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EXPANSIONS OF MONOMIAL IDEALS AND MULTIGRADED MODULES

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ABSTRACT. We introduce an exact functor defined on multigraded modules which we call the expansion functor and study its homological properties. The expansion functor applied to a monomial ideal amounts to substitute the variables by monomial prime ideals and to apply this substitution to the generators of the ideal. This operation naturally occurs in various combinatorial contexts.

Introduction. In this paper we first study an operator defined on monomial ideals and their multigraded free resolution which we call the *expansion operator*. The definition of this operator is motivated by constructions in various combinatorial contexts. For example, let G be a finite simple graph with vertex set $V(G) = [n]$ and edge set $E(G)$, and let $I(G)$ be its edge ideal in $S = K[x_1, \ldots, x_n]$. We fix a vertex j of G. Then a new graph G' is defined by duplicating j, that is, $V(G') = V(G) \cup \{j'\}$ and

$$E(G') = E(G) \cup \{\{i, j'\} : \{i, j\} \in E(G)\}$$

where j' is a new vertex. It follows that $I(G') = I(G) + (x_i x_{j'} : \{i, j\} \in E(G))$. This duplication can be iterated. We denote by $G^{(i_1,\ldots,i_n)}$ the graph which is obtained from G by i_j duplications of j. Then the edge ideal of $G^{(i_1,\ldots,i_n)}$ can be described as follows: let S^* be the polynomial ring over K in the variables

$$x_{11}, \ldots, x_{1i_1}, x_{21}, \ldots, x_{2i_2}, \ldots, x_{n1}, \ldots, x_{ni_n},$$

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and consider the monomial prime ideal $P_j = (x_{j1}, \ldots, x_{ji_j})$ in S^*. Then

$$I(G^{(i_1,\ldots,i_n)}) = \sum_{\{v_i,v_j\}\in E(G)} P_iP_j.$$

We say the ideal $I(G^{(i_1,\ldots,i_n)})$ is obtained from $I(G)$ by expansion with respect to the n-tuple $(i_1, \ldots, i_n)$ with positive integer entries.

More generally, let $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}) \subset S$ be any monomial ideal and $(i_1, \ldots, i_n)$ an n-tuple with positive integer entries. Then we define the *expansion of I with respect to* $(i_1, \ldots, i_n)$ as the monomial ideal

$$I^* = \sum_{i=1}^{m} P_1^{a_i(1)} \cdots P_n^{a_i(n)} \subseteq S^*,$$

where $\mathbf{a}_i = (a_i(1), \ldots, a_i(n))$. A similar construction by a sequence of duplications of vertices of a simplicial complex is applied in **[7]** to derive an equivalent condition for vertex cover algebras of simplicial complexes to be standard graded. In this case, the cover ideal of the constructed simplicial complex is an expansion of the cover ideal of the original one. This construction also appears in the so-called parallelization process in graph theory which is a sequence of duplications and deletions, the duplications being special expansions in our terminology. The corresponding algebraic operations have been considered, for example, by Martinez-Bernal, Morey and Villarreal in **[10]**. There is also an operator on hypergraphs which is called an s-expansion, see **[4, 6]**, which applied to the edge ideal of a hypergraph yields a different result than the expansion operator defined in this paper. There is still another instance, known to us, where expansions of monomial ideals (in our sense) naturally appear. Indeed, let $\mathcal{M}$ be any vector matroid on the ground set E with a set of bases $\mathcal{B}$, that is, E is a finite subset of nonzero vectors of a vector space V over a field F and $\mathcal{B}$ is the set of maximal linearly independent subsets of E. We define on E the following equivalence relation: for $v, w \in E$, we set $v \sim w$ if and only if v and w are linearly dependent. Next, we choose one representative of each equivalence class, say, $v_1, \ldots, v_r$. Let $\mathcal{M}'$ be the vector matroid on the ground set $E' = \{v_1, \ldots, v_r\}$. Then, obviously, the matroidal ideal attached to $\mathcal{M}$ is an expansion of the matroidal ideal attached to $\mathcal{M}'$.

In the first section, we present the basic properties of the expansion operator and show that this operator commutes with the standard algebraic operations on ideals; see Lemma 1.1 and Corollary 1.4. It also commutes with primary decompositions of monomial ideals as shown in Proposition 1.2. As a consequence, one obtains that $\operatorname{Ass}(S^*/I^*) = \{P^* : P \in \operatorname{Ass}(S/I)\}$. Also, it is not so hard to see that I^* has linear quotients if I has linear quotients. In view of this fact it is natural to ask whether I^* has a linear resolution if I has a linear resolution. This question has a positive answer, as is shown in the following sections where we study the homological properties of I^*.

Let S be equipped with the standard multigraded structure. It turns out, as shown in Section 2, that the expansion operator can be made an exact functor from the category of finitely generated multigraded S-modules to the category of finitely generated multigraded S^*-modules. Let M be a finitely generated multigraded S-module. Applying this functor to a multigraded free resolution $\mathbb{F}$ of M we obtain an acyclic complex $\mathbb{F}^*$ with $H_0(\mathbb{F}^*) \cong M^*$. Unfortunately, $\mathbb{F}^*$ is not a free resolution of M^* because the expansion functor applied to a free module does not necessarily yield a free module. To remedy this problem, we construct in Section 3, starting from $\mathbb{F}^*$, a double complex which provides a minimal multigraded free resolution of M^*. In the last section, we use this construction to show that M and M^* have the same regularity. We also compute the graded Betti numbers of M^* in terms of multigraded free resolution of M.

1. Definition and basic properties of the expansion operator. In this section, we define the expansion operator on monomial ideals and compare some algebraic properties of the constructed ideal with the original one.

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K in the variables $x_1, \ldots, x_n$. Fix an ordered n-tuple $(i_1, \ldots, i_n)$ of positive integers, and consider the polynomial ring $S^{(i_1,\ldots,i_n)}$ over K in the variables

$$x_{11}, \ldots, x_{1i_1}, x_{21}, \ldots, x_{2i_2}, \ldots, x_{n1}, \ldots, x_{ni_n}.$$

Let P_j be the monomial prime ideal $(x_{j1}, x_{j2}, \ldots, x_{ji_j}) \subseteq S^{(i_1,\ldots,i_n)}$. Attached to each monomial ideal I with a set of monomial generators $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, we define the *expansion of I with respect to the n-tuple*

$(i_1, \ldots, i_n)$, denoted by $I^{(i_1,\ldots,i_n)}$, to be the monomial ideal

$$I^{(i_1,\ldots,i_n)} = \sum_{i=1}^{m} P_1^{a_i(1)} \cdots P_n^{a_i(n)} \subseteq S^{(i_1,\ldots,i_n)}.$$

Here $a_i(j)$ denotes the j-th component of the vector $\mathbf{a}_i$. Throughout the rest of this paper we work with the fixed n-tuple $(i_1, \ldots, i_n)$. So we simply write S^* and I^*, respectively, rather than $S^{(i_1,\ldots,i_n)}$ and $I^{(i_1,\ldots,i_n)}$.

For monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ in S if $\mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{a}}$, then $P_1^{a_i(1)} \cdots P_n^{a_i(n)} \subseteq P_1^{b_i(1)} \cdots P_n^{b_i(n)}$. So the definition of I^* does not depend on the choice of the set of monomial generators of I.

For example, consider $S = K[x_1, x_2, x_3]$ and the ordered 3-tuple $(1, 3, 2)$. Then we have $P_1 = (x_{11})$, $P_2 = (x_{21}, , x_{22}, x_{23})$ and $P_3 = (x_{31}, x_{32})$. So, for the monomial ideal $I = (x_1x_2, x_3^2)$, the ideal $I^* \subseteq K[x_{11}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}]$ is $P_1P_2 + P_3^2$, namely,

$$I^* = (x_{11}x_{21}, x_{11}x_{22}, x_{11}x_{23}, x_{31}^2, x_{31}x_{32}, x_{32}^2).$$

We define the K-algebra homomorphism $\pi : S^* \to S$ by

$$\pi(x_{ij}) = x_i \quad \text{for all } i, j.$$

As usual, we denote by $G(I)$ the unique minimal set of monomial generators of a monomial ideal I. Some basic properties of the expansion operator are given in the following lemma.

