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ON HIGH RANK $\pi/3$ AND $2\pi/3$-CONGRUENT NUMBER ELLIPTIC CURVES

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ABSTRACT. Consider the elliptic curves given by

$$E_{n,\theta} : y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x$$

where $0 < \theta < \pi$, $\cos(\theta) = s/r$ is rational with $0 \leq |s| < r$ and $\gcd(r, s) = 1$. These elliptic curves are related to the θ-congruent number problem as a generalization of the congruent number problem. For fixed θ, this family corresponds to the quadratic twist by n of the curve $E_\theta : y^2 = x^3 + 2sx^2 - (r^2 - s^2)x$. We study two special cases: $\theta = \pi/3$ and $\theta = 2\pi/3$. We have found a subfamily of $n = n(w)$ having rank at least 3 over $\mathbb{Q}(w)$ and a subfamily with rank 4 parametrized by points of an elliptic curve with positive rank. We also found examples of n such that $E_{n,\theta}$ has rank up to 7 over $\mathbb{Q}$ in both cases.

1. Introduction. The construction of high rank elliptic curves is an important problem concerning elliptic curves. Dujella [**5**] collected a list of high rank elliptic curves with prescribed torsion groups. The largest known rank, found by Elkies [**8**] in 2006, is 28. In this work we search for high ranks in the family of elliptic curves related to the $\pi/3$ and $2\pi/3$ congruent problems.

Let us briefly describe the problem. Consider $0 < \theta < \pi$ such that $\cos(\theta) = s/r$ with r and s in $\mathbb{Q}$, $0 \leq |s| < r$ and $\gcd(r, s) = 1$. A positive integer n is called a θ-congruent number if there exists a triangle with rational sides and area equal to $n\alpha_\theta$, where $\alpha_\theta = \sqrt{r^2 - s^2}$. It is clear that, if a positive integer n is θ-congruent, then so is nt^2, for any integer t, so we concentrate on square-free positive integers.

The problem of determining θ-congruent numbers is related to the problem of finding non-2-torsion points on the family of elliptic curves

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which are called θ-congruent number elliptic curves

$$E_{n,\theta} : y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x,$$

where r and s are as above, see **[26]**. Observe that this curve is the quadratic twist by n of the curve $E_{1,\theta}$.

This family of elliptic curves was introduced by Koblitz in **[13,** Section I.2, Exercise 3], and systematically studied by Fujiwara **[9, 10]**. Let $E_{n,\theta}(\mathbb{Q})$ be the group of rational points on $E_{n,\theta}$, and denote by $r_\theta(n)$ its (algebraic) rank.

An ordinary congruent number is nothing but a $\pi/2$-congruent number and hence a congruent number elliptic curve is just a $\pi/2$-congruent number elliptic curve. Rogers **[19, 20]** and Dujella, Janfada and Salami **[6]**, recently exhibited a list of congruent number elliptic curves with $r_{\pi/2}(n)$ up to 7.

We restrict our search for high rank θ-congruent number elliptic curves to the cases $\theta = \pi/3$ and $2\pi/3$.

In this paper we present a family of values of $n = n(w)$ such that the curves $E_{n(w),2\pi/3}$ have rank at least 3 over $\mathbb{Q}(w)$. An equivalent result is valid for the $\pi/3$ case. We also exhibit examples of curves with rank up to 7 in both cases, $\pi/3$ and $2\pi/3$.

Yoshida **[27]** proved that $r_{\pi/3}(6) = 1$, $r_{\pi/3}(39) = 2$ and also $r_{2\pi/3}(5) = 1$, $r_{2\pi/3}(14) = 2$. These are the smallest positive integers corresponding to the given Mordell-Weil ranks. In this paper, we find the smallest positive integers n for which $r_{\pi/3}(n) = 3, 4, 5$ in one case and $r_{2\pi/3}(n) = 3, 4$ in the other (the result for $r_{\pi/3}(n) = 4$ is conditional, assuming the BSD and GRH).

In our computations, we use the Pari/Gp software **[18]**, William Stein's SAGE software **[24]** and Cremona's mwrank program **[4]** as well as the program package Magma **[2]**.

2. Preliminary results. In this section we recall some results about θ-congruent number elliptic curves, in particular, a criterion for a square-free positive integer to be a θ-congruent number. This, jointly with the subfamilies mentioned before, are the starting point for our search of good candidates for high rank curves.

