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SCALING BY 5 ON A $\frac{1}{4}$–CANTOR MEASURE

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To the memory of William B. Arveson

ABSTRACT. Each Cantor measure μ with scaling factor $\frac{1}{2n}$ has at least one associated orthonormal basis of exponential functions (ONB) for $L^2(\mu)$. In the particular case where the scaling constant for the Cantor measure is $\frac{1}{4}$ and two specific ONBs are selected for $L^2(\mu_{1/4})$, there is a unitary operator U defined by mapping one ONB to the other. This paper focuses on the case in which one ONB Γ is the original Jorgensen-Pedersen ONB for the Cantor measure $\mu_{1/4}$ and the other ONB is 5Γ. The main theorem of the paper states that the corresponding operator U is ergodic in the sense that only the constant functions are fixed by U.

1. Introduction.

The factor 4 *is a gift of God* (*or of the other party*).
—John von Neumann to Edward Teller, 1946

Infinite Bernoulli convolutions are special cases of affine self-similarity systems, also called *iterated function systems* (IFSs). Thus, IFS measures generalize distributions of Bernoulli convolutions; Bernoulli convolutions in turn generalize Cantor measures. For over a decade, it has been known that a subclass of IFS measures μ have associated Fourier bases for $L^2(\mu)$ **[27]**. If $L^2(\mu)$ does have a Fourier ONB with Fourier frequencies $\Gamma \subset \mathbb{R}$, we then say that (μ, Γ) is a *spectral pair*. In the

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case that a set of Fourier frequencies exist for $L^2(\mu)$, we say that Γ is a *spectrum* for μ; we say μ is a *spectral measure*. The goal of this paper is to examine the operator U which scales one spectrum into another spectrum. We observe how the intrinsic scaling (by 4) which arises in our set Γ interacts with the spectral scaling (to 5Γ) that defines U. We call U an operator-fractal due to its self-similarity, which is described in detail in [**26**].

The self-similarity property makes the spectrum of U interesting. The main theorem in this paper is Theorem 4.6, which states that the only functions which are fixed by U are the constant functions—in other words, U is an ergodic operator in the sense of Halmos [**19**].

The spectral connection to scaling factors begun in [**27**] can be highly non-intuitive. For example, when the scaling factor is $\frac{1}{3}$—that is, $\mu_{\frac{1}{3}}$ is the Cantor-Bernoulli measure for the omitted third Cantor set construction—there is no Fourier basis. In other words, there is no Fourier series representation in $L^2(\mu_{1/3})$. In fact, there can be at most two orthogonal Fourier frequencies in $L^2(\mu_{1/3})$ [**27**]. But if we modify the Cantor-Bernoulli construction, using scale $\frac{1}{4}$, as opposed to $\frac{1}{3}$, then the authors of [**27**] proved that a Fourier basis does exist in $L^2(\mu_{1/4})$. They showed much more: *each* of the Cantor-Bernoulli measures $\mu_{1/2n}$ with $n \in \mathbb{N}$ has a Fourier basis. For each of these measures, there is a canonical choice for a Fourier spectrum $\Gamma_{1/2n}$.

We consider here a particular additional symmetry relation for the subclass of Cantor-Bernoulli measures that form spectral pairs. Starting with a spectral pair (μ, Γ), we consider an action which scales the set Γ. In the special case of $\mu_{1/4}$, we scale Γ by 5. Scaling by 5 induces a natural unitary operator U in $L^2(\mu_{1/4})$, and we study the spectral-theoretic properties of U.

1.1. Bernoulli convolution measures. The Bernoulli convolution measure with scaling factor λ, denoted μ_λ, can be constructed with an *iterated function system* (IFS) of two affine maps

$$\tau_+(x) = \lambda(x+1) \quad \text{and} \quad \tau_-(x) = \lambda(x-1). \tag{1}$$

By Banach's fixed point theorem, there exists a compact subset of the line, denoted X_λ and called the *attractor* of the IFS, which satisfies the invariance property

$$X_\lambda = \tau_+(X_\lambda) \cup \tau_-(X_\lambda). \tag{2}$$

Hutchinson proved that there exists a unique measure μ_λ corresponding to the IFS (1), which is supported on X_λ and is invariant in the sense that

$$\mu_\lambda = \frac{1}{2}\big(\mu_\lambda \circ \tau_+^{-1}\big) + \frac{1}{2}\big(\mu_\lambda \circ \tau_-^{-1}\big), \tag{3}$$

[**21,** Theorems 3.3(3) and 4.4(1)]. The property in equation (3) defines the measure μ_λ and can be used to compute its Fourier transform. The Fourier transform of μ_λ is a Riesz-type product:

$$\widehat{\mu}_\lambda(t) = \prod_{k=1}^{\infty} \cos(2\pi\lambda^k t). \tag{4}$$

Bernoulli convolution measures have been studied in various settings, long before IFS theory was developed. Some of the earliest papers on Bernoulli convolution measures date to the 1930s and work with an infinite convolution definition for μ_λ; they include [**12, 22, 29, 39**]. The history of Bernoulli convolutions up until 1998 is detailed in [**36**].