Lemma 1.1. *Let I and J be monomial ideals. Then*

(i) *$f \in I^*$ if and only $\pi(f) \in I$, for all $f \in S^*$;*
(ii) *$(I + J)^* = I^* + J^*$;*
(iii) *$(IJ)^* = I^*J^*$;*
(iv) *$(I \cap J)^* = I^* \cap J^*$;*
(v) *$(I\ J)^* = I^* : J^*$;*
(vi) *$\sqrt{I^*} = (\sqrt{I})^*$;*
(vii) *If the monomial ideal Q is P-primary, then Q^* is P^*-primary.*

Proof. (i) Since I and I^* are monomial ideals, it is enough to show the statement holds for all monomials $u \in S^*$. So let $u \in S^*$ be a monomial. Then $u \in I^*$ if and only if for some monomial $\mathbf{x}^{\mathbf{a}} \in G(I)$ we have $u \in P_1^{a(1)} \cdots P_n^{a(n)}$, and this is the case if and only if $\mathbf{x}^{\mathbf{a}}$ divides $\pi(u)$.

(ii) is trivial.

(iii) Let $A = \{x^{\mathbf{a}_1} \dots x^{\mathbf{a}_m}\}$ and $B = \{x^{\mathbf{b}_1} \dots x^{\mathbf{b}_{m'}}\}$ be, respectively, a set of monomial generators of I and J. The assertion follows from the fact that $\{x^{\mathbf{a}_i} x^{\mathbf{b}_j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq m'\}$ is a set of monomial generators for IJ.

(iv) With the same notation as (ii), one has the set of monomial generators $\{\mathrm{lcm}(x^{\mathbf{a}_i}, x^{\mathbf{b}_j}) : 1 \leq i \leq m \text{ and } 1 \leq j \leq m'\}$ of $I \cap J$ where $\mathrm{lcm}(x^{\mathbf{a}_i}, x^{\mathbf{b}_j})$ is the least common multiple of $x^{\mathbf{a}_i}$ and $x^{\mathbf{b}_j}$. Therefore,

$$\begin{aligned} I^* \cap J^* &= \left(\sum_{i=1}^{m} P_1^{a_i(1)} \cdots P_n^{a_i(n)} \right) \cap \left(\sum_{j=1}^{m'} P_1^{b_j(1)} \cdots P_n^{b_j(n)} \right) \\ &= \sum_{1 \leq j \leq m'} \sum_{1 \leq i \leq m} (P_1^{a_i(1)} \cdots P_n^{a_i(n)}) \cap (P_1^{b_j(1)} \cdots P_n^{b_j(n)}) \\ &= (I \cap J)^*. \end{aligned}$$

The second equality holds because the summands are all monomial ideals.

(v) Since $I\ J = \bigcap_{u \in G(J)} I\ (u)$, by statement (iv) it is enough to show that $(I\ (u))^* = I^*\ (u)^*$ for all monomials $u \in S$. Observe that, if $u = \mathbf{x}^{\mathbf{a}} \in S$, then $(u)^* = P_1^{a_i(1)} \cdots P_n^{a_i(n)} \subseteq S^*$. In addition, by properties of the ideal quotient, one has that $I\ L_1 L_2 = (I : L_1) : L_2$ for all ideals L_1, L_2 of S. Therefore, we only need to show that, for each variable $x_j \in S$, the equality $(I\ x_j)^* = I^* : P_j$ holds where we set $P_j = (x_{j1}, \dots, x_{ji_j})$ as before. So let $f \in S^*$. Then, by (i), one has $f x_{j\ell} \in I^*$ for all ℓ if and only if $\pi(f) x_j = \pi(f x_{j\ell}) \in I$. Hence, $(I\ x_j)^* = I^*\ P_j$.

(vi) One should only notice that, in general, for any two ideals L_1 and L_2, we have $\sqrt{L_1 + L_2} = \sqrt{\sqrt{L_1} + \sqrt{L_2}}$. With this observation, the definition of I^* implies (vi).

(vii) Let, for two monomials $u, v \in S^*$, $uv \in Q^*$ but $u \notin Q^*$. Then, by (i), we have $\pi(uv) = \pi(u)\pi(v) \in Q$ and $\pi(u) \notin Q$. Hence, $\pi(v) \in \sqrt{Q} = P$ because Q is a primary ideal. Now the result follows from (i) and (vi). $\square$

Proposition 1.2. *Let I be a monomial ideal, and consider an (irredundant) primary decomposition $I = Q_1 \cap \cdots \cap Q_m$ of I. Then $I^* = Q_1^* \cap \cdots \cap Q_m^*$ is an (irredundant) primary decomposition of I^*.*

In particular, $\mathrm{Ass}(S^*/I^*) = \{P^* : P \in \mathrm{Ass}(S/I)\}$.

Proof. The statement follows from Lemma 1.1 (iv) and (vii). □

By a result of Brodmann [**1**], in any Noetherian ring R, the set $\mathrm{Ass}(R/I^s)$ stabilizes for $s \gg 0$, that is, there exists a number s_0 such that $\mathrm{Ass}(R/I^s) = \mathrm{Ass}(R/I^{s_0})$ for all $s \geq s_0$. This stable set $\mathrm{Ass}(R/I^{s_0})$ is denoted by $\mathrm{Ass}^\infty(I)$.

Corollary 1.3. *Let I be a monomial ideal. Then* $\mathrm{Ass}^\infty(I^*) = \{P^* : P \in \mathrm{Ass}^\infty(I)\}$.

Proof. By part (iii) of Lemma 1.1 one has $(I^k)^* = (I^*)^k$. Hence, $\mathrm{Ass}(S^*/(I^*)^k) = \{P^* : P \in \mathrm{Ass}(S/I^k)\}$ by Proposition 1.2. □

Let I be a monomial ideal with $\mathrm{Min}(I) = \{P_1, \ldots, P_r\}$. Given an integer $k \geq 1$, the k-th symbolic power $I^{(k)}$ of I is defined to be

$$I^{(k)} = Q_1 \cap \ldots \cap Q_r,$$

where Q_i is the P_i-primary component of I^k.

Corollary 1.4. *Let I be a monomial ideal. Then $\left(I^{(k)}\right)^* = (I^*)^{(k)}$ for all $k \geq 1$.*

Proposition 1.5. *Let I be a monomial ideal. If I has linear quotients, then I^* also has linear quotients.*

Proof. Let I have linear quotients with respect to the ordering

$$u_1, u_2, \ldots, u_m \tag{1}$$

of $G(I)$. We show that I^* has linear quotients with respect to the ordering

$$u_{11}, \ldots, u_{1r_1}, \ldots, u_{m1}, \ldots, u_{mr_m} \tag{2}$$

of $G(I^*)$ where, for all i,j, we have $\pi(u_{ij}) = u_i$, and $u_{i1} <_{\mathrm{Lex}} \cdots <_{\mathrm{Lex}} u_{ir_i}$ by the ordering of

$$x_{11} > \cdots > x_{1i_1} > \cdots > x_{n1} > \cdots > x_{ni_n}.$$

Let $v, w \in G(I^*)$ be two monomials such that in (2) the monomial v appears before w. In order to prove that I^* has linear quotients with respect to the above-mentioned order, we must show that there exist a variable x_{ij} and a monomial $v' \in G(I^*)$ such that $x_{ij} | (v/\gcd(v,w))$, in (2) the monomial v' comes before w, and $x_{ij} = v'/\gcd(v',w)$.

First assume that $\pi(v) = \pi(w)$, and x_{ij} is the greatest variable with respect to the given order on the variables such that $x_{ij}|(v/\gcd(v,w))$. Let $v' \in G(I^*)$ be the monomial with $\pi(v') = \pi(w)$ and $v' = x_{x_{ij}} \gcd(w, v')$. By the choice of x_{ij} and, since the order of monomials in (2) are given by the lexicographic order, the monomial v' comes before w in (2), as desired.