We use the Mestre-Nagao sum, the Mestre's conditional upper bound for the rank of elliptic curves over $\mathbb{Q}$ and the root number as sieving tools in order to reduce the size of the lists and to select only the best candidates for high rank. We briefly describe these items below.

It is known that, for the usual congruent numbers, there exist a close relation with elliptic curves, and in fact, the following classical result holds: n is a congruent number if and only if $r_{\pi/2}(n) > 0$, see e.g., [**13,** Section I.9, Proposition 18]. A similar theorem was proved by Fujiwara, see [**9**], for θ-congruent numbers.

Theorem 2.1. *Let n be an arbitrary square-free positive integer, and consider the elliptic curve $E_{n,\theta}$ as above. Then*:

(i) *n is a θ-congruent number if and only if there exists a non-2-torsion point in $E_{n,\theta}(\mathbb{Q})$;*
(ii) *for $n \neq 1, 2, 3, 6$, n is a θ-congruent number if and only if $r_\theta(n) > 0$.*

Kan [**12**] proved the following result which gives a family of θ-congruent numbers for every $0 < \theta < \pi$.

Lemma 2.2. *A square-free positive integer n is a θ-congruent number if and only if n is the square-free part of*

$$pq(p+q)(2rq + p(r-s)), \tag{1}$$

for some positive integers p, q with $\gcd(p, q) = 1$.

Yoshida [**27, 28**] proved important results concerning θ-congruent numbers. In particular, in [**27**], he gave the root numbers for the cases $\pi/3$ and $2\pi/3$ (see Table 1).

TABLE 1. Root-numbers.

		$2\pi/3$	$\pi/3$
$n \equiv 1, 2, 3, 6, 7, 11, 13, 14, 18$	(mod 24)	$+1$	-1
$n \equiv 5, 9, 10, 15, 17, 19, 21, 22, 23$	(mod 24)	-1	$+1$

Now we recall the Mestre-Nagao sum for an elliptic curve E over $\mathbb{Q}$. Reduce E modulo a prime p, and suppose that N_p is the number of points on E with coordinates on F_p. For any positive integer t, let

$\mathbf{P}_t$ be the set of all primes less than t and $a_p = p + 1 - N_p$. The Mestre-Nagao sum is defined by

$$S(t,E) = \sum_{p\in\mathbf{P}_t}\left(1-\frac{p-1}{N_p}\right)\log p = \sum_{p\in\mathbf{P}_t}\frac{-a_p+2}{N_p}\log p.$$

It is experimentally known [**14, 17**] that high rank curves have large values $S(t,E)$. We cite [**3**] for a heuristic argument which links the Mestre-Nagao sum to the Birch and Swinnerton-Dyer conjecture [**1**].

Now we describe Mestre's conditional (assuming the Birch and Swinnerton-Dyer conjecture and GRH) upper bound (see [**7, 15**]) for the rank of an elliptic curve over $\mathbb{Q}$. Let E be an elliptic curve with conductor N. For an integer $m \geq 1$, let

$$b(p^m) = \begin{cases} 0 & \text{if } p|N, \\ \alpha_p^m + {\alpha'}_p^m & \text{if } p\nmid N, \end{cases}$$

where α_p and α'_p are the roots of $x^2 - a_p x + p$. Let

$$F(x) = \begin{cases} (1-x)\cos(\pi x) + \sin(\pi x)/\pi & \text{if } x\in[0,1], \\ 0 & \text{if } x > 1. \end{cases}$$

Take a positive real number λ, and write

$$M(\lambda) = 2\big(\log(2\pi) + \int_0^\infty \big(F(x/\lambda)/(e^x-1) - e^{-x}/x\big)\,dx\big).$$

Mestre's conditional upper bound for the rank of E is defined as

$$M(\lambda,E) = \frac{\pi^2}{8\lambda}\left(\log(N) - 2\sum_{p^m\leq e^\lambda} b(p^m)F(m\log(p)/\lambda)\frac{\log(p)}{p^m} - M(\lambda)\right).$$

3. A family with generic rank at least 3.

3.1. Twists. Observe that, once θ is fixed, the curve

$$E_{n,\theta}: \quad y^2 = x^3 + 2snx^2 - (r^2-s^2)n^2x$$

is the quadratic twist with parameter n of the curve $E_{1,\theta}: y^2 = x^3 + 2sx^2 - (r^2-s^2)x$. General results about twists can be applied for any θ, and we can find families of rank at least 2 over $\mathbb{Q}(r,s)$ by direct application of results given in Mestre [**16**] or Rubin and Silverberg [**21**], [**22**] (see also [**11, 25**]).

