is a compactification isomorphism. The diagram above now holds with $\varphi = \overline{\pi}$, $(\theta, S') = (\varepsilon_{_A}, S^A)$ and $(\psi, T') = (\varepsilon_{_B}, T^B)$. Note that

$$\overline{\pi}^{-1}(\varepsilon_{_A}(s), \varepsilon_{_B}(t)) = \varepsilon_{_F}(s,e) \cdot \varepsilon_{_F}(e,t) = \overline{q}_{_S}(\varepsilon_{_A}(s)) \cdot \overline{q}_{_T}(\varepsilon_{_B}(t));$$

hence $\overline{\pi}^{-1}(x,y) = \overline{q}_{_S}(x) \cdot \overline{q}_{_T}(y)$, and therefore $(x,y) = \big(\overline{\pi} \circ \overline{q}_{_S}(x)\big) \cdot \big(\overline{\pi} \circ \overline{q}_{_T}(y)\big)$. But, for $s \in S$, $\overline{\pi}(\overline{q}_{_S}(\varepsilon_{_A}(s))) = \overline{\pi}(\varepsilon_{_F}(s,e)) = (\varepsilon_{_A}(s), \varepsilon_{_B}(e)) = q_{_{SA}}(\varepsilon_{_A}(s))$; hence, $\overline{\pi} \circ \overline{q}_{_S}(x) = q_{_{SA}}(x) = r_{_{SA}}(x,y)$ for all $(x,y) \in S^A \times T^B$. A similar identity holds for $\overline{\pi} \circ \overline{q}_{_T}$. Therefore, $q_{_{SA}}$ and

$q_{_{T^B}}$ are homomorphisms and $r_{_{S^A}} \cdot r_{_{T^B}}$ is the identity mapping. That $S^A \times T^B$ is a Zappa product now follows from Lemma 3.1. $\square$

Corollary 3.4. *The maximal factor and the minimal extension of a family $\mathcal{G} := \{(\psi_i, G^{(i)}) \mid i \in I\}$ of Zappa product compactifications of G are Zappa product compactifications of G. Thus, the collection of Zappa compactifications of G is a complete sublattice of the lattice of all right topological semigroup compactifications of G.*

Proof. The maximal factor of the family $\mathcal{G}$ is G^H, where $H = \bigcap_i F_i$. Clearly, H satisfies conditions (a) and (b) of the theorem; hence, G^H is a Zappa product.

The minimal extension of $\mathcal{G}$ is G^F, where F is the intersection of all m-admissible algebras containing $K := \bigcup_i \psi^* C(G^{(i)})$. Let F' denote the set of all $f \in F$ such that $r^*_{_A}(g)$, $r^*_{_B}(g) \in F$ and $g(S, \cdot)$ is relatively compact for every $g \in L(G)\overline{R(G)}^p f$. Then F' is an m-admissible algebra containing K and therefore equals F. By the theorem, then, G^F is a Zappa product. $\square$

4. LC and WAP compactifications of G.

Theorem 4.1. *Let S be a compact topological group, T a topological semigroup, and let $F = LC(G)$ or $F = K(G)$. Then $G^F \cong S' \times_{_Z} T'$ for some compactification $S' \lesssim S^{SAP}$ of S and compactification $T' \lesssim T^{F_T}$ of T, where F_T is the corresponding space of functions on T.*

Proof. Let $f \in F$ and $\widehat{f} \in C(G^F)$ with $\varepsilon_{_F}(\widehat{f}) = f$. Since S is compact and the mapping $s \mapsto L(s, e)f$ is norm continuous, condition (b) of Theorem 3.3 holds. Thus, it remains to show that $r^*_{_S}(f) \in F$ and $r^*_{_T}(f) \in F$. We prove this for $F = LC(G)$, the proof for $K(G)$ being similar.