Next assume that $\pi(v) \neq \pi(w)$. Since I has linear quotients, there exists a monomial $u \in G(I)$, coming before $\pi(w)$ in (1), for which there exists a variable x_i such that $x_i \mid (\pi(v)/\gcd(\pi(v), \pi(w))$ and $x_i = u/\gcd(u, \pi(w))$. Let x_{ij} be the variable which divides $v/\gcd(v,w)$. Then there exists a monomial $v' \in G(I^*)$ with $\pi(v') = u$, and $v' = x_{ij} \gcd(v', w)$. Since $\pi(v')$ comes before $\pi(w)$ in (1), the monomial v' also appears before w in (2). Thus, v' is the desired element. $\square$

2. The expansion functor. Throughout this paper, we consider the standard multigraded structure for polynomial rings, that is, the graded components are the one-dimensional K-spaces spanned by the monomials. Let $\mathcal{C}(S)$ denote the category of finitely generated multigraded S-modules whose morphisms are the multigraded S-module homomorphisms. We fix an n-tuple $(i_1, \ldots, i_n)$ and define the expansion functor which assigns a multigraded module M^* over the standard multigraded polynomial ring S^* to each $M \in \mathcal{C}(S)$.

In the first step we define the expansion functor on multigraded free modules and multigraded maps between them.

Let $F = \bigoplus_i S(-\mathbf{a}_i)$ be a multigraded free S-module. Then F, as a multigraded module, is isomorphic to the direct sum $\bigoplus_i (\mathbf{x}^{\mathbf{a}_i})$ of principal monomial ideals of S. This presentation of F by principal monomial ideals is unique. We define the S^*-module F^* to be

$$F^* = \bigoplus_i (\mathbf{x}^{\mathbf{a}_i})^*.$$

Observe that, in general, F^* is no longer a free S^*-module. Let

$$\alpha \bigoplus_{i=1}^{r} S(-\mathbf{a}_i) \longrightarrow \bigoplus_{j=1}^{s} S(-\mathbf{b}_j)$$

be a multigraded S-module homomorphism. Then, for the restriction $\alpha_{i,j} : S(-\mathbf{a}_i) \to S(-\mathbf{b}_j)$ of α, we have

$$\alpha_{i,j}(f) = \lambda_{ji}\mathbf{x}^{\mathbf{a}_i - \mathbf{b}_j} f \quad \text{for all } f \in S(-\mathbf{a}_i)$$

with $\lambda_{ji} \in K$, and $\lambda_{ji} = 0$ if $\mathbf{x}^{\mathbf{b}_j}$ does not divide $\mathbf{x}^{\mathbf{a}_i}$. The $(s \times r)$-matrix $[\lambda_{ji}]$ is called the *monomial matrix expression* of α.

Notice that, if $\mathbf{x}^{\mathbf{b}_j}$ divides $\mathbf{x}^{\mathbf{a}_i}$, then $(\mathbf{x}^{\mathbf{a}_i})^* \subseteq (\mathbf{x}^{\mathbf{b}_j})^*$. So we define a multigraded S^*-module homomorphism $\alpha^* : \bigoplus_{i=1}^r (\mathbf{x}^{\mathbf{a}_i})^* \to \bigoplus_{j=1}^s (\mathbf{x}^{\mathbf{b}_j})^*$ associated to α whose restriction $\alpha^*_{ij} : (\mathbf{x}^{\mathbf{a}_i})^* \to (\mathbf{x}^{\mathbf{b}_j})^*$ is defined as follows: if $\mathbf{x}^{\mathbf{b}_j} \mid \mathbf{x}^{\mathbf{a}_i}$, then

$$\alpha^*_{ij}(f) = \lambda_{ji} f \quad \text{for all } f \in (\mathbf{x}^{\mathbf{a}_i})^*,$$

and α^*_{ij} is the zero map if $\mathbf{x}^{\mathbf{b}_j}$ does not divide $\mathbf{x}^{\mathbf{a}_i}$.

Obviously, one has $(\mathrm{id}_F)^* = \mathrm{id}_{F^*}$ for each finitely generated multigraded free S-module F.

Lemma 2.1. *Let*

$$\bigoplus_{i=1}^{r} S(-\mathbf{a}_i) \xrightarrow{\alpha} \bigoplus_{j=1}^{s} S(-\mathbf{b}_j) \xrightarrow{\beta} \bigoplus_{k=1}^{t} S(-\mathbf{c}_k) \tag{3}$$

be a sequence of multigraded S-module homomorphisms. Then the following statements hold:

(i) $(\beta\alpha)^* = \beta^*\alpha^*$;

(ii) *If the sequence* (3) *is exact, then*

$$\bigoplus_{i=1}^{r} (\mathbf{x}^{\mathbf{a}_i})^* \xrightarrow{\alpha^*} \bigoplus_{j=1}^{s} (\mathbf{x}^{\mathbf{b}_j})^* \xrightarrow{\beta^*} \bigoplus_{k=1}^{t} (\mathbf{x}^{\mathbf{c}_k})^* \tag{4}$$

is also an exact sequence of multigraded S^*-modules.

Proof. Let $[\lambda_{ji}]$ and $[\mu_{kj}]$ be, respectively, monomial matrix expressions of α and β.

(i) follows from the definition and the fact that the product $[\mu_{kj}][\lambda_{ji}]$ is the monomial expression of $\beta\alpha$.

(ii) By abuse of notation, we may consider the following sequence which is isomorphic to the sequence (3):

$$\bigoplus_{i=1}^{r}(\mathbf{x}^{\mathbf{a}_i}) \xrightarrow{\alpha} \bigoplus_{j=1}^{s}(\mathbf{x}^{\mathbf{b}_j}) \xrightarrow{\beta} \bigoplus_{k=1}^{t}(\mathbf{x}^{\mathbf{c}_k}).$$

Here the maps are given by the matrices $[\lambda_{ji}]$ and $[\mu_{kj}]$. In order to show that the sequence (4) is exact, it is enough to show that it is exact in each multidegree. For this purpose, first consider the following commutative diagram:

$$\begin{array}{ccccc} \bigoplus_{i=1}^{r}(\mathbf{x}^{\mathbf{a}_i})^* & \xrightarrow{\alpha^*} & \bigoplus_{j=1}^{s}(\mathbf{x}^{\mathbf{b}_j})^* & \xrightarrow{\beta^*} & \bigoplus_{k=1}^{t}(\mathbf{x}^{\mathbf{c}_k})^* \\ {\scriptstyle\overline{\pi}}\downarrow & & {\scriptstyle\overline{\pi}}\downarrow & & {\scriptstyle\overline{\pi}}\downarrow \\ \bigoplus_{i=1}^{r}(\mathbf{x}^{\mathbf{a}_i}) & \xrightarrow{\alpha} & \bigoplus_{j=1}^{s}(\mathbf{x}^{\mathbf{b}_j}) & \xrightarrow{\beta} & \bigoplus_{k=1}^{t}(\mathbf{x}^{\mathbf{c}_k}) \end{array} \tag{5}$$

where the vertical maps $\overline{\pi}$ are defined as follows: if $J_1, \dots, J_m$ are monomial ideals of S, then $\overline{\pi} : J_1^* \oplus \cdots \oplus J_m^* \to J_1 \oplus \cdots \oplus J_m$ is defined by

$$\overline{\pi}(f_1, \dots, f_m) = (\pi(f_1), \dots, \pi(f_m));$$

here π is the K-algebra homomorphism introduced in the first section. In order to see that Diagram (5) is indeed commutative we note that, for all $f \in (\mathbf{x}^{\mathbf{a}_i})^*$, we have

$$\overline{\pi}(\alpha_{ij}^*(f)) = \overline{\pi}(\lambda_{ji}f) = \lambda_{ji}\pi(f) = \alpha_{ij}(\pi(f)).$$

This shows that the left hand square in the diagram is commutative. The same argument shows that the right hand square is commutative, as well.