In our particular cases, $\theta = \pi/3$ corresponds to $s = 1$ and $r = 2$ and $\theta = 2\pi/3$ to $s = -1$ and $r = 2$. This leads us to study the quadratic twists of the curves

$$E_{\pi/3} : y^2 = x^3 + 2x^2 - 3x, \qquad E_{2\pi/3} : y^2 = x^3 - 2x^2 - 3x.$$

Each curve is the twist of the other by -1 so their twists can be studied jointly.

3.2. A family of twists for $\theta = 2\pi/3$ with rank ≥ 3.

3.2.1. Rank 1. We start with the twists of the curve

$$E_{2\pi/3} : y^2 = x^3 - 2x^2 - 3x$$

with parameter $(u + a)(u + b)(u + c)$, so we have the family of twists $y^2 = x^3 + Ax^2 + Bx$, where

$$A = -2(u+a)(u+b)(u+c), \qquad B = -3(u+a)^2(u+b)^2(u+c)^2.$$

Now we impose $-(b + u)(c + u)^2$ as the x-coordinate of a new point. This is the same as choosing

$$c = \frac{-3a - 4u + abw^2 + auw^2}{1 + bw^2 + uw^2}.$$

With this choice, we get a family of twists with rank at least 1 over $\mathbb{Q}(b, u, w)$ which, after clearing denominators, can be written as $y^2 = x^3 + A_1x^2 + B_1x$, with

$$\begin{aligned} A_1 &= -2(b + u)(-3 + bw^2 + uw^2)(1 + bw^2 + uw^2), \\ B_1 &= -3(b + u)^2(-3 + bw^2 + uw^2)^2(1 + bw^2 + uw^2)^2. \end{aligned}$$

The x-coordinate of the infinite order point is $x_1 = -(b+u)(-3+bw^2+uw^2)^2$.

3.2.2. Rank 2. We proceed by forcing $3(b + u)(1 + bw^2 + uw^2)$ to be the x-coordinate of a new point in the previous rank 1 family of twists. For this purpose, it is enough to choose

$$b = -\frac{-4 + u^2 + uw^2 + u^3w^2}{(1 + u^2)w^2}.$$

Now the new family of twists can be written as $y^2 = x^3 + A_2x^2 + B_2x$ with

$$A_2 = -10(-2+u)(2+u)(-1+2u)(1+2u)(1+u^2),$$
$$B_2 = -75(-2+u)^2(2+u)^2(-1+2u)^2(1+2u)^2(1+u^2)^2.$$

The x-coordinates of the two infinite order points are

$$x_1 = (-2+u)(2+u)(-1+2u)^2(1+2u)^2(1+u^2),$$
$$x_2 = -15(-2+u)(2+u)(1+u^2)^2.$$

These two points are independent, so the new family has rank at least 2 over $\mathbb{Q}(u)$.

3.2.3. Rank 3. Finally, we choose

$$u = -\frac{70-10w+w^2}{3(5+w^2)}$$

in order to get $5(-2+u)^2(-1+2u)^2(1+u^2)$ as the x-coordinate of a new point in the rank 2 family. In this way, we get $y^2 = x^3 + A_3x^2 + B_3x$ with

$$\begin{aligned} A_3 = {} & -2(-5+w)(-2+w)(4+w)(25+w)(31-4w+w^2) \\ & (100-10w+7w^2)(1025-280w+66w^2-4w^3+2w^4), \\ B_3 = {} & -3(-5+w)^2(-2+w)^2(4+w)^2(25+w)^2(31-4w+w^2)^2 \\ & (100-10w+7w^2)^2(1025-280w+66w^2-4w^3+2w^4)^2. \end{aligned}$$

The x-coordinates of the three independent points are given by

$$\begin{aligned} x_1 = {} & \frac{-1}{9(5+w^2)^2}(-5+w)^2(-2+w)(4+w)(25+w)^2(31-4w+w^2)^2 \\ & (100-10w+7w^2)(1025-280w+66w^2-4w^3+2w^4), \end{aligned}$$

$$\begin{aligned} x_2 = {} & 3(-2+w)(4+w)(100-10w+7w^2) \\ & (1025-280w+66w^2-4w^3+2w^4)^2, \\ x_3 = {} & (31-4w+w^2)^2(100-10w+7w^2)^2 \\ & (1025-280w+66w^2-4w^3+2w^4). \end{aligned}$$