1.2. Notation and terminology. We will use the notation $e_t(\cdot)$ to denote the complex exponential function $e^{2\pi i t(\cdot)}$. Given a set $\Gamma \subseteq \mathbb{R}$, we denote by $E(\Gamma)$ the set $\{e_\gamma : \gamma \in \Gamma\}$. Throughout, we fix $\lambda = \frac{1}{4}$ and work exclusively with the Bernoulli convolution measure $\mu_{1/4}$, which we often will denote just as μ. We will work with the set Γ from [**27**]:

$$\begin{aligned} \Gamma &= \left\{ \sum_{i=0}^{m} a_i 4^i : a_i \in \{0,1\}, m \text{ finite} \right\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \ldots\}. \end{aligned} \tag{5}$$

Jorgensen and Pedersen showed that Γ is a spectrum for μ—that is, the set of exponential functions $E(\Gamma)$ is an orthonormal basis for $L^2(\mu)$ [**27,** Theorem 5.6 and Corollary 5.9].

It is known that other scaling symmetries are possible in $L^2(\mu)$; examples are given in [**8, 25, 33**]. In particular, Dutkay and Jorgensen have shown that the ONB property is preserved under scaling by powers of 5—that is, for each $n \in \mathbb{N}$, each scaled set $5^n\Gamma$ is also a spectrum for a Fourier basis for $L^2(\mu)$ [**8,** Proposition 5.1]. This result may be counterintuitive since the resulting scaled set (6) of Fourier frequencies appears quite "thin." In this paper, we will restrict our attention to the case $n = 1$:

$$5\Gamma = \{0, 5, 20, 25, 80, 85, 100, 105, 320, \ldots\}. \tag{6}$$

The 5–scaling property for the ONB (5) induces a unitary operator U in $L^2(\mu)$, as given in the next definition.

Definition 1.1. Define the operator U on the orthonormal basis $E(\Gamma)$ by

$$U(e_\gamma) := e_{5\gamma} \quad \text{for all } \gamma \in \Gamma. \tag{7}$$

In [**26**], we gave operators such as U the name *operator-fractals* due to the self-similarity they exhibit. Due to this self-similar structure, the spectral representation and the spectral resolution for U are surprisingly subtle. Despite this, we are able to establish ergodic and spectral-theoretic properties of the unitary operator U.

1.3. Organization of the paper. We begin in Section 1 with a background discussion of Fourier bases on Cantor measures and motivate our interest in the operator-fractal U. In Section 2, we list some of the standard results from spectral theory which will be used later in the paper. Section 3 presents some of the unique properties of the unitary operator U. Our main theorem—Theorem 4.6—demonstrates that the only functions fixed by U are constant functions. In other words, U is an ergodic operator. Theorem 4.6 is proved in Section 4. In Section 5, we explore various aspects of the relationships of the scaling factors $(\times 4)$ and $(\times 5)$ inherent in the operator U.

1.4. Recent developments and associated literature. The paper which started much of the work considered here is [**27**]. Since then, a large literature on duality and spectral theory for affine dynamical systems has evolved. Here, we point out just a few of the most recent developments in the field. First, the papers of Li study orthogonal exponential functions with respect to invariant measures [**30, 31, 32**]; the papers [**20, 24, 28, 42**] also fit into this framework. Dutkay, Jorgensen and their coauthors have a range of work pertaining to Fourier duality: [**4, 5, 7, 9, 10, 11**]. Spectral measures for affine IFSs are also studied in the works [**13, 14, 34**]. The relationship of wavelets and frames to self-similar measures is explored in [**2, 6**]. The works of Gabardo and his coauthors are also highly relevant: [**15, 16, 17, 41**].

2. The spectral theorem and some of its consequences. Starting with the spectral pair (μ, Γ), where Γ is given in (5), we study the unitary operator U in $L^2(\mu)$ corresponding to a scaling of Γ by 5 in detail. In order to understand U, we ask for information about its spectrum. For reference, we list here some results from spectral theory which will be used in the later proofs. Details can be found in [**1,** Chapters IX and X], Dunford and Schwartz [**3,** Chapter X] and Nelson [**35,** Chapter 6].

Theorem 2.1. (The spectral theorem for unitary operators) [**1,** Theorem 10.10, page 200]. *Let U be a unitary operator on $\mathcal{H}$. Then there exists a unique Borel p.v.m. E^U on the Borel space $(\mathbb{T}, \mathcal{B})$ such that*

$$U = \int_{\sigma(U)} z \, \mathrm{d}E^U(z). \tag{8}$$

The measure E^U is supported on the spectrum of U, $\sigma(U) \subseteq \mathbb{T}$.

Next, we recall the *functional calculus* associated with the spectral theorem. Given a Borel function ϕ on $\mathbb{T}$, we can study the associated operator $\phi(U)$. The construction of $\phi(U)$ begins with the case where ϕ is a polynomial (with both positive and negative powers) and then extends to continuous functions and Borel functions. The next lemma, which holds for E^U-essentially bounded functions $\phi : \mathbb{T} \to \mathbb{C}$, can be extended to suitable Borel functions ϕ by Lemma 2.4. See both [**1,** Theorem 10.9] and [**3,** Chapter X.2], especially Corollaries X.2.8 and X.2.9 and the material between the two corollaries, for more information about the following lemma.