Let $\eta = \sum_{j=1}^{n} i_j$. Consider (5) in each multidegree

$$\mathbf{d}^* = (d_{1,1}, \dots, d_{1,i_1}, \dots, d_{n,1}, \dots, d_{n,i_n}) \in \mathbb{N}^\eta,$$

and let $\mathbf{d} = (d_1, \ldots, d_n)$ where $d_j = d_{j,1} + \cdots + d_{j,i_j}$ for all j, namely,
(6)

$$\begin{array}{ccccc}
(\bigoplus_{i=1}^{r}(\mathbf{x}^{\mathbf{a}_i})^*)_{\mathbf{d}^*} & \xrightarrow{\alpha^*} & (\bigoplus_{j=1}^{s}(\mathbf{x}^{\mathbf{b}_j})^*)_{\mathbf{d}^*} & \xrightarrow{\beta^*} & (\bigoplus_{k=1}^{t}(\mathbf{x}^{\mathbf{c}_k})^*)_{\mathbf{d}^*} \\
\pi \downarrow \cong & & \pi \downarrow \cong & & \pi \downarrow \cong \\
(\bigoplus_{i=1}^{r}(\mathbf{x}^{\mathbf{a}_i}))_{\mathbf{d}} & \xrightarrow{\alpha} & (\bigoplus_{j=1}^{s}(\mathbf{x}^{\mathbf{b}_j}))_{\mathbf{d}} & \xrightarrow{\beta} & (\bigoplus_{k=1}^{t}(\mathbf{x}^{\mathbf{c}_k}))_{\mathbf{d}}
\end{array}$$

The above diagram is commutative, since Diagram (5) is commutative. Furthermore, the restriction of $\pi : S^* \to S$ to the multidegree d^*, namely, $\pi : S^*_{d^*} \to S_d$, is an isomorphism of K-vector spaces. Thus, for all a, π induces an isomorphism $((\mathbf{x}^{\mathbf{a}})^*)_{\mathbf{d}^*} \to (\mathbf{x}^{\mathbf{a}})_{\mathbf{d}}$. Therefore, the given restrictions of π in (6) are isomorphisms of K-vector spaces. Hence, the exactness of the second row in (6) implies the exactness of the first row. $\square$

Now we are ready to define the expansion functor $\mathcal{C}(S) \to \mathcal{C}(S^*)$. For each $M \in \mathcal{C}(S)$, we choose a multigraded free presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0,$$

and set $M^* = \operatorname{Coker} \varphi^*$. Let $\alpha : M \to N$ be a morphism in the category $\mathcal{C}(S)$, and let $F_1 \xrightarrow{\varphi} F_0 \to M \to 0$ and $G_1 \xrightarrow{\psi} G_0 \to N \to 0$ be, respectively, a multigraded free presentation of M and N. We choose a lifting of α:

$$\begin{array}{ccccccc}
F_1 & \xrightarrow{\varphi} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
\alpha_1 \downarrow & & \alpha_0 \downarrow & & \downarrow \alpha & & \\
G_1 & \xrightarrow{\psi} & G_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Let α^* be the map induced by maps α_0^* and α_1^*. It is easily seen that this assignment defines an additive functor and, as a consequence of Lemma 2.1, such defined expansion functor is exact. In particular, we have:

Theorem 2.2. *Let M be a finitely generated multigraded S-module with a multigraded free resolution*

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0.$$

Then

$$\mathbb{F}^* : 0 \longrightarrow F_p^* \xrightarrow{\varphi_p^*} F_{p-1}^* \longrightarrow \cdots \longrightarrow F_1^* \xrightarrow{\varphi_1^*} F_0^* \longrightarrow 0$$

is an acyclic sequence of multigraded S^*-modules with $H_0(\mathbb{F}^*) = M^*$.

Let $I \subseteq S$ be a monomial ideal. The monomial ideal I^* as defined in Section 1 is isomorphic, as a multigraded S^*-module, to the module which is obtained by applying the expansion functor to I. Therefore, there is no ambiguity in our notation. Indeed, applying the expansion functor to the exact sequence of S-modules,

$$0 \longrightarrow I \hookrightarrow S \longrightarrow S/I \longrightarrow 0,$$

and using the fact that it is an exact functor, we obtain that $(S/I)^* \cong S^*/I^*$. On the other hand, in order to compute $(S/I)^*$, we choose a multigraded free presentation

$$F_1 \xrightarrow{\varphi_1} S \longrightarrow S/I \longrightarrow 0.$$

Then $(S/I)^* \cong \mathrm{Coker}(\varphi_1^*)$. Since the image of φ_1^* coincides with the ideal I^* as defined in Section 1, the assertion follows.

3. The free S^*-resolution of M^*. Let M be a finitely generated multigraded S-module, and let $\mathbb{F}$ be a multigraded free resolution of M. We are going to construct a multigraded free resolution of M^*, the expansion of M with respect to the n-tuple $(i_1, \ldots, i_n)$, by the resolution $\mathbb{F}$. To have a better perspective on this construction, let

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0,$$

and consider the acyclic complex $\mathbb{F}^*$, see Theorem 2.2,

$$\mathbb{F}^* : 0 \longrightarrow F_p^* \xrightarrow{\varphi_p^*} F_{p-1}^* \longrightarrow \cdots \longrightarrow F_1^* \xrightarrow{\varphi_1^*} F_0^* \longrightarrow 0.$$

We will first construct a minimal multigraded free resolution $\mathbb{G}_i$ of each F_i^*, and a natural lifting $\mathbb{G}_i \to \mathbb{G}_{i-1}$ of each map φ_i^* to obtain a double complex $\mathbb{C}$. Then we show that the total complex of $\mathbb{C}$ is a free resolution of M^*, and it is minimal if the free resolution $\mathbb{F}$ is minimal.

To obtain the desired resolution $\mathbb{G}_i$ of F_i^* and a suitable lifting of φ_i^* we need some preparation. For that purpose, we first construct for each of the modules F_i^* an explicit free resolution $\mathbb{G}_i$ based on mapping cone constructions as described in **[9]**. Then, in a second step, the liftings φ_i^* will be explicitly described by their action on the basis elements of the free modules of the resolution $\mathbb{G}_i$.

Let $J \subseteq S = K[x_1, \ldots, x_n]$ be a monomial ideal for which $G(J) = \{u_1, \ldots, u_m\}$ is the minimal set of monomial generators with a non-decreasing ordering with respect to their total degree. Suppose that J has linear quotients with respect to the ordering $u_1, \ldots, u_m$ of generators. Then $\operatorname{set}(u_j)$ is defined to be

$$\operatorname{set}(u_j) = \{k \in [n] : x_k \in (u_1, \ldots, u_{j-1}) : u_j\} \quad \text{for all } j = 1, \ldots, m.$$

In **[9,** Lemma 1.5] a minimal multigraded free resolution $\mathbb{G}(J)$ of J is given as follows: the S-module $G_i(J)$ in homological degree i of $\mathbb{G}(J)$ is the multigraded free S-module whose basis is formed by the symbols

$$f(\sigma; u) \quad \text{with} \quad u \in G(J),\ \sigma \subseteq \operatorname{set}(u),\ \text{and } |\sigma| = i.$$

Here $\deg f(\sigma; u) = \sigma + \deg u$ where σ is identified with the $(0,1)$-vector in $\mathbb{N}^n$ whose k-th component is 1 if and only if $k \in \sigma$. The augmentation map $\varepsilon : G_0 \to J$ is defined by $\varepsilon(f(\emptyset; u)) = u$.

In **[9]**, the chain map of $\mathbb{G}(J)$ for a class of such ideals is described. We first recall some definitions needed to present this chain map. Let $M(J)$ be the set of all monomials belonging to J. The map $g : M(J) \to G(J)$ is defined as follows: $g(u) = u_j$ if j is the smallest number such that $u \in (u_1, \ldots, u_j)$. This map is called the *decomposition function of* J. The *complementary factor* $c(u)$ is defined to be the monomial for which $u = c(u)g(u)$.