Let $x_\alpha := (u_\alpha, v_\alpha) \to x := (u, v)$ in G. By the joint continuity of $[\cdot, \cdot]$ and the compactness of S, $[v_\alpha, s] \to [v, s]$ uniformly in $s \in S$. It follows that $L(x_\alpha) r^*_{_S} f(s, t) = f(u_\alpha [v_\alpha, s], e)$ converges uniformly in (s, t), so $r^*_{_S} f \in LC(G)$.

Since $r_S^*(F) \subset F$, r_S has a continuous extension $\overline{r}_S : G^F \mapsto G^F$. From $r_S \cdot r_T = \mathrm{id}_G$ we then have

$$L(x_\alpha)r_T^* f(s,t) = \widehat{f}\left(\left\{\overline{r}_S\left(\varepsilon(x_\alpha)\varepsilon(s,t)\right)\right\}^{-1}\varepsilon(x_\alpha)\varepsilon(s,t)\right),$$

the inverse taken in the compact group $H := \varepsilon\big(S \times \{e\}\big)$. We claim that $L(x_\alpha)r_T^* f(s,t)$ converges uniformly in (s,t) to $L(x)r_T^* f(s,t)$. If not, then there exist $r > 0$, a subnet $\{x_\beta\}$, and a net $\{y_\beta := (s_\beta, t_\beta)\}$ with $\varepsilon(y_\beta) \to b \in G^{LC}$ such that, for all β,

$$\Big|\widehat{f}\left(\left\{\overline{r}_S\left(\varepsilon(x_\beta)\varepsilon(y_\beta)\right)\right\}^{-1}\varepsilon(x_\beta)\varepsilon(y_\beta)\right) - \widehat{f}\left(\left\{\overline{r}_S\left(\varepsilon(x)\varepsilon(y_\beta)\right)\right\}^{-1}\varepsilon(x)\varepsilon(y_\beta)\right)\Big| \geq r.$$

But this is impossible because of the restricted joint continuity property of multiplication in G^F, since the terms in braces converge in $H \subseteq \varepsilon(G^F)$ and $\varepsilon(x_\beta)\varepsilon(y_\beta)$, $\varepsilon(x)\varepsilon(y_\beta) \to \varepsilon(x)b$. Therefore, $r_T^* f \in LC(G)$.

That $T' \lesssim T^{F_T}$ follows from the inclusion $q_T^*(F) \subseteq F_T$, this because $q_T : T \mapsto G$ is a continuous homomorphism. □

Theorem 4.2. *Let S be a compact topological group, T a topological semigroup, and let $F = LCWAP(G)$ or $F = KWAP(G)$. Then $G^F \cong S' \times_Z T'$ for some compactifications S' of S and T' of T if and only if $r_S^*(F) \subseteq F$. In this case, $S' \lesssim S^{SAP}$ and $T' \lesssim T^{F_T}$, where F_T is the corresponding space of functions on T.*

Proof. The necessity is clear. For the sufficiency, let $f \in F$ and $\widehat{f} \in C(G^F)$ with $\varepsilon_F(\widehat{f}) = f$. Since S is compact and the mapping $s \mapsto L(s,e)f$ is norm continuous, condition (b) of Theorem 3.3 holds. Thus, it suffices to show that if $r_S^*(f) \in F \Rightarrow r_T^*(f) \in F$. We do this for the case $F = LCWAP(G)$, the other case being similar.

From Theorem 4.1, $r_T^*(f) \in LC(G)$. To show that $r_T^*(f) \in WAP(G)$ we use Grothendieck's criterion. Let $\{(s_m, t_m)\}$ and $\{(s_n, t_n)\}$ be sequences in G such that the limits

$$\ell_1 = \lim_m \lim_n f \circ r_T\big((s_n, t_n)(s_m, t_m)\big)$$

and

$$\ell_2 = \lim_n \lim_m f \circ r_T\big((s_n, t_n)(s_m, t_m)\big)$$

exist. We need to show that $\ell_1 = \ell_2$. Set $\varepsilon = \varepsilon_F$. By hypothesis, r_S has an extension $\overline{r}_S : G^F \to G^F$. Thus,