Lemma 2.2. *Suppose U is a unitary operator on the Hilbert space $\mathcal{H}$ with associated p.v.m. E^U, so that*

$$U = \int_{\sigma(U)} z \, E^U(dz).$$

Suppose $\phi, \phi_1, \phi_2 : \mathbb{T} \to \mathbb{C}$ *are* E^U*-essentially bounded, Borel-measurable functions. Define*

$$\pi_U(\phi) = \phi(U) = \int_{\sigma(U)} \phi(z)\, E^U(dz). \tag{9}$$

Then

(i) $[\phi(U)]^* = \overline{\phi}(U)$. *In other words,* π_U *is a* $*$*-homomorphism.*
(ii) $\pi_U(\phi_1\phi_2) = \pi_U(\phi_1)\pi_U(\phi_2)$, *and as a result, the operators* $\phi_1(U)$ *and* $\phi_2(U)$ *commute.*
(iii) *If* $\phi(z) \equiv 1$, *then* $\phi(U)$ *is the identity operator.*
(iv) *The operator* $\phi(U)$ *is bounded.*

We note that the converse of (iv) is true as well: if $\phi(U)$ is bounded, then the function ϕ is E^U-essentially bounded. Finally, Lemma 2.2 is also true for normal operators N, with $\mathbb{T}$ being replaced by $\mathbb{C}$.

For each vector $v \in \mathcal{H}$, there exists a real-valued Borel measure m_v supported on $\mathbb{T}$ such that

$$m_v(A) = \langle E^U(A)v, v\rangle_{\mathcal{H}}, \tag{10}$$

where $E^U(A)$ is the projection $\int_{\sigma(U)} \chi_A(z)E^U(dz)$. When v is a unit vector, note that m_v is a probability measure [**38,** (2), page 302].

There is an important isometric connection between operators of the form $\phi(U)$ and the measures m_v, which we state as the next lemma.

Lemma 2.3. [**3,** Corollary X.2.9]. *Suppose* U *is a unitary operator on the Hilbert space* $\mathcal{H}$ *with associated p.v.m.* E^U. *Let* m_v *be the Borel measure on* $\mathcal{H}$ *defined in equation* (10). *Suppose* $\phi : \mathbb{T} \to \mathbb{C}$ *is an* E^U*-essentially bounded, Borel-measurable function. Then*

$$\|\phi(U)v\|^2_{\mathcal{H}} = \int_{\sigma(U)} |\phi(z)|^2 \,\mathrm{d}m_v(z). \tag{11}$$

For any m_v-integrable function ϕ on $\mathbb{T}$,

$$\begin{aligned}\int_{\mathbb{T}} \phi(z)\,\mathrm{d}m_v(z) &= \left\langle \int_{\mathbb{T}} \phi(z)E^U(dz)v, v\right\rangle_{\mathcal{H}} \\ &= \langle \phi(U)v, v\rangle_{\mathcal{H}}.\end{aligned} \tag{12}$$

We noted earlier that $\phi(U)$ is a bounded operator if and only if ϕ is E^U-essentially bounded. However, $\phi(U)$ can be a well-defined unbounded operator for some *unbounded* Borel functions $\phi : \mathbb{T} \to \mathbb{C}$. In the case that $\phi(U)$ is an unbounded operator, we need to be especially vigilant about the domain of $\phi(U)$. When $\phi(U)$ is a well-defined unbounded operator, the usual formulas discussed in the bounded case do, in fact, carry over. Moreover, it is also known that the domain of the operator $\phi(U)$ is determined by the measures m_v, as described in Lemma 2.4.

Lemma 2.4. *Suppose ϕ is a Borel measurable function on $\mathbb{T}$ and N is a normal operator on the Hilbert space $\mathcal{H}$. Let $\phi(N)$ be the operator defined by*

$$\phi(N) = \int_{\mathbb{T}} \phi(z) E^N(dz). \tag{13}$$

Then $\phi(N)$ is a densely defined operator, and $v \in dom(\phi(N))$ if and only if $\phi \in L^2(m_v)$. In this case, the isometry in (11) *holds*:

$$\|\phi(N)v\|^2 = \int_{\mathbb{T}} |\phi(z)|^2 \, dm_v(z) = \|\phi\|^2_{L^2(m_v)}. \tag{14}$$

From Lemma 2.4, the results of Lemmas 2.2 and 2.3 can be extended to suitable Borel (not necessarily essentially bounded) functions ϕ on $\mathbb{T}$.

3. The unitary operator U. In this section we present more details about the unitary operator U on $\mathcal{H} = L^2(\mu)$ defined by $Ue_\gamma = e_{5\gamma}$ for all $\gamma \in \Gamma$. We observe that U has a number of intriguing properties which make it of interest from the spectral theoretic point of view.

3.1. Properties of U. Given the definition of U on exponential functions, one might ask whether there is a straightforward way to compute powers of U. Equations (5) and (6) show that 5Γ is not contained in Γ, though, so it would be surprising if U behaved well with respect to iteration—and in fact, it does not.