The decomposition function g is called *regular* if $\operatorname{set}(g(x_s u)) \subseteq \operatorname{set}(u)$ for all $s \in \operatorname{set}(u)$ and $u \in G(J)$. By **[9,** Theorem 1.12], if the decomposition function of J is regular, then the chain map ∂ of $\mathbb{G}(J)$ is given by

$$\begin{aligned}\partial(f(\sigma; u)) = &-\sum_{t \in \sigma} (-1)^{\alpha(\sigma,t)} x_t f(\sigma \setminus t; u) \\ &+ \sum_{t \in \sigma} (-1)^{\alpha(\sigma,t)} \frac{x_t u}{g(x_t u)} f(\sigma \setminus t; g(x_t u)).\end{aligned}$$

Here the definition is extended by setting $f(\sigma; u) = 0$ if $\sigma \not\subseteq \operatorname{set}(u)$, and

$$\alpha(\sigma, t) = |\{s \in \sigma : s < t\}|.$$

Now let $P = (x_{i_1}, \ldots, x_{i_r}) \subseteq S$ be a monomial prime ideal and a a positive integer. Considering the lexicographic order on the minimal set of monomial generators of P^a by the ordering $x_1 > \cdots > x_n$, the ideal P^a has linear quotients. Moreover, $\mathrm{set}(u) = \{i_1, \ldots, i_{m(u)-1}\}$ for all u in the minimal set of monomial generators of P^a, where

$$m(u) = \max\{j \in [r] : x_{i_j} | u\}.$$

We denote the decomposition function of P^a by g_a and the complementary factor of each monomial u by $c_a(u)$. The decomposition function g_a is regular. Hence, one can apply the above result to find the chain map of the minimal free resolution of P^a. We denote by $(\mathbb{G}(P^a), \partial)$ the resolution of the a-th power of a monomial prime ideal P obtained in this way.

Let $u \in P^a$ be a monomial. We may write u uniquely in the form $u = v \prod_{k=1}^{d} x_{j_k}$ such that $v \notin P$, $x_{j_k} \in P$ for all k, and $j_1 \leq j_2 \leq \cdots \leq j_d$. Observe that, with the above-mentioned ordering on the minimal set of monomial generators of P^a, we obtain

$$g_a(u) = \prod_{k=1}^{a} x_{j_k}. \tag{7}$$

Lemma 3.1. *Let P be a monomial prime ideal and u a monomial in the minimal set of monomial generators of P^a. If $a \geq b$, then*

$$g_b(g_a(x_t u)) = g_b(x_t g_b(u)) \quad \textit{for all } t.$$

Proof. Let $u = \prod_{k=1}^{a} x_{j_k}$ with $j_1 \leq \cdots \leq j_a$. Then, using formula (7) for g_a and g_b one has that $g_b(g_a(x_t u))$ and $g_b(x_t g_b(u))$ are both equal to $x_t(\prod_{k=1}^{b-1} x_{j_k})$ if $x_t \in P$ and $t \leq j_b$. Otherwise, they are both equal to $\prod_{k=1}^{b} x_{j_k}$. $\square$

Let $P \subseteq S$ be a monomial prime ideal, and consider the resolutions $\mathbb{G}(P^a)$ and $\mathbb{G}(P^b)$. Suppose that $a \geq b$. For each s, we define the map $\varphi_s^{a,b} : G_s(P^a) \to G_s(P^b)$, between the modules in homological degree s of $\mathbb{G}(P^a)$ and $\mathbb{G}(P^b)$, to be

$$\varphi_s^{a,b}(f(\sigma; u)) = c_b(u) f(\sigma; g_b(u)).$$

Here, as before, we set $f(\sigma; g_b(u)) = 0$ if $\sigma \not\subseteq \mathrm{set}(g_b(u))$. We have the following result:

Proposition 3.2. *The map $\varphi^{a,b} = (\varphi_s^{a,b}) : \mathbb{G}(P^a) \to \mathbb{G}(P^b)$ is a lifting of the inclusion S-homomorphism $\iota : P^a \to P^b$, that is, a complex homomorphism with $H_0(\varphi^{a,b}) = \iota$.*

Proof. In this proof, for simplicity, we denote $\varphi_s^{a,b}$ by φ_s. First observe that the following diagram is commutative:

$$\begin{array}{ccc} G_0(P^a) & \xrightarrow{\varepsilon} & P^a \\ {\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\iota} \\ G_0(P^b) & \xrightarrow{\varepsilon} & P^b \end{array}$$

In fact, for each element $f(\emptyset; u)$ in the basis of $G_0(P^a)$, we have

$$\varepsilon\varphi_0(f(\emptyset; u)) = \varepsilon(c_b(u)f(\emptyset; g_b(u))) = c_b(u)g_b(u) = u,$$

and also $\iota\varepsilon(f(\emptyset; u)) = u$. Therefore, the above diagram is commutative.

Next, we show that the following diagram is commutative for all $s \geq 1$:

$$\begin{array}{ccc} G_s(P^a) & \xrightarrow{\partial} & G_{s-1}(P^a) \\ {\scriptstyle\varphi_s}\downarrow & & \downarrow{\scriptstyle\varphi_{s-1}} \\ G_s(P^b) & \xrightarrow{\partial} & G_{s-1}(P^b) \end{array}$$

For each element $f(\sigma; u)$ in the basis of $G_s(P^a)$, one has

$$\begin{aligned} \varphi_{s-1}(\partial(f(\sigma; u))) &= \varphi_{s-1}\Bigg(- \sum_{t\in\sigma} (-1)^{\alpha(\sigma,t)} x_t f(\sigma \setminus t; u) \\ &\qquad + \sum_{t\in\sigma} (-1)^{\alpha(\sigma,t)} \frac{x_t u}{g_a(x_t u)} f(\sigma \setminus t; g_a(x_t u))\Bigg) \\ &= - \sum_{t\in\sigma} (-1)^{\alpha(\sigma,t)} x_t c_b(u) f(\sigma \setminus t; g_b(u)) \\ &\quad + \sum_{t\in\sigma} (-1)^{\alpha(\sigma,t)} \frac{x_t u}{g_a(x_t u)} c_b(g_a(x_t u)) f(\sigma \setminus t; g_b(g_a(x_t u))), \end{aligned}$$

and, on the other hand,

$$\begin{aligned}\partial(\varphi_s(f(\sigma;u))) &= \partial(c_b(u)f(\sigma;g_b(u))) \\ &= -\sum_{t\in\sigma}(-1)^{\alpha(\sigma,t)}x_t c_b(u)f(\sigma\setminus t;g_b(u)) \\ &\quad+\sum_{t\in\sigma}(-1)^{\alpha(\sigma,t)}c_b(u)\frac{x_t g_b(u)}{g_b(x_t g_b(u))}f(\sigma\setminus t;g_b(x_t g_b(u)).\end{aligned}$$