For $w = 10$, after reducing coefficients, we get the rank 3 curve given by $y^2 = x^3 - 442x^2 - 146523x$. The specialized points are

$$P_1 = \{-2873/81, 1562912/729\},\ P_2 = \{867, 13872\},\ P_3 = \{2873/4, 48841/8\}.$$

A calculation with mwrank [**4**] shows that these three points are independent. An argument of specialization [**23**] proves that this family has rank at least 3 over $\mathbb{Q}(w)$.

Observe that the parameter for the rank 3 family of twists can be made both positive and negative for infinitely many values of w, so we get a family of rank 3 twist for both the $2\pi/3$ and $\pi/3$ congruent number problems.

3.3. A subfamily with rank $\geq$ 4. We can find a subfamily with rank ≥ 4 in the family $y^2 = x^3 + A_3x^2 + B_3x$ by forcing

$$-\frac{1}{4}(-5+w)^2(-2+w)^2(4+w)^2(25+w)^2(31-4w+w^2)(100-10w+7w^2)$$

to be the x-coordinate of a point on the curve. We get the condition

$$25w^4 - 26w^3 + 699w^2 - 3770w + 13300 = z^2. \tag{2}$$

It can be transformed to the elliptic curve

$$Y^2 = X^3 + X^2 - 17220X - 352800$$

with positive rank (rank is equal to 2 with generators $[255, 3450]$, $[-22, 126]$, corresponding to the points $(w, z) = [315/74, 695275/5476]$ and $[8, -342]$ on the quartic (2)). Hence, we get infinitely many rational parameters w for which the curve $y^2 = x^3 + A_3x^2 + B_3x$ has the rank ≥ 4.

4. Strategies and results.

4.1. General setting. Now we attempt to find high rank elliptic curves $E_{n,\theta}$ in two cases $\theta = \pi/3$ and $2\pi/3$. We will use the expression (1) and the families given in the previous section as sources for good candidates for high rank curves. We shall use the following notations: $r_\theta(n)$ for the rank and $s_\theta(n)$ for the 2-Selmer rank (see, e.g., [**6**]), which is an upper bound for the rank; that is, $r_\theta(n) \leq s_\theta(n)$.

We proceed in three steps, depending on the range and the form of the square-free positive integers n.

Step (I). In this step we take all the square-free positive integers $n \leq 5 \times 10^6$. By a direct computation with mwrank, we find the 2-Selmer rank of $E_{n,\theta}$ for all square-free n in that range and in each case $\theta = \pi/3$ and $2\pi/3$. Our computations show that there are no integers n with $s_\theta(n) \geq 6$. Table 2 presents the distribution of the number of these square-free integers according to the values of $s_\theta(n)$. Finally, we compute directly rank $r_\theta(n)$ with mwrank to find the smallest n's with $r_{\pi/3}(n) = 3, 4, 5$ as well as the smallest n's with $r_{2\pi/3}(n) = 3, 4$.

TABLE 2. Distribution of $s_\theta(n)$

$s_\theta(n)$	0	1	2	3	4	5	≥ 6	Total
$\theta = \pi/3$	783043	1401045	734290	116158	5045	52	0	3039633
$\theta = 2\pi/3$	760511	1374165	751192	144641	9038	86	0	3039633

Step (II). We consider all square-free θ-congruent numbers $n > 5 \times 10^6$ of the form (1) in Lemma 2.2 with $1 < p, q \leq 10^4$, $\gcd(p, q) = 1$, and having at least 4 odd prime factors. We get a list with more than 7×10^6 elements for each of the cases $\theta = \pi/3$ and $\theta = 2\pi/3$. Using the Mestre-Nagao sum, we reduce by Pari/Gp program the length of this list. In fact, we choose the n with $S(10^3, E_{n,\theta}) > 15$, $S(10^4, E_{n,\theta}) > 20$, $S(10^5, E_{n,\theta}) > 40$, for which $s_{\pi/3}(n) \geq 6$, and $s_{2\pi/3}(n) \geq 5$. After computing the values of $r_\theta(n)$ for these candidates by mwrank, we finally select the n with $r_{\pi/3}(n) = 6, 7$ and the n with $r_{2\pi/3}(n) = 5, 6$. In the cases in which mwrank do not give exact value $r_\theta(n)$, we compute Mestre's conditional upper bound $M(\lambda, E_{n,\theta})$ for $r_\theta(n)$ with $15 \leq \lambda < 24$.