Proposition 3.1. *The formula $U^k e_\gamma = e_{5^k\gamma}$ does not hold in general.*

Proof. It is sufficient to prove inequality for a specific example: consider the case $\gamma = 1$ and $k = 3$. We have $U(e_1) = e_5$, and since $5 \in \Gamma$, $U^2(e_1) = e_{25}$. However, $25 \notin \Gamma$, so we expand e_{25} in terms of $E(\Gamma)$ to compute $U(e_{25})$:

$$\begin{aligned} U(e_{25}) &= U\left(\sum_{\gamma\in\Gamma} \widehat{\mu}_{1/4}(25-\gamma)e_\gamma\right) = \sum_{\gamma\in\Gamma} \widehat{\mu}_{1/4}(25-\gamma)e_{5\gamma} \\ &= \sum_{\gamma\in\Gamma} \widehat{\mu}_{1/4}(25-\gamma)\left(\sum_{\xi\in\Gamma} \widehat{\mu}_{1/4}(5\gamma-\xi)e_\xi\right) \\ &= \sum_{\xi,\gamma\in\Gamma} \widehat{\mu}_{1/4}(25-\gamma)\widehat{\mu}_{1/4}(5\gamma-\xi)e_\xi. \end{aligned} \tag{15}$$

On the other hand,

$$e_{125} = \sum_{\xi\in\Gamma} \widehat{\mu}_{1/4}(125-\xi)e_\xi. \tag{16}$$

Now compare the $\xi = 5$ term in equations (15) and (16). In equation (16), the coefficient of e_5 is $\widehat{\mu}_{1/4}(120) = \widehat{\mu}_{1/4}(30) \approx 0.50$. In equation (15), the coefficient of e_5 is

$$\sum_{\gamma\in\Gamma} \widehat{\mu}_{1/4}(25-\gamma)\widehat{\mu}_{1/4}(5\gamma-5) \approx 0.58.$$

The approximations were made with 512 terms of $\Gamma(1/4)$ in *Mathematica*. $\square$

Corollary 3.2. *U is not implemented by a transformation of the form $U(f) = f \circ \tau$ where $\tau(x) = 5x \pmod 1$.*

In Section 5, we will further see that the operator U cannot be spatially implemented by any point transformation. The distinction between the behavior of unitary operators which are implemented by such a transformation τ and the behavior of the unitary operator U is one of the motivations for why we study U in detail. Theorem 4.6 states that the only functions fixed by U are the constant functions. While our unitary operator U is not spatially implemented, we can still form Cesaro means of its iterations, and one of the corollaries of Theorem 4.6 is an application of the von Neumann ergodic theorem in Section 5.

Another motivation to study this particular operator U comes from the relationship U has with the representation of the Cuntz algebra $\mathcal{O}_2$, which is realized by the two operators

$$S_0(e_\gamma) = e_{4\gamma} \text{ and } S_1(e_\gamma) = e_{4\gamma+1}$$

defined on the ONB $E(\Gamma)$. The operator U commutes with S_0 but does not commute with S_1. The fact that U does not commute with S_1 makes its spectral theory harder to understand, but the commuting with S_0 gives us a foothold into its spectral theory. The relationship between U and operators forming the representation of $\mathcal{O}_2$ is studied in detail in [**26**].

We make a preliminary observation about how U scales elements of the ONB $E(\Gamma)$.

Lemma 3.3. *Suppose $\gamma \in \Gamma$ and $\lambda \in \mathbb{T}$ are such that*

$$Ue_\gamma = \lambda e_\gamma \in L^2(\mu_{1/4}). \tag{17}$$

Then $\gamma = 0$ and $\lambda = 1$.

Proof. Suppose $\gamma \in \Gamma \backslash \{0\}$. If $Ue_\gamma = \lambda e_\gamma$, then

$$\begin{aligned} 0 &= \|Ue_\gamma - \lambda e_\gamma\|^2_{L^2(\mu)} \\ &= \|e_{5\gamma} - \lambda e_\gamma\|^2_{L^2(\mu)} \\ &= \|e_{4\gamma} - \lambda e_0\|^2_{L^2(\mu)} = 2 \end{aligned} \tag{18}$$

since e_0 and $e_{4\gamma}$ are distinct elements of the ONB $E(\Gamma)$ and $|\lambda| = 1$. Therefore, we have a contradiction. □

Remark 1. We note here the special property that the scaling factor 5 plays in the argument above, in particular that $(5-1)e_\gamma$ is always also an element of Γ. This sets 5 apart from many other odd scale factors. Lemma 3.3 replicated for another odd integer p requires both that $p\Gamma$ also be a spectrum for μ (which is not always the case and is not generally easy to check) and that p be of the form $p = 4^k + 1$. Thus, we have chosen to concentrate on the generic case $p = 5$.

3.2. Cyclic subspaces of U. A key component in the proof of Theorem 4.6 is the correspondence between the U-cyclic subspaces of $L^2(\mu)$ and the Hilbert spaces $L^2(m_v)$ with respect to the scalar measures generated by U. We now define the cyclic subspaces of the unitary operator U and present this correspondence with the spaces $L^2(m_v)$. For more details about the structure of the cyclic subspaces, see [**35, 37**].

Definition 3.4. Let $v \in L^2(\mu)$. Then

$$H(v) = \overline{\operatorname{span}}_{L^2(\mu)}\{U^k v \mid k \in \mathbb{Z}\}. \tag{19}$$

In other words, if $\phi(z)$ is the polynomial $\phi(z) = \sum_{k=-N}^{N} c_k z^k$ where $c_k \in \mathbb{C}$, $k = -N, \ldots, N$ and $z \in \mathbb{T}$, then the vectors $\phi(U)v$ are dense in $H(v)$.