By Lemma 3.1, we have

$$f(\sigma\setminus t;g_b(g_a(x_t u))) = f(\sigma\setminus t;g_b(x_t g_b(u)).$$

Hence, we only need to show that

$$\frac{uc_b(g_a(x_t u))}{g_a(x_t u)} = \frac{c_b(u)g_b(u)}{g_b(x_t g_b(u))}.$$

Using the facts $u = c_b(u)g_b(u)$ and $g_a(x_t u) = c_b(g_a(x_t u))g_b(g_a(x_t u))$, one has

$$\frac{u\,c_b(g_a(x_t u))}{g_a(x_t u)} = \frac{c_b(u)g_b(u)\,c_b(g_a(x_t u))}{c_b(g_a(x_t u))\,g_b(g_a(x_t u))} = \frac{c_b(u)g_b(u)}{g_b(g_a(x_t u))} = \frac{c_b(u)g_b(u)}{g_b(x_t g_b(u))},$$

where the last equality is obtained by Lemma 3.1. □

The following simple example demonstrates the lifting $\varphi^{a,b}$ considered in Proposition 3.2: let $P = (x_1, x_2, x_3) \subseteq K[x_1, x_2, x_3]$, and let $\iota\, P^3 \to P^2$ be the inclusion map. Then, by Proposition 3.2, a lifting of ι is given by $\varphi^{3,2} = (\varphi_s^{3,2}) : \mathbb{G}(P^3) \to \mathbb{G}(P^2)$ where, for each s and for each $u = x_{j_1}x_{j_2}x_{j_3}$ in the minimal set of monomial generators of P^3 with $j_1 \le j_2 \le j_3$, one has

$$\varphi_s^{3,2}(f(\sigma;u)) = x_{j_3}f(\sigma;x_{j_1}x_{j_2}).$$

For example, $\mathrm{set}(x_1x_3^2) = \{1,2\}$. Then $\varphi_2^{3,2}(f(\{1,2\};x_1x_3^2)) = x_3f(\{1,2\};x_1x_3)$, In this example, $f(\{1,2\};x_1x_3) \neq 0$, since $\{1,2\} \subseteq \mathrm{set}(x_1x_3) = \{1,2\}$.

Proposition 3.3. *Let $a \ge b \ge c$ be nonnegative integers. Then* $\varphi^{a,c} = \varphi^{b,c} \circ \varphi^{a,b}$.

Proof. We must show that, for all s if $f(\sigma; u)$ is an element in the basis of $G_s(P^a)$, then

$$c_c(u)f(\sigma; g_c(u)) = c_b(u)\, c_c(g_b(u))f(\sigma; g_c(g_b(u))).$$

First observe that, by the formula given in (7),

$$g_c(g_b(u)) = g_c(u).$$

Moreover,

$$c_b(u)\, c_c(g_b(u)) = c_b(u)\frac{g_b(u)}{g_c(g_b(u))} = \frac{u}{g_c(g_b(u))} = \frac{u}{g_c(u)} = c_c(u). \qquad \square$$

Let $\mathbf{x}^{\mathbf{a}} \in S = K[x_1, \dots, x_n]$ be a monomial. Recall that the expansion of ideals are given with respect to the fixed n-tuple $(i_1, \dots, i_n)$, and as before, we set $P_j = (x_{j1}, x_{j2}, \dots, x_{ji_j}) \subseteq S^*$. We define the complex $\mathbb{G}^{\mathbf{a}}$ to be

$$\mathbb{G}^{\mathbf{a}} = \bigotimes_{j=1}^{n} \mathbb{G}(P_j^{a(j)}).$$

Proposition 3.4. *The complex $\mathbb{G}^{\mathbf{a}}$ is a minimal free resolution of $(\mathbf{x}^{\mathbf{a}})^*$.*

Proof. The ideal $(\mathbf{x}^{\mathbf{a}})^*$ is the product of the ideals $P_j^{a(j)}$. Since the minimal generators of the ideals P_j are in pairwise distinct sets of variables, the assertion follows from the next simple fact. $\square$

Lemma 3.5. *Let $J_1, \dots, J_r$ be (multi)graded ideals of a polynomial ring S with the property that*

$$\operatorname{Tor}_i^S(S/(J_1 \cdots J_{k-1}), S/J_k) = 0 \quad \textit{for all } k = 2, \dots, r \textit{ and all } i > 0.$$

Let $\mathbb{G}_k$ be a (multi)graded minimal free resolution of J_k for $k = 1, \dots, r$. Then $\bigotimes_{k=1}^{r} \mathbb{G}_k$ is a (multi)graded minimal free resolution of $J_1 \cdots J_k$.

Proof. Let I and J be graded ideals of S with $\operatorname{Tor}_i^S(S/I, S/J) = 0$ for all $i > 0$, and let $\mathbb{F}$ and $\mathbb{G}$ be, respectively, graded minimal free resolutions of S/I and S/J. Suppose that $\widetilde{\mathbb{F}}$ is the minimal free resolution of I obtained by $\mathbb{F}$, that is, we have $\widetilde{F}_i = F_{i+1}$ in each homological degree $i \geq 0$ of $\widetilde{\mathbb{F}}$. In the same way, we consider the resolution $\widetilde{\mathbb{G}}$ for J. We will show that $\widetilde{\mathbb{F}} \otimes \widetilde{\mathbb{G}}$ is a graded minimal free resolution of IJ. Then, by induction on the number of ideals $J_1, \ldots, J_r$, we conclude the desired statement.

First observe that $F_0 \cong S \cong G_0$ because $\mathbb{F}$ and $\mathbb{G}$ are graded minimal free resolutions of S/I and S/J. Now let $\mathbb{D}$ be the subcomplex of $\mathbb{F} \otimes \mathbb{G}$ whose k-th component is $F_0 \otimes G_k \oplus F_k \otimes G_0$ for all $k > 0$ and its 0-th component is $F_0 \otimes G_0$. Hence, $\widetilde{\mathbb{F}} \otimes \widetilde{\mathbb{G}} \cong ((\mathbb{F} \otimes \mathbb{G})/\mathbb{D})[2]$. If $\iota : \mathbb{D} \to \mathbb{F} \otimes \mathbb{G}$ is the natural inclusion map, then we have the following short exact sequence of the complexes:

$$0 \longrightarrow \mathbb{D} \xrightarrow{\iota} \mathbb{F} \otimes \mathbb{G} \longrightarrow (\widetilde{\mathbb{F}} \otimes \widetilde{\mathbb{G}})[-2] \longrightarrow 0. \tag{8}$$

Since $\mathrm{Tor}_i^S(S/I, S/J) = 0$ for all $i > 0$, we have $H_i(\mathbb{F} \otimes \mathbb{G}) = 0$ if $i > 0$; see, for example, [**11,** Theorem 10.22]. Furthermore, $H_i(\mathbb{D}) = 0$ for all $i \geq 2$. Hence, by the long exact sequence of homology modules arising from (8), one has $H_i(\widetilde{\mathbb{F}} \otimes \widetilde{\mathbb{G}}) = 0$ if $i \neq 0$, and $H_0(\widetilde{\mathbb{F}} \otimes \widetilde{\mathbb{G}}) = H_2(\widetilde{\mathbb{F}} \otimes \widetilde{\mathbb{G}})[-2] \cong H_1(\mathbb{D})$. So we only need to show that $H_1(\mathbb{D}) \cong IJ$.

The complex $F_0 \otimes \mathbb{G}$ may be considered as a subcomplex of $\mathbb{D}$ by the natural inclusion map $\iota' : F_0 \otimes \mathbb{G} \to \mathbb{D}$. Then $\widetilde{\mathbb{F}} \otimes G_0 \cong (\mathbb{D}/(F_0 \otimes \mathbb{G}))[1]$, and we have the following short exact sequence of complexes:

$$0 \longrightarrow F_0 \otimes \mathbb{G} \xrightarrow{\iota'} \mathbb{D} \longrightarrow (\widetilde{\mathbb{F}} \otimes G_0)[-1] \longrightarrow 0. \tag{9}$$

The short exact sequence (9) yields the following exact sequence of homology modules

$$0 \longrightarrow H_1(\mathbb{D}) \longrightarrow H_1((\widetilde{\mathbb{F}} \otimes G_0)[-1]) \longrightarrow H_0(F_0 \otimes \mathbb{G}) \longrightarrow H_0(\mathbb{D}) \longrightarrow 0,$$

which is isomorphic to

$$0 \longrightarrow H_1(\mathbb{D}) \longrightarrow I \longrightarrow S/J \longrightarrow S/(I + J) \longrightarrow 0.$$

Thus, $H_1(\mathbb{D}) \cong I \cap J$. Now $(I \cap J)/(IJ) = \mathrm{Tor}_1^S(S/I, S/J) = 0$; see, for example, [**11,** Proposition 10.20]. Hence, $I \cap J = IJ$ and consequently $H_1(\mathbb{D}) \cong IJ$, as desired. $\square$

Let $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in S$ be monomials such that $\mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{a}}$, that is, $b(j) \leq a(j)$ for all $j = 1, \ldots, n$. The complex homomorphisms $\varphi^{a(j),b(j)}$: $\mathbb{G}(P_j^{a(j)}) \to \mathbb{G}(P_j^{b(j)})$ induce the complex homomorphism $\varphi^{\mathbf{a},\mathbf{b}} : \mathbb{G}^{\mathbf{a}} \to$

$\mathbb{G}^{\mathbf{b}}$ by $\varphi^{\mathbf{a},\mathbf{b}} = \bigotimes_{j=1}^{n} \varphi^{a(j),b(j)}$. Then $\varphi^{\mathbf{a},\mathbf{b}}$ is a lifting of the inclusion map $\iota\ (\mathbf{x}^{\mathbf{a}})^* \to (\mathbf{x}^{\mathbf{b}})^*$.