$$\varepsilon \circ r_S((s_n, t_n)(s_m, t_m)) = \varepsilon(s_n[t_n, s_m], e) = \overline{r}_S\big(\varepsilon(s_n, t_n)\varepsilon(s_m, t_m)\big) \tag{6}$$

and

$$\begin{aligned}\varepsilon \circ r_T((s_n, t_n)(s_m, t_m)) \\ = \{\overline{r}_S(\varepsilon(s_n, t_n)\varepsilon(s_m, t_m))\}^{-1}\,\varepsilon(s_n, t_n)\varepsilon(s_m, t_m),\end{aligned} \tag{7}$$

the inverse taken in the compact group $H := \varepsilon\big(S \times \{e\}\big)$. Choose subnets such that $\varepsilon(s_{n_\alpha}, t_{n_\alpha}) \to a$ and $\varepsilon(s_{m_\beta}, t_{m_\beta}) \to b$ in G^F. From (7),

$$\begin{aligned}&\lim_\beta \lim_\alpha \varepsilon \circ r_T((s_{n_\alpha}, t_{n_\alpha})(s_{m_\beta}, t_{m_\beta}))\\ &\quad= \lim_\beta \lim_\alpha \big\{\overline{r}_S(\varepsilon(s_{n_\alpha}, t_{n_\alpha})\varepsilon(s_{m_\beta}, t_{m_\beta}))\big\}^{-1}\,\varepsilon(s_{n_\alpha}, t_{n_\alpha})\varepsilon(s_{m_\beta}, t_{m_\beta})\\ &\quad= \{\overline{r}_S(ab)\}^{-1}ab\\ &\quad= \lim_\alpha \lim_\beta \varepsilon \circ r_T((s_{n_\alpha}, t_{n_\alpha})(s_{m_\beta}, t_{m_\beta})),\end{aligned}$$

the convergence of the term in braces takes place in H thereby allowing use of the restricted joint continuity property of G^F (see Section 2). Thus, $\ell_1 = \ell_2$; hence, $r_T^*(f) \in WAP(G)$. □

Corollary 4.3. *Let S be a compact topological group, T a topological semigroup, and let $F = LCWAP(G)$ or $F = KWAP(G)$. Suppose that the Zappa product mapping $[\cdot,\cdot]$ has the property that, for each pair of sequences $\{s_n\}$ in S and $\{t_n\}$ in T, there exist subnets such that*

$$\lim_\beta \lim_\alpha [t_{n_\beta}, s_{n_\alpha}] = \lim_\alpha \lim_\beta [t_{n_\beta}, s_{n_\alpha}]. \tag{8}$$

Then $G^F \cong S' \times_Z T'$ for some compactifications of $S' \lesssim S^{SAP}$ and $T' \lesssim T^{F_T}$. In particular, this holds if G is a semidirect product with $[\cdot,\cdot]$ trivial.

Proof. By Theorem 4.2, it suffices to show that $r_S^*(F) \subseteq F$. We do this only for $F = LCWAP(G)$. Let $f \in F$. From Theorem 4.1, $r_S^*(f) \in LC(G)$. To show that $r_S^*(f) \in WAP(G)$, let $\{(s_m, t_m)\}$ and $\{(s_n, t_n)\}$ be sequences in G such that the limits

$$\ell_1 = \lim_m \lim_n f\big(r_S((s_n, t_n)(s_m, t_m))\big)$$

and

$$\ell_2 = \lim_n \lim_m f\big(r_S((s_n, t_n)(s_m, t_m))\big)$$

exist. Using the first equality in (6), the compactness of S, and the double limit hypothesis (8), we may choose subnets such that

$$\lim_\beta \lim_\alpha \varepsilon \circ r_S\big((s_{n_\alpha}, t_{n_\alpha})(s_{m_\beta}, t_{m_\beta})\big) = \lim_\alpha \lim_\beta \varepsilon \circ r_S\big((s_{n_\alpha}, t_{n_\alpha})(s_{m_\beta}, t_{m_\beta})\big).$$

Therefore, $\ell_1 = \ell_2$; hence, $r_S^*(f) \in WAP(G)$. □

Note that, if T is a topological group, then $WAP(G) \subseteq LC(G)$; hence, Theorem 4.2 and Corollary 4.3 reduce to assertions about weakly almost periodic compactifications.