The U-cyclic subspace $H(v)$ is the smallest closed subspace which contains v and is invariant under U and U^* [**35**]. There is also another characterization of the cyclic subspaces which directly connects $H(v)$ to the space $L^2(m_v)$. Recall that $\phi \in L^2(m_v)$ if and only if v belongs to the domain of $\phi(U)$, by Lemma 2.4. The map from $\phi \in L^2(m_v)$ to the vector $\phi(U)v \in L^2(\mu)$ is an isometry, and in fact, is bijective [**37,** Thm. 2.1, Cor. 2.5].

Theorem 3.5. *Given $v \in L^2(\mu)$ with $\|v\| = 1$, the map $\phi \mapsto \phi(U)v$ is an isometric isomorphism between the Hilbert space $L^2(m_v)$ and the cyclic subspace $H(v)$.*

Every element w of the cyclic subspace $H(v)$ can therefore be written uniquely in the form $w = \psi(U)v$. Further, given $w \in H(v)$, we also know the corresponding function ψ given Theorem 3.5.

Theorem 3.6. [**37,** Corollary 2.5] *Let $w \in H(v)$, and let $\psi \in L^2(m_v)$ be such that $w = \psi(U)v$. Then*

$$\psi = \sqrt{\frac{dm_w}{dm_v}}. \tag{20}$$

4. Spectral properties of U. In this section, we prove that the unitary operator U on $L^2(\mu)$ defined from the 5-scaled ONB 5Γ acts ergodically, with ergodicity defined relative to μ in the sense of Halmos **[19]**. Specifically, only the constant functions are invariant under U.

Lemma 4.1. *Let U be a unitary operator on a Hilbert space $\mathcal{H}$, and let $v \in \mathcal{H}$ with $\|v\| = 1$. The spectral measure m_v with respect to U and v is a Dirac mass supported at 1 if and only if $Uv = v$.*

Proof. ($\Rightarrow$). Assume $m_v = \delta_1$, and consider the norm $\|Uv - v\|^2$. By separating the inner product, we get

$$\begin{aligned} \|v - Uv\|^2 &= \Big\langle v - Uv, v - Uv \Big\rangle \\ &= \|Uv\|^2 + \|v\|^2 - \langle v, Uv\rangle - \overline{\langle v, Uv\rangle}. \end{aligned} \tag{21}$$

Since U is unitary and $\|v\| = 1$, we have

$$\|v - Uv\|^2 = 2 - \langle v, Uv\rangle - \overline{\langle v, Uv\rangle}. \tag{22}$$

Now we take advantage of the measure m_v:

$$\begin{aligned} \|v - Uv\|^2 &= 2 - \langle v, Uv\rangle - \overline{\langle v, Uv\rangle} \\ &= 2 - \int z\, dm_v(z) - \int \overline{z}\, dm_v(z) \\ &= 2 - \int z\, d\delta_1(z) - \int \overline{z}\, d\delta_1(z) \\ &= 2 - 1 - \overline{1} = 0. \end{aligned} \tag{23}$$

Therefore, $\|v - Uv\| = 0$, hence $v = Uv$.

($\Leftarrow$). Suppose $Uv = v$. For any $f \in L^2(m_v)$,

$$f(U) = \int f(z) E^U(dz). \tag{24}$$

Since $Uv = v$, we find that

$$f(U)v = f(1)v. \tag{25}$$

To see this, start with a polynomial: if $f(U) = \sum_{k=-\ell}^{\ell} c_k U^k$, then

$$f(U)v = \left(\sum_{k=-\ell}^{\ell} c_k U^k\right) v = \sum_{k=-\ell}^{\ell} c_k U^k v = \left(\sum_{k=-\ell}^{\ell} c_k\right) v = f(1)v.$$

We then use the polynomials as the starting point for approximating all other functions in $L^2(m_v)$. In particular, let f be a characteristic function χ_A for a Borel subset A of the circle $\mathbb{T}$. The right hand side above is $\chi_A(1)$. Then by equation (25),

$$m_v(A) = \langle \chi_A(1)v, v\rangle = \begin{cases} 1 & 1 \in A \\ 0 & 1 \notin A. \end{cases}$$

In other words, m_v is the Dirac mass δ_1. □

Corollary 4.2. *Let U be unitary on $\mathcal{H}$, and suppose $v \in \mathcal{H}$ with $\|v\| = 1$. Then $Uv = \lambda v$ for some $\lambda \in \mathbb{T}$ if and only if m_v is a Dirac mass supported at λ.*

Proof. Replace "1" in the proof above by "λ". □

Assume now that v is a non-constant function which is fixed by U. By Lemma 3.3, v cannot actually be one of the ONB elements e_γ for any $\gamma \in \Gamma\backslash\{0\}$. We next consider whether v can belong to a cyclic subspace generated by one of the e_γ functions.

Proposition 4.3. *Let $U : e_\gamma \mapsto e_{5\gamma}$ for all $\gamma \in \Gamma$. Suppose $v \in L^2(\mu)$ is a nonconstant function with $\|v\| = 1$. Choose any $\gamma \in \Gamma\backslash\{0\}$ such that $\langle v, e_\gamma\rangle \neq 0$. If $Uv = v$, then v is not in the U-cyclic subspace $H(e_\gamma)$ generated by e_γ.*

Proof. By Lemma 3.3, we know $v \neq e_\gamma$; hence, $0 < |\langle v, e_\gamma\rangle| < 1$. Assume that $v \in H(e_\gamma)$—i.e., by Theorem 3.5,

$$v = f(U)e_\gamma, \tag{26}$$

for a unique $f \in L^2(m_{e_\gamma})$. Since $Uv = v$, we have $Uf(U)e_\gamma = f(U)e_\gamma$. Therefore, using the isometric isomorphism between $H(e_\gamma)$ and $L^2(m_{e_\gamma})$, we have

$$zf(z) = f(z), \quad \text{or} \quad f(z)(z-1) = 0 \tag{27}$$

almost everywhere m_{e_γ} on $\mathbb{T}$. In other words, since U fixes v, we know that f is fixed by multiplication by z.