Lemma 3.6. *Let $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}} \in S$ be monomials such that $\mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{c}} \mid \mathbf{x}^{\mathbf{b}}$. Then*

$$(\varphi^{\mathbf{b},\mathbf{c}}) \circ (\varphi^{\mathbf{a},\mathbf{b}}) = \varphi^{\mathbf{a},\mathbf{c}}.$$

Proof. This is a consequence of Proposition 3.3. □

Lemma 3.7. *Let $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in S$ be monomials such that $\mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{a}}$ and for some $t \in [n]$, $b(t) < a(t)$. Then each component of $\varphi^{\mathbf{a},\mathbf{b}}$ is minimal, i.e.,*

$$\varphi_s^{\mathbf{a},\mathbf{b}}(G_s^{\mathbf{a}}) \subseteq \mathfrak{m}^* G_s^{\mathbf{b}}$$

for all s where $\mathfrak{m}^*$ is the graded maximal ideal of S^*.

Proof. Consider the number t for which $b(t) < a(t)$. For each s, one has

$$\varphi_s^{a(t),b(t)}(f(\sigma;u)) = c_{b(t)}(u) f(\sigma; g_{b(t)}(u)) \tag{10}$$

for all $f(\sigma;u)$ in the basis of $G_s({P_t}^{a(t)})$. Since $b(t) < a(t)$, formula (7) implies that $g_{b(t)}(u) \neq u$ in (10). Hence, $c_{b(t)}(u) \in \mathfrak{m}^*$, and consequently

$$\varphi_s^{a(t),b(t)}(f(\sigma;u)) \in \mathfrak{m}^* G_s({P_t}^{b(t)}).$$

Now the desired result follows from the definition of $\mathbb{G}^{\mathbf{a}}$, $\mathbb{G}^{\mathbf{b}}$ and $\varphi^{\mathbf{a},\mathbf{b}}$. □

Let M be a finitely generated multigraded S-module, and suppose that

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

is a minimal multigraded free resolution of M with $F_i = \bigoplus_j S(-\mathbf{a}_{ij})$. By Theorem 2.2, we obtain the acyclic complex

$$\mathbb{F}^* : 0 \longrightarrow F_p^* \xrightarrow{\varphi_p^*} F_{p-1}^* \longrightarrow \cdots \longrightarrow F_1^* \xrightarrow{\varphi_1^*} F_0^* \longrightarrow 0.$$

Now, by Proposition 3.4, the complex $\mathbb{G}_i = \bigoplus_j \mathbb{G}^{\mathbf{a}_{ij}}$ is a minimal multigraded free resolution of $F_i^* = \bigoplus_j (\mathbf{x}^{\mathbf{a}_{ij}})^*$. For each i, if $[\lambda_{\ell k}]$ is the monomial expression of φ_i, then we set $d : \bigoplus_j \mathbb{G}^{\mathbf{a}_{ij}} \to \bigoplus_j \mathbb{G}^{\mathbf{a}_{(i-1)j}}$ to be the complex homomorphism whose restriction to $\mathbb{G}^{\mathbf{a}_{ik}}$ and $\mathbb{G}^{\mathbf{a}_{(i-1),\ell}}$ is $\lambda_{\ell k}\varphi^{\mathbf{a}_{ik},\mathbf{a}_{(i-1),\ell}}$. Thus, d is a lifting of φ_i^* and, moreover, by Lemma 3.6, we have a double complex $\mathbb{C}$ with $C_{ij} = G_{ij}$, namely, the module in homological degree j of the complex $\mathbb{G}_i$, whose column differentials are given by the differentials ∂ of each $\mathbb{G}_i$ and row differentials are given by d.

$$\begin{array}{ccccccc}
0 \to F_p^* & \xrightarrow{\varphi_p^*} & F_{p-1}^* & \to \cdots \to & F_1^* & \xrightarrow{\varphi_1^*} & F_0^* \to 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
G_{p0} & \xrightarrow{d} & G_{p-1\,0} & \to \cdots \to & G_{10} & \xrightarrow{d} & G_{00} \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
G_{p1} & \xrightarrow{d} & G_{p-1\,1} & \to \cdots \to & G_{11} & \xrightarrow{d} & G_{01} \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
G_{p2} & \xrightarrow{d} & G_{p-1\,2} & \to \cdots \to & G_{12} & \xrightarrow{d} & G_{02} \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial
\end{array}$$

We have the following result:

Theorem 3.8. *The total complex $T(\mathbb{C})$ of $\mathbb{C}$ is a minimal multigraded free resolution of M^*.*

Proof. By Theorem 2.2, the following sequence of multigraded modules is acyclic

$$\mathbb{F}^* : 0 \longrightarrow F_p^* \xrightarrow{\varphi_p^*} F_{p-1}^* \longrightarrow \cdots \longrightarrow F_1^* \xrightarrow{\varphi_1^*} F_0^* \longrightarrow 0,$$

and $H_0(\mathbb{F}^*) \cong M^*$. For each i, the complex $\mathbb{G}_i = \bigoplus_j \mathbb{G}^{\mathbf{a}_{ij}}$ is a minimal multigraded free resolution of F_i^* because each $\mathbb{G}^{\mathbf{a}_{ij}}$ is a minimal free resolution of $(\mathbf{x}^{\mathbf{a}_{ij}})^*$, as we know by Proposition 3.4.

We compute the spectral sequences with respect to the column filtration of the double complex $\mathbb{C}$. So $E^1_{r,s} = H^v_s(\mathbb{C}_{r,\bullet}) = H_s(\mathbb{G}_r)$. Since each $\mathbb{G}_r$ is an acyclic sequence with $H_0(\mathbb{G}_r) \cong F_r^*$, one has $E^1_{r,s} = 0$ if $s \neq 0$, and $E^1_{r,0} \cong F_r^*$. Next, we consider $E^2_{r,s} = H^h_r(H^v_s(\mathbb{C}))$. So $E^2_{r,s} = 0$ if $s \neq 0$, and $E^2_{r,0} = H_r(\mathbb{F}^*)$. By Theorem 2.2, we have

$H_r(\mathbb{F}^*) = 0$ for $r \neq 0$ and $H_0(\mathbb{F}^*) = M^*$. Thus, we see that $E^2_{r,s} = 0$ for $(r, s) \neq (0, 0)$ and $E^2_{0,0} = M^*$. This implies that the total complex $T(\mathbb{C})$ of $\mathbb{C}$ is a multigraded free resolution of M^*; see, for example, [**11,** Proposition 10.17].

Since each $\mathbb{G}_i$ is a minimal free resolution of F_i^*, and moreover by Lemma 3.7, each component of d is also minimal, we conclude that the free resolution $T(\mathbb{C})$ of M^* is minimal. $\square$

4. Homological properties of M^*. This section concerns some homological properties of the expansion of a finitely generated multigraded S-module M with respect to a fixed n-tuple $(i_1, \ldots, i_n)$. We call

$$\mathcal{P}_M(t) = \sum_{j=0}^{p} \beta_j t^j$$

the *Betti polynomial* of M, where β_j is the j-th Betti number of M and p is its projective dimension.