5. The minimal ideal as a Zappa product.

Theorem 5.1. *Let S be a compact topological group, T a topological semigroup, and F an m-admissible and right amenable subalgebra of $WAP(G)$ such that $G^F \cong S' \times_Z T'$ for some compactifications (θ, S') of S and (ψ, T') of T. Then $B := q_T^*(F)$ is amenable if and only if F is amenable and $M(G^F) \cong S' \times_Z M(T')$ (under the restricted Zappa product mappings).*

Proof. Assume that B is amenable so that $M(T')$ is a compact topological group. Since $L := S' \times M(T')$ is a left ideal it contains a minimal idempotent (a, b). Let d' $(= \theta(e))$ be the identity of S' and e' the identity of $M(T')$. From $(a, b)^2 = (a, b)$, we have $a = a[b, a]$ and $\langle b, a\rangle b = b$; hence, $[b, a] = d'$. Also, from (2) and (4),

$$\langle b, a\rangle = \langle e'b, a\rangle = \langle e', [b, a]\rangle\langle b, a\rangle = \langle e', d'\rangle\langle b, a\rangle = e'\langle b, a\rangle,$$

which shows that $\langle b,a\rangle \in M(T')$. Since $\langle b,a\rangle b = b$, we see that $b \in M(T')$ and $\langle b,a\rangle = e'$. Thus, for $y \in M(T')$, from (3) and (4)

$$\langle yb,a\rangle = \langle y,[b,a]\rangle\langle b,a\rangle = \langle y,d'\rangle e' = y$$

and

$$[yb,a] = \big[y,[b,a]\big] = [y,d'] = d'.$$

Taking $y = b^{-1}$, the inverse of b in $M(T')$, we have $\langle e',a\rangle = b^{-1}$ and $[e',a] = d'$; hence, $(d',e')(a,b) = ([e',a],\langle e',a\rangle b) = (d',e')$. It follows that (d',e') is contained in the minimal left ideal $G^F(a,b)$ and is therefore a minimal idempotent. Since $G^F(d',e') = S' \times T'e' = L$, $L \subseteq M(G^F)$. But, by the right amenability of F, $M(G^F)$ is the unique minimal left ideal of G^F. Therefore, as sets, $S' \times M(T') = M(G^F)$. Since $(x,y) \in M(G^F) \Rightarrow (d',y) \in M(G^F) \Rightarrow ([y,x],\langle y,x\rangle) = (d',y)(x,\psi(e)) \in M(G^F) \Rightarrow \langle y,x\rangle \in M(T')$, we see that $M(G^F) = S' \times_z M(T')$. Since S' and $M(T')$ are groups, $M(G^F)$ must be a group; hence, F is amenable.

Conversely, suppose that F is amenable and $M(G^F) = S' \times_z M(T')$. Let b be any idempotent in $M(T')$. Then $(d',b)(d',b) = (d',b^2) = (d',b)$; hence, (d',b) must be the identity of the group $M(G^F)$. Therefore, for any $y \in M(T')$,

$$(d',y) = (d',b)(d',y) = (d',by)$$

and

$$(d',y) = (d',y)(d',b) = (d',yb),$$

which shows that b is an identity for $M(T')$. Thus, $M(T')$ is a group; hence, B is amenable. □