We know that f is a nonzero function from $L^2(m_{e_\gamma})$, so $f(z)$ must be nonzero at $z = 1$, and f is 0 almost everywhere m_{e_γ} on $\mathbb{T} \setminus \{1\}$. This also implies that $m_{e_\gamma}(\{1\}) > 0$.

Claim 4.4. ([**37,** Corollary 2.5]). *The measures* m_{e_γ} *and* m_v *satisfy*:

$$|f(z)|^2 dm_{e_\gamma}(z) = dm_v(z) = d\delta_1(z). \tag{28}$$

To verify the claim, let $\phi \in C(\mathbb{T})$. Then

$$\begin{aligned}\int_{\mathbb{T}} |f(z)|^2 \phi(z) dm_{e_\gamma}(z) &= \langle f(U)\overline{f(U)}\phi(U)e_\gamma, e_\gamma\rangle_{L^2(\mu)}\\ &= \langle \phi(U)f(U)e_\gamma, f(U)e_\gamma\rangle_{L^2(\mu)}\\ &= \langle \phi(U)v, v\rangle_{L^2(\mu)}\\ &= \int \phi(z)\, dm_v(z).\end{aligned} \tag{29}$$

Then, by Lemma 4.1, $m_v = \delta_1$.

Since $|f|^2 m_{e_\gamma} = \delta_1$ is a probability measure supported at 1 and f is zero almost everywhere m_{e_γ} except at $z = 1$, we know that

$$\int_{\mathbb{T}} f(z)\, dm_{e_\gamma} = f(1), \tag{30}$$

and

$$|f(1)|^2 = 1. \tag{31}$$

Next, let $k(z) = f(z) - f(1)$ for $z \in \mathbb{T}$.

$$\|f(U)e_\gamma - f(1)e_\gamma\|^2_{L^2(\mu))} = \|k(U)e_\gamma\|^2_{L^2(\mu)} = \|k(z)\|^2_{L^2(m_{e_\gamma})} \tag{32}$$

Now, we look at the inner product defining $\|k(z)\|^2_{L^2(m_{e_\gamma})}$:

$$\begin{aligned}&\int_{\mathbb{T}} k(z)\overline{k(z)}\, dm_{e_\gamma}\\ &= \int_{\mathbb{T}} |f(z)|^2 dm_{e_\gamma} + \int_{\mathbb{T}} |f(1)|^2 dm_{e_\gamma} - 2\mathrm{Re}\overline{f(1)} \int_{\mathbb{T}} f(z)\, dm_{e_\gamma}\\ &= \underbrace{1}_{\text{Claim}} + \underbrace{1}_{\text{Eqn (31)}} - \underbrace{2\mathrm{Re}\overline{f(1)}f(1)}_{\text{Eqn (30)}} = 0.\end{aligned} \tag{33}$$

This proves that, if we assume $Uv = v$ and $v \in H(e_\gamma)$, then we must have

$$v = f(1)e_\gamma.$$

But U cannot fix any scalar multiple of an exponential function for $\gamma \neq 0$ by Lemma 3.3. Therefore, v cannot belong to any cyclic subspace $H(e_\gamma)$ for $\gamma \in \Gamma \setminus \{0\}$. $\square$

Lemma 4.5. *Suppose U is a unitary operator on $L^2(\mu)$. Suppose $v, w \in L^2(\mu)$. If $v \perp H(w)$, then $H(v) \perp H(w)$.*

Proof. Let $f \in L^2(m_w)$, and let $x = f(U)w$. Let $k \in \mathbb{Z}$. Consider the inner product

$$\langle U^k v, f(U)w \rangle = \langle v, U^{-k} f(U)w \rangle. \tag{34}$$

Since $f \in L^2(m_w)$ and m_w is supported on the circle $\mathbb{T}$, we also have $z^{-k} f(z) \in L^2(m_w)$. Theorem 3.5 then gives $U^{-k} f(U)w \in H(w)$. Since v is orthogonal to $H(w)$,

$$\langle U^k v, f(U)w \rangle = \langle v, U^{-k} f(U)w \rangle = 0. \tag{35}$$

By linearity, every vector of the form $g(U)v$ where g has the form

$$g(z) = \sum_{n=-k}^{k} c_k z^k \tag{36}$$

is orthogonal to $H(w)$. Since the functions g in equation (36) are dense in $L^2(m_v)$, we can conclude that $H(v) \perp H(w)$. $\square$

We remark here that, given $v \neq 0$, we can define a real-valued probability measure on $\mathbb{T}$ with $\widetilde{m}_v = m_v / \|v\|^2$, where

$$\widetilde{m}_v(A) = \frac{m_v}{\|v\|^2}(A) = \frac{1}{\|v\|^2} \langle E^U(A)v, v \rangle \tag{37}$$

for any Borel set $A \subseteq \mathbb{T}$.