Step (III). In this part we use the families in Section 3 in order to search for good candidates for high rank. Since curves in the families with rank 3 and 4 have large coefficients, we find the family with rank 2 the most suitable for our purpose. The search for rank 6 curves is conducted upon the rank 2 family with $u = p/q$ for $1 < p < q < 4000$ with sieving conditions $S(523, E_{n,\theta}) > 18$, $S(1979, E_{n,\theta}) > 28$ and the Selmer rank ≥ 6.

The search for rank 7 is made in the same family of twists with $u = p/q$ for $1 < p < q < 13000$ with the following conditions, root number equal to -1, $S(523, E_{n,\theta}) > 20$, $S(1979, E_{n,\theta}) > 30$ and the Selmer rank ≥ 7. The ranks are calculated with mwrank. For the case $2\pi/3$ for $p = 4127$ and $q = 10004$, i.e., for $n = 127486974129099 16241$, the corresponding curve has rank 7. In this case, the direct application of mwrank gives only $6 \leq$ rank ≤ 7, but applying mwrank to an isogenous curve give the seventh independent point.

In the next subsections, we collect the results. We find the smallest integers n such that $r_{\pi/3}(n) = 3, 4, 5$ and $r_{2\pi/3}(n) = 3, 4$, and we exhibit examples of curves with rank up to 7 in both cases.

4.2. The case $\theta = \pi/3$. *Rank* 3. The integers 407 and 646 are the two smallest ones among 116158 integers n less than 5×10^6 with $s_{\pi/3}(n) = 3$. We have $r_{\pi/3}(646) = 3$, while for $n = 407$, Magma gives that the analytic rank is 1, so by Kolyvagin's theorem $r_{\pi/3}(407) = 1$. Therefore, the value $n = 646$ is the minimum value producing a curve with rank 3.

Rank 4. The smallest n that we have found with rank 4 is $n = 172081$. There are 63 integers n less than 172081 with $s_{\pi/3}(n) = 4$. For 29 cases mwrank gives $0 \leq r_{\pi/3}(n) \leq 4$, and for all these cases the 4-descent implemented in Magma gives that the rank is ≤ 2. In the remaining 34 cases, mwrank gives $2 \leq r_{\pi/3}(n) \leq 4$. In most of these cases, the 4-descent shows that rank is equal to 2. However, in three cases: $n = 31622$, 143222, 150866, we are not able to show rank < 4 unconditionally. In these cases, we use Mestre's conditional upper bound (with $\lambda = 11$), which gives $r_{\pi/3}(n) \leq 2$, so $r_{\pi/3}(n) = 2$ (conditionally). Thus, the value $n = 172081$ is, conditionally (assuming BSD and GRH), the minimum value giving a curve with rank 4.

Rank 5. The direct computation shows that $n = 221746$ is the smallest among 52 integers n in the observed range with $s_{\pi/3}(n) = 5$, and, since $r_{\pi/3}(221746) = 5$, $n = 221746$ is the smallest positive integer giving rank 5.

Rank 6. The smallest n that we have found with rank 6 is $n = 11229594411$. We do not know if it is the smallest one with this property. The values of n given in Table 3 also give curves with rank 6.

Rank 7. The only n that we have found giving rank 7 is $n = 365803464586$. We do not know if it is the smallest one.

TABLE 3. Case $\pi/3$. Other n with rank 6.

40004232681,	158763281079,	167514827545,	198606002595,
251819173095,	271314827665,	3302971161265,	3492293850595,
5144668978371,	6634009064865,	17073273800095,	40582123000419,
45563330326345,	7658263493840940211.		

TABLE 4. Ranks in the cases $\theta = \pi/3$.