Corollary 5.2. *Let S and T be topological groups with S compact, and let F be an m-admissible subalgebra of $WAP(G)$. Suppose that the Zappa product mapping $[\cdot,\cdot]$ has the double limit property* (8). *Then $M(G^F) = S' \times_z M(T')$.*

6. AP Compactifications of G.

Theorem 6.1. *Let S and T be topological semigroups. Then $G^{SAP} \cong S' \times_Z T'$ for some topological group compactifications S' of S and T' of T if and only if either of the inclusions* (i) $r_S^*(SAP(G)) \subseteq SAP(G)$ *or* (ii) $r_T^*(SAP(G)) \subseteq SAP(G)$ *holds, in which case both hold. In particular, if G is a semidirect product, then $G^{SAP} \cong S' \times_Z T'$.*

Proof. Let $f \in SAP(G)$ and $\widehat{f} \in C(G^{SAP})$ such that $\varepsilon^*(\widehat{f}) = f$,where $\varepsilon := \varepsilon_{SAP}$. The identity $f(s,t) = L(s,e)f(e,t)$ and the relative compactness $L(G)f$ imply that condition (b) of Theorem 3.3 holds. By symmetry, it remains to show that (i) implies (ii). If (i) holds, then r_S has an extension $\overline{r}_S : G^{SAP} \to G^{SAP}$. The identity

$$\varepsilon\big(r_T(s,t)\big) = \big\{\overline{r}_S\big(\varepsilon(s,t)\big)\big\}^{-1}\varepsilon\big(s,t\big) \tag{9}$$

implies that

$$L(s,t)r_T^*(f)(s',t') = \widehat{f}\left(\big\{\overline{r}_S\big(\varepsilon((s,t),(s',t'))\big)\big\}^{-1}\varepsilon((s,t),(s',t'))\right),$$

and hence that $r_T^*(f) \in AP(G)$. Therefore, r_T has an extension $\overline{r}_T : G^{AP} \to G^{SAP}$. If $\varphi : G^{AP} \to G^{SAP}$ denotes the canonical homomorphism, then from (9),

$$\overline{r}_T(u) = \big\{\overline{r}_S\big(\varphi(u)\big)\big\}^{-1}\varphi(u), \quad u \in G^{AP}.$$

It follows easily that $\overline{r}_T$ has the distal property

$$\overline{r}_T(uvw) = \overline{r}_T(uw), \quad u, v^2 = v,\ w \in G^{AP},$$

which implies that $r_T^*(SAP(G)) \subseteq D(G) \cap AP(G) = SAP(G)$.

In the case of a semidirect product, either r_S or r_T is a homomorphism; hence, (i) or (ii) holds. □

Theorem 6.2. *Let S be a compact topological group and T a topological semigroup such that $WAP(G)$ is amenable. If the Zappa product mapping $[\cdot,\cdot]$ has the double limit property* (8)*, then $G^{SAP} \cong S'' \times_Z T''$ for some topological group compactifications of S and T. In particular, if G is a semidirect product with $[\cdot,\cdot]$ trivial, then G^{SAP} is a semidirect product.*

Proof. Let $F = LCWAP(G)$. By Corollary 4.3, $G^F \cong S' \times_Z T'$ for some compactifications of S and T. By Theorem 5.1, $M(G^F) \cong S' \times_Z M(T')$ is a group with identity (d', e'); hence, $[e', x] = x$ for all $x \in S'$. Moreover, since G^F is a Zappa product, $r_S^*(F) \subseteq F$; hence, r_S has an extension $\overline{r}_S : G^F \mapsto G^F$.