We now come to our main result, in which we prove that in fact only the constant functions are fixed by U.

Theorem 4.6. *Let $U : e_\gamma \mapsto e_{5\gamma}$ for all $\gamma \in \Gamma$. If $Uv = v$, with $\|v\| = 1$, then $v = \alpha e_0$ for some $\alpha \in \mathbb{T}$, i.e., U is an ergodic operator.*

Proof. Assume there exists $v \in L^2(\mu) \ominus \overline{\text{span}\{e_0\}}$ such that $Uv = v$ and $\|v\| = 1$. Choose $\gamma \in \Gamma \setminus \{0\}$ such that $\langle v, e_\gamma \rangle_{L^2(\mu)} \neq 0$. Let Q be the orthogonal projection onto $H(e_\gamma)$.

Let $v = v_1 + v_2 = Qv + v_2$, where v_2 is orthogonal to v_1. By Proposition 4.3, we know both v_1 and v_2 are nonzero. Because v_2 is orthogonal to $H(e_\gamma)$, $H(v_2) \perp H(e_\gamma)$ by Lemma 4.5.

Let A be a Borel set in $\mathbb{T}$. Recall that E^U is the projection-valued measure associated to U via the spectral theorem. We compute $m_v(A)$:

$$\begin{aligned} m_v(A) &= \langle E^U(A)v, v\rangle = \langle E^U(A)(v_1+v_2), v_1+v_2\rangle \\ &= \langle E^U(A)v_1, v_1\rangle + \langle E^U(A)v_1, v_2\rangle \\ &\quad + \langle E^U(A)v_2, v_1\rangle + \langle E^U(A)v_2, v_2\rangle. \end{aligned} \tag{38}$$

Now, $E^U(A)v_1$ is an element of $H(v_1)$ because

$$E^U(A) = \int_{\mathbb{T}} \chi_A(z)\, dE^U(z) = \chi_A(U).$$

Since $v_1 \in H(e_\gamma)$, we can write $v_1 = f(U)e_\gamma$ for f a function in $L^2(m_{e_\gamma})$. Then,

$$\langle E^U(A)v_1, v_2\rangle = \langle \chi_A(U)f(U)e_\gamma, v_2\rangle.$$

The product $\chi_A \cdot f$ is again a function in $L^2(m_{e_\gamma})$, so Theorem 3.5 shows that $E^U v_1 \in H(e_\gamma)$. Thus we have that the term $\langle E^U(A)v_1, v_2\rangle = 0$ since v_2 is orthogonal to $H(e_\gamma)$. Similarly,

$$\langle E^U(A)v_2, v_1\rangle = \langle v_2, E^U(A)v_1\rangle = \langle v_2, \chi_A(U)f(U)e_\gamma\rangle = 0.$$

This gives, for any Borel subset $A \subseteq \mathbb{T}$,

$$\begin{aligned} m_v(A) &= \langle E^U(A)v_1, v_1\rangle + \langle E^U(A)v_2, v_2\rangle \\ &= m_{v_1}(A) + m_{v_2}(A) \\ &= \|v_1\|^2 \widetilde{m}_{v_1}(A) + \|v_2\|^2 \widetilde{m}_{v_2}(A). \end{aligned}$$

We have shown in the above that m_v is a convex combination of the probability measures $\widetilde{m}_{v_1}$ and $\widetilde{m}_{v_2}$. The coefficients are both nonzero since the vectors v_1 and v_2 are both nonzero. But this contradicts the fact from Lemma 4.1 that $m_v = \delta_1$ since Dirac measures are extreme points in the convex space of probability measures. With this contradiction, we find that v must be a unit vector in the span of the vector e_0. Therefore, the operator U is ergodic. $\square$

5. The mixed scales 4 **and** 5**.** In this section, we study the two different scales $\times 4$ and $\times 5$—scaling by 4 and scaling by 5. We have devoted most of the paper to the scale $\times 5$ because $\times 5$ maps one ONB of $L^2(\mu)$ to another. However, the "natural" scale inherent in $L^2(\mu)$ is $\times 4$. For example, if $\tau_n : [0,1] \to [0,1]$ is defined by $\tau_n(x) = nx \pmod 1$, then

$$\mu \circ \tau_4^{-1} = \mu.$$

We will see that it is difficult to obtain positive results for the corresponding measure

$$\mu \circ \tau_5^{-1}.$$

Let $U_n : L^2(\mu) \to L^2(\mu)$ be defined on the ONB $E(\Gamma)$ by

$$U_n(e_\gamma) = e_{n\gamma}. \tag{39}$$

As we have seen, U_5 is ergodic (Theorem 4.6), and U_4 is an isometry but not unitary.

First, we will study the spatial implementation of $U = U_5$ and U_4. Then we compare ergodic theorems for the operators U_5 and U_4. Finally, we compare the spectral measures from U_5 to the measure μ itself. Our results about the scaling pair $(\times 4, \times 5)$ fit into the setting of the paper **[23]**, which explores occurrence and non-occurrence of mixed scaling in ergodic theory.

5.1. Spatial implementation. Recall from the discussion in subsection 3.1, equations (15) and (16), that although it is tempting to think that $U_5^k e_\gamma = e_{5^k\gamma}$, this equation does not hold in general. However, such an equation certainly holds for U_4.