$r_\theta(n)$	n	Generators of $E_{n,\theta} : y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x$
3	646	[-722,34656], [6137,521645], [-1216,40432].
4	172081	[-505141,-61627202], [-58621,-78669382], [-440076,-143244738], [224175,92987790].
5	221746	[345450,207822720], [-15792,49357896], [994896,1130036040], [-13254,-45063600], [-386575,-255989965].
6	11229594411	[904103532759/25,-992069570757491352/125], [1541731888897/16,2090318638263775025/64], [265444083202036/2025,4636387440736982658134/91125], [719501508201/64,408734174250222581/512], [13006760076899764/269361, 1693181585331404000267498/139798359], [50286669020153449/278784, 11896090671289659453790795/147197952].
7	365803464586	[433764757524,212456676940982628], [1291274050073,-1689545579159165609], [-59335333874904423/3644281, -570541659890431976790514695/6956932429], [11954902524369/4,-45277466996084516865/8], [2138828658027602/5329, 56890395483549429623312/389017], [786769181014433554/80089, 721982407380536692088852160/22665187], [-562236028164373765342/540237049, 3617165210435366625559445197360/12556729729907].

4.3. The case $\theta = 2\pi/3$. *Rank* 3. The smallest n with $s_{2\pi/3}(n) = 3$ is $n = 221$. Since $r_{2\pi/3}(221) = 3$, we conclude that $n = 211$ is the smallest n for which the rank is 3.

Rank 4. The smallest n that we have found with rank 4 is $n = 12710$. There are two smaller positive integers with Selmer rank equal to 4 ($n = 4718$ and $n = 6398$) but having analytic rank 0, so by Kolyvagin's theorem, the algebraic rank is also 0. Thus, the minimality of $n = 12710$ follows.

Rank 5. The smallest n that we have found with rank 5 is $n = 16470069$. We do not know if it is the smallest one with this property.

Rank 6. We have found several positive integers n with $r_{2\pi/3}(n) = 6$ where $n = 456249066$ is the smallest one. Other values are given in Table 5.

Rank 7. The integer $n = 127486974129099 16241$ with $r_{2\pi/3}(n) = 7$ has been found within the family of rank 2 of Section 3. It corresponds to $u = 4127/10004$ in such a family. The data for this curve are too large to fit in the table, so we give them here. The rank and independent points were found by applying mwrank [4] to one of its 2-isogenous curves. The curve is

$$y^2 = x^3 - 25497394825819832482x^2 - 48758785717780797419512465244890671 0243x$$

and x-coordinates of 7 independent points are:

$$-3478204633589378700, -1168594544971913 3341,$$

$$6574179551855299730183742058990161459509481/5755988368777969859700 25,$$

$$25824931965925746931591311990861035046105913 21/6468722004446965722331 1844,$$

$$93780507427270339924086066690295941912574093056 1/18861375626453019864153493504,$$

$$1104659735365516010974660708851802130043678065196866538846789294168732798650875054944604 13809/117483357503782515880827567198396424934300531958532372966821126108 98176,$$

$$604677170128352690791956712052363555568843154944505048349318846158389503896191687 7433 002192769879034594342389788 9681/242330170223125876400553676591840836189420653457595 17 265987410838049572492835809955736272757 2100.$$

TABLE 5. Case $2\pi/3$. Other n with rank 6.

764046470,	902472906,	5062245006,	9667090290,
11801899970,	19969987310,	20240772006,	23819599518,
24080567966,	30834423438,	39360775454,	58181539130,
64256704710,	98708770590,	106366008126,	148280772990,
181684390314,	292826163630,	309000045354,	333515184002,
554883184814,	653918457570,	685374515826,	713465075246,
860842004286,	1185986591790,	1248260820170,	2004510092970,
2743972777910,	10745486363210,	55967962170246,	90952836208430,
104732378607110,	177348563238770,	219163751391326,	1459584795789354,
29410732919116094,	40315634933149394,	30375400815771401390.	

TABLE 6. Ranks in the cases $\theta = 2\pi/3$.

$r_\theta(n)$	n	Generators of $E_{n,\theta} : y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x$
3	221	[-204,1734], [-169,2704], [4131,-249696].
4	12710	[-310,384400], [-9920,-1153200], [48050,5381600], [76880,16337000].
5	16470069	[-3115959/4,-198146948769/8], [-16255958103/1024,-813789518594283/32768], [118172745075/1849,-21701053829180880/79507], [174895662711/3481,-10850526914590440/205379], [18013358979/361,-275820552686448/6859].
6	4562490669	[1372171206,2930957696016], [24303608784,3714988879700280], [1677715326,-33259028622624], [3635049873,-183588193835865], [27273656667348/18769, 39342846732689875284/2571353], [36967427406/25,2217080599939296/125].

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