Now let $f \in F$ and $g = r_S^*(f)$. Set $\varepsilon = \varepsilon_F$, and let $\varepsilon^*(\widehat{f}) = f$ and $\varepsilon^*(\widehat{g}) = g$, so that $\widehat{g} = \overline{r}_S^*(\widehat{f})$. For $(x, y), (u, v) \in G^F$, we have

$$\begin{aligned} \overline{r}_S\big((x,y)(d',e')(u,v)\big) &= \overline{r}_S\big((x,y)([e',u], \langle e',u\rangle v)\big) \\ &= \big(x[y,u], \psi(e)\big) \\ &= \overline{r}_S\big((x,y)(u,v)\big); \end{aligned}$$

hence, $\widehat{g}\big((x,y)(d',e')(u,v)\big) = \widehat{g}\big((x,y)(u,v)\big)$. Therefore, g has the distal property; hence, $g \in SAP(G)$. The conclusion now follows from Theorem 6.1. □

7. Examples. (a) Let $\mathbb{C}$ denote the complex numbers under addition and $\mathbb{C}^*$ the nonzero complex numbers under multiplication. Then

$$G = S \times T = (\mathbb{C} \times \mathbb{C}^*) \times (\mathbb{C} \times \mathbb{C}^*)$$

is a Zappa product under multiplication

$$(z, a, w, b)(z', a', w', b') = (z + bz', aa', a'w + w', bb').$$

Since $(0, 1, w, b)(z, a, 0, 1) = (bz, a, aw, b)$,

$$[(w, b), (z, a)] = (bz, a)$$

and

$$\langle (w, b), (z, a)\rangle = (aw, b) = [(z, a), (w, b)].$$

We show that G^{AP} is isomorphic to the direct product $\mathbb{C}^{*AP} \times \mathbb{C}^{*AP} = (\mathbb{C}^* \times \mathbb{C}^*)^{AP}$.

Given a net $\{(s_\alpha, t_\alpha)\}$ in G with $s_\alpha = (z_\alpha, a_\alpha)$ we have, for $t = (w, b)$,

$$\begin{aligned} r_S\big((s,t)(s_\alpha,t_\alpha)\big) &= (s,e)([t,s_\alpha],e) \\ &= (s,e)(bz_\beta, a_\beta, 0, 1) \\ &= (s,e)(0,a_\alpha,0,1)(0,1,0,b)(z_\alpha,1,0,1)(0,1,0,b^{-1}). \end{aligned}$$

It follows easily that $\varepsilon_{_{AP}}(r_{_S}((s,t)(s_\alpha,t_\alpha)))$ has a subnet that converges in G^{AP} uniformly in (s,t). Therefore, $r_{_T}^*(AP(G)) \subseteq AP(G)$; hence, by Theorem 6.1, $G^{AP} = S^A \times_{_Z} T^B$ for $A = q_{_S}^*(AP(G))$ and $B = q_{_T}^*(AP(G))$.

Since $AP(G)$ is generated by coefficients $f_{\xi\zeta}(s,t) := (U(s,t)\xi,\zeta)$ of continuous finite dimensional unitary representations U of G, A is generated by the coefficients $f_{\xi\zeta}(\cdot,e)$. Since $r_{_S}^*(f_{\xi\zeta}) \in AP(G)$, U has the convergence property that each sequence $\{(s_n,t_n)\}$ in G has a subsequence $\{(s_k,t_k)\}$ such that $U \circ r_{_S}((s_k,t_k)(s,t)) = U(s_k[t_k,s],e)$ converges uniformly in s. Since S is commutative, the unitary representation $U(\cdot,e)$ is a direct sum of characters, each of which has this convergence property. Therefore, A is generated by the continuous characters $\chi(z,a) = \chi_1(z)\chi_2(a)$ of $S = \mathbb{C} \times \mathbb{C}^*$ with the property that each sequence $(s_n,t_n) = (z_n,a_n,w_n,b_n)$ has a subsequence (z_k,a_k,w_k,b_k) such that $\chi(s_k[t_k,s]) = \chi(z_k + b_k z, a_k a)$ converges uniformly in $s = (z,a)$. Taking $a_n = a = 1$ and b_n real, we see that, for some θ_1, $\theta_2 \in \mathbb{R}$ and all k, the sequence

$$\chi(s_k[t_k,s]) = \chi_1(z_k + b_k z) = \exp\big[i(\theta_1(x_k + b_k x) + \theta_2(y_k + b_k y))\big]$$

converges uniformly in x and $y \in \mathbb{R}$. This is possible only if $\theta_1 = \theta_2 = 0$. Therefore, the characters χ_1 are trivial and A is generated by the characters χ_2 of $\mathbb{C}^*$ so that $S^A = \mathbb{C}^{*AP}$. Similarly for T^B.