Definition 5.1. We say that the operator T is *spatially implemented* if there exists a point transformation $\tau : [0,1] \to [0,1]$ such that $Tf = f \circ \tau$.

Proposition 5.2. *The operator $U_5 : L^2(\mu) \to L^2(\mu)$ is not spatially implemented.*

Proof. Suppose there were such a transformation $\tau : [0,1] \to [0,1]$ such that $U_5 f = f \circ \tau$. Then $U_5(fg) = U_5(f)U_5(g)$, and as a specific consequence,

$$U_5(e_1 \cdot e_1) = U_5(e_1) \cdot U_5(e_1) = e_5 \cdot e_5 = e_{10} \\ = \widehat{\mu}(10)e_0 + \sum_{\xi \neq 0} \widehat{\mu}(10-\xi)e_\xi. \tag{40}$$

On the other hand, $e_1 \cdot e_1 = e_2$, and

$$\begin{aligned} U_5(e_2) &= \sum_{\gamma \in \Gamma} \widehat{\mu}(2-\gamma)e_{5\gamma} = \sum_{\gamma,\xi \in \Gamma} \widehat{\mu}(2-\gamma)\widehat{\mu}(5-\xi)e_\xi \\ &= \sum_{\gamma \in \Gamma} \widehat{\mu}(2-\gamma)\widehat{\mu}(5)e_0 + \sum_{\substack{\gamma \in \Gamma \\ \xi \neq 0}} \widehat{\mu}(2-\gamma)\widehat{\mu}(5-\xi)e_\xi \\ &= 0\, e_0 + \sum_{\substack{\gamma \in \Gamma \\ \xi \neq 0}} \widehat{\mu}(2-\gamma)\widehat{\mu}(5-\xi)e_\xi. \end{aligned} \tag{41}$$

By comparing the constant terms in equations (40) and (41), we see that the two expressions cannot be the same, since $\widehat{\mu}(10) \neq 0$. Therefore U_5 is not spatially implemented. □

The operator U_4, on the other hand, is readily seen to be spatially implemented by the map $\tau_4(x) = 4x \pmod 1$.

5.2. Averaging. With Theorem 4.6 in hand, we can study averaging with respect to U_5 and U_4. Suppose $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator, and

$$Q = \{f \in \mathcal{H} : Tf = f\}.$$

Let P_Q be the orthogonal projection onto Q. The ergodic theorem of von Neumann states that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} T^k f = P_Q(f), \tag{42}$$

[40]. In the special case of $U_5 : L^2(\mu) \to L^2(\mu)$, the subspace Q is the one-dimensional space spanned by the constant function e_0 by

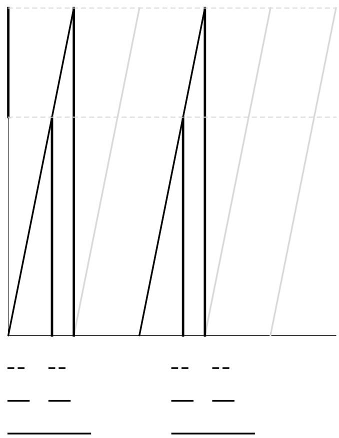

FIGURE 1. The graph of τ_5 on $[0,1] \times [0,1]$ sits above the first two approximations of the Cantor set $X_{1/4}$. The set $(2/3, 1]$ is pulled back by two branches of τ_5^{-1}. The left-most branch of τ_5^{-1} pulls $[2/3, 1)$ back to a set on the horizontal axis which contains a scaled copy of $X_{1/4}$.

Theorem 4.6. Therefore,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} U_5^k f = \langle f, e_0 \rangle e_0 = \left(\int f(x)\, \mathrm{d}\mu_{1/4} \right) e_0. \tag{43}$$

If we think of the Cesaro mean of the iterations of U as a "time average" and think of the integral with respect to the measure μ as a "space average," then we have now shown that the time average applied to functions in $L^2(\mu)$ equals the space average.

By contrast, we note that the isometry $U_4(e_\gamma) := e_{4\gamma}$ is spatially implemented and that it is induced by $\tau_4(x) = 4x \pmod 1$. We also know that μ is invariant under τ_4. Because U_4 can be realized as a shift on the underlying digit space, it is not hard to see that the only functions fixed by U_4 are also the constant functions:

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} U_4^k f = \langle f, e_0 \rangle e_0 = \left(\int f(x)\, \mathrm{d}\mu_{1/4} \right) e_0. \tag{44}$$

Again, the time average applied to U_4 on the function f equals the same space average of f. But the result for U_5 is much deeper than that of U_4. For background references on ergodic transformations, see **[19, 40]**, and for references on multiplicity theory, see **[18, 35]**.

There is no clean relation between the two measures μ and $\mu \circ \tau_5^{-1}$.

Proposition 5.3. *The measures μ and $\mu \circ \tau_5^{-1}$ are not equivalent.*

Proof. Set $A = (2/3, 1]$. Then $\mu(A) = 0$, since A is not contained in the Cantor set $X_{1/4}$, but $\mu \circ \tau_5^{-1}((A) > 1/8$, as demonstrated in Figure 1. Therefore, $\mu \circ \tau_5^{-1}$ is not absolutely continuous with respect to μ. Neither are the two measures concentrated on disjoint sets: both measures assign positive values to the set $[0, 1/2]$. $\square$

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