Analogous results hold for the subgroups of G obtained by replacing one or both occurrences of $\mathbb{C}^*$ by the torus $\mathbb{T}$. Also, one may obviously replace $\mathbb{C}$ and $\mathbb{C}^*$ by $\mathbb{R}$ and $\mathbb{R}^*$, respectively. □

(b) Consider, $\mathbb{Z}/q\mathbb{Z}$, the ring of integers mod $q \in \mathbb{N}$. Let $H := (\mathbb{Z}/q\mathbb{Z},+)$, and let J be a subsemigroup of $(\mathbb{Z}/q\mathbb{Z},\cdot)$ containing 1. Then

$$G = S \times_{_Z} T := (\mathbb{T} \times H) \times_{_Z} (J \times \mathbb{C})$$

is a Zappa product semigroup under multiplication

$$(a,n,m,z)(a',n',m',z') = (aa', n'm + n, mm', za' + z'),$$

where

$$[(m,z),(a,n)] = (a,mn) \quad \text{and} \quad \langle (m,z),(a,n)\rangle = (m,az).$$

Clearly, $[\cdot,\cdot]$ satisfies the double limit property (8); hence, by Corollary 4.3,

$$(S \times_z T)^{LCWAP} = (\mathbb{T} \times H)' \times_z (J \times \mathbb{C})'.$$

Moreover, by Theorem 4.1,

$$(S \times_z T)^{LC} = (\mathbb{T} \times H)'' \times_z (J \times \mathbb{C})''. \qquad \square$$

(c) Consider $H_4 = \mathbb{Z}^4$ with multiplication

$$\begin{aligned}(j,k,m,n)&(j',k',m',n')\\ &= (j+j'+nk'+m'n(n-1)/2, k+k'+nm', m+m', n+n').\end{aligned}$$

Since $(j,k,m,n) = (0,k,m,0)(j,0,0,n) =: st$ and

$$ts = (j,0,0,n)(0,k,m,0) = (0,k{+}nm,m,0)(j{+}nk{+}mn(n{-}1)/2,0,0,n),$$

H_4 is a Zappa product

$$(0,\mathbb{Z},\mathbb{Z},0)\cdot(\mathbb{Z},0,0,\mathbb{Z}) \cong \mathbb{Z}^2 \times_z \mathbb{Z}^2$$

with

$$[t,s] = (0,k+nm,m',0)$$

and

$$\langle t,s\rangle = (j+nk+mn(n-1)/2,0,0,n).$$

Since $(j,k,m,n) = (j,k,m,0)(0,0,0,n) =: s't'$, H_4 is also a semidirect product

$$(\mathbb{Z},\mathbb{Z},\mathbb{Z},0)(0,0,0,\mathbb{Z}) \cong \mathbb{Z}^3 \times_z \mathbb{Z}$$

with $\langle s',t'\rangle$ trivial and

$$[t',s'] = (j+nk+mn(n-1/2,k+mn,m,0).$$

Therefore, $H_4^{AP} \cong (\mathbb{Z}^3)^A \times_z \mathbb{Z}^{AP}$, where A is generated by the characters $\chi(a,b,c) = \exp\left(2\pi i(\theta_1 a + \theta_2 b + \theta_3 c)\right)$ of $\mathbb{Z}^3$ with the property that each net

$$\chi\big(s'_\alpha[t'_\alpha,s']\big) = \exp\left[2\pi i(\theta_1 a_\alpha + \theta_2 b_\alpha + \theta_3 c_\alpha)\right]$$

where $a_\alpha = j_\alpha + j + n_\alpha k + mn_\alpha(n_\alpha - 1)/2$, $b_\alpha = k_\alpha + k + mn_\alpha$, and $c_\alpha = m_\alpha + m$, has a subnet that converges uniformly in (j,k,m). This clearly forces θ_1 and θ_2 rational, so H_4^{AP} is the semidirect product

$$\big((\mathbb{Z}^B)^2 \times \mathbb{Z}^{AP}\big) \times_z \mathbb{Z}^{AP},$$

where B is generated by the characters $n \mapsto e^{2\pi i\theta n}$ with θ rational. It follows as above that H_4^{AP} is a Zappa product

$$(\mathbb{Z}^B \times \mathbb{Z}^{AP}) \times_z (\mathbb{Z}^B \times \mathbb{Z}^{AP}). \qquad \square$$

(d) The group $G = \mathbb{T} \times \mathbb{Z}^3$ with multiplication

$$(\zeta, k, m, n)(\zeta', k', m', n') = (\zeta\zeta'\lambda^{nk'+m'n(n-1)/2}, k + k' + m'n, m + m', n + n'),$$

where $\lambda = e^{2\pi i\theta}$ is fixed with θ irrational, is a Zappa product

$$(1, \mathbb{Z}, \mathbb{Z}, 0) \cdot (\mathbb{T}, 0, 0, \mathbb{Z}) = \mathbb{Z}^2 \times_z (\mathbb{T} \times \mathbb{Z})$$

and contains a dense isomorphic copy of H_4 under $\phi(j, k, m, n) = (\lambda^j, k, m, n)$. Arguing as above, G^{AP} is a Zappa product

$$(\mathbb{Z}^B \times \mathbb{Z}^{AP}) \times_z (1 \times \mathbb{Z}^{AP}) = (\mathbb{Z}^B \times \mathbb{Z}^{AP}) \times_z \mathbb{Z}^{AP},$$

and the induced homomorphism $\phi^{AP} : H_4^{AP} \to G^{AP}$ maps the third factor of $(\mathbb{Z}^B \times \mathbb{Z}^{AP}) \times_z (\mathbb{Z}^B \times \mathbb{Z}^{AP})$ onto 1 and leaves the other coordinates fixed. $\square$

REFERENCES

1. J. Berglund, H. Junghenn and P. Milnes, *Analysis on semigroups*: *Function spaces, compactifications, representations*, John Wiley and Sons, New York, 1989.

2. K. de Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63–97.

3. ———, *Almost periodic functions on semigroups*, Acta Math. **105** (1961), 99–140.

4. H. Junghenn, *Distal compactifications of semigroups*, Trans. Amer. Math. Soc. **274** (1982), 379–397.

5. H. Junghenn and B. Lerner, *Semigroup compactifications of semidirect products*, Trans. Amer. Math. Soc. **265** (1981), 393–404.

6. M. Kunze, *Lineare Parallelrechner*, Interner Bericht, Tech. Hochschule, Darmstadt, 1982.

7. M. Landstad, *On the Bohr compactification of a transformation group*, Math. Z. **127** (1972), 167–178.

8. G. Mackey, *Products of subgroups and projective multipliers*, Coll. Math. Soc. Janos Bolyai **5** (1970), Hilbert Space Operators, Tihany, Hungary.

9. J. Szep, *On the structure of groups which can be represented as the product of two subgroups*, Acta Sci. Math. Szeged **12** (1950), 57–61.

10. G. Zappa, *Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili traloro*, At. Sec. Cong. Mat. Ital., Bologna, 1942.

Department of Mathematics, The George Washington University, Washington, DC, 20052
Email address: **hdj@gwu.edu**

Department of Mathematics, University of Western Ontario, London, Ontario (Emeritus)
Email address: **milnes@uwo.ca**