First we need to have the Betti numbers of I^* for the case that $I = (\mathbf{x}^{\mathbf{a}})$ is a principal ideal, and hence I^* is a product of powers of monomial prime ideals in pairwise distinct sets of variables. We have the following fact:

Proposition 4.1. *Let* $I = (\mathbf{x}^{\mathbf{a}})$ *with* $\mathbf{a} = (a_1, \ldots, a_n)$. *Then* I^* *has a linear resolution with*

$$\mathcal{P}_{I^*}(t) = \prod_{j \in \operatorname{supp}(\mathbf{a})} P_j(t),$$

where

$$P_j(t) = \sum_{i=0}^{i_j - 1} \binom{i_j + a_j - 1}{i_j - i - 1} \binom{a_j + i - 1}{i} t^i$$

In particular, $\operatorname{proj\,dim} I^* = \sum_{j \in \operatorname{supp}(\mathbf{a})} (i_j - 1)$.

Proof. By Lemma 3.5, it is enough to compute the Betti numbers for powers of monomial prime ideals. So let $R = K[y_1, \ldots, y_r]$ be a polynomial ring, and consider $J = (y_1, \ldots, y_r)^s$ for some positive integer s. Then the Eagon-Northcott complex resolving J gives the Betti numbers; see, for example, [**2**]. Alternatively, one can use the

Herzog-Kühl formula to obtain the Betti numbers [**8,** Theorem 1]. In fact, first observe that J is Cohen-Macaulay and it has linear resolution; see [**3,** Theorem 3.1]. Next by the Auslander-Buchsbaum formula, one has $\operatorname{proj}\dim(J) = r-1$. Therefore, by the Herzog-Kühl formula one obtains that

$$\begin{aligned}\beta_j(J) &= \frac{s(s+1)\cdots(s+j-1)\widehat{(s+j)}(s+j+1)\cdots(s+r-1)}{j!\times(r-j-1)!}\\ &= \frac{(s+r-1)!}{(r-j-1)!(s+j)!}\times\frac{(s+j-1)!}{j!(s-1)!}\\ &= \binom{r+s-1}{r-j-1}\binom{s+j-1}{j}.\end{aligned}$$

□

Theorem 4.2. *Let M be a finitely generated multigraded S-module with the minimal multigraded free resolution*

$$\mathbb{F}: 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0,$$

where $F_i = \bigoplus_j S(-\mathbf{a}_{ij})$ for $i = 0,\dots,p$. Let M^* be the expansion of M with respect to $(i_1, i_2, \dots, i_n)$. Then

$$\beta_{jk}(M^*) = \sum_{i=0}^{p} \beta_{j-i,k}(F_i^*) \quad \textit{for all } j \textit{ and } k,$$

Moreover, $\operatorname{reg}(M) = \operatorname{reg}(M^*)$ *and*

$$\operatorname{proj}\dim M^* = \max_{i,j}\{i + \sum_{k\in \operatorname{supp}(\mathbf{a}_{ij})}(i_k - 1)\}.$$

Proof. By Theorem 3.8, we have

$$\beta_{jk}(M^*) = \sum_{i=0}^{p} \beta_{j-i,k}(F_i^*) \quad \text{for all } j \text{ and } k.$$

Next we show that $\operatorname{reg}(M) = \operatorname{reg}(M^*)$. If $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{N}^n$ is a vector, we denote $\sum_{i=1}^n a_i$ by $|a|$. For each $F_i = \bigoplus_{j=1}^{\beta_i} S(-\mathbf{a}_{ij})$, we choose $\mathbf{a}_{i\ell_i}$ such that

$$|\mathbf{a}_{i\ell_i}| = \max\{|\mathbf{a}_{ij}| : j = 1,\dots,\beta_i\}.$$

Then $\operatorname{reg}(M) = \max\{|\mathbf{a}_{i\ell_i}| - i : i = 0,\dots,p\}$.

On the other hand, $\mathbb{G}_i = \bigoplus_{j=1}^{\beta_i} \mathbb{G}^{\mathbf{a}_{ij}}$, and $\mathbb{G}^{\mathbf{a}_{ij}}$ is a $|\mathbf{a}_{ij}|$-linear resolution of $(\mathbf{x}^{\mathbf{a}_{ij}})^*$. Hence, if, for a fixed j, we set $r_j = \max\{k - j : \beta_{jk}(M^*) \neq 0\}$, then

$$\begin{aligned} r_j &= \max\{(|\mathbf{a}_{i\ell_i}| + (j - i)) - j : i = 0, \ldots, j\} \\ &= \max\{|\mathbf{a}_{i\ell_i}| - i : i = 0, \ldots, j\}, \end{aligned}$$

and $\operatorname{reg}(M^*) = \max\{r_j\ j = 0, \ldots, p\}$. This yields $\operatorname{reg}(M) = \operatorname{reg}(M^*)$.

By Theorem 3.8, the complex $T(\mathbb{C})$ is a minimal free resolution of M^*. Therefore,

$$\operatorname{proj\,dim} M^* = \max_i \{i + \operatorname{proj\,dim} F_i^* : i = 0, \ldots, p\}.$$

On the other hand, since $F_i^* = \bigoplus_j (\mathbf{x}^{\mathbf{a}_{ij}})^*$, we conclude that $\operatorname{proj\,dim} F_i^*$ is the maximum number among the numbers $\operatorname{proj\,dim}(\mathbf{x}^{\mathbf{a}_{ij}})^*$. Hence,

$$\operatorname{proj\,dim} M^* = \max_{i,j} \{i + \operatorname{proj\,dim}(\mathbf{x}^{\mathbf{a}_{ij}})^*\}$$

Now, by Proposition 4.1,

$$\operatorname{proj\,dim} M^* = \max_{i,j} \left\{ i + \sum_{k \in \operatorname{supp}(\mathbf{a}_{ij})} (i_k - 1) \right\}. \qquad \square$$

As a result of Theorem 4.2, we obtain the following:

Corollary 4.3. *Let M be a finitely generated multigraded S-module. Then M has a d-linear resolution if and only if M^* has a d-linear resolution.*

Remark 4.4. In Theorem 4.2, the Betti numbers of M^* are given in terms of the Betti numbers of S^*-modules F_i^*. Each F_i^* is a direct sum of ideals of the form $(\mathbf{x}^{\mathbf{a}})^*$. So one can apply Proposition 4.1 to obtain a more explicit formula for Betti numbers of M^*.

Let M be a finitely generated multigraded S-module with a minimal multigraded free resolution

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0,$$

where $F_i = \bigoplus_j S(-\mathbf{a}_{ij})$ for $i = 0, \ldots, p$. We call a shift $\mathbf{a}_{ij}$ in $\mathbb{F}$ an *extremal multigraded shift* of M if, for all multigraded shifts $\mathbf{a}_{k\ell}$

in $\mathbb{F}$ with $k > i$, the monomial $\mathbf{x}^{\mathbf{a}_{ij}}$ does not divide $\mathbf{x}^{\mathbf{a}_{k\ell}}$. Then, by Theorem 4.2,

$$\operatorname{proj\,dim} M^* = \max\left\{ i + \sum_{k \in \operatorname{supp}(\mathbf{a}_{ij})} (i_k - 1) : \mathbf{a}_{ij} \text{ is an extremal shift of } \mathbb{F} \right\}.$$

It is not always the case that the extremal shifts of $\mathbb{F}$ only appear in F_p. For example, consider the ideal $I = (x_1x_4x_6, x_2x_4x_6, x_3x_4x_5, x_3x_4x_6)$ and its minimal multigraded free resolution $\mathbb{F}$. Here, for simplicity, we write monomials rather than shifts:

$$\begin{aligned}\mathbb{F} : 0 \longrightarrow (x_1x_2x_3x_4x_6) &\longrightarrow (x_1x_2x_4x_6) \oplus (x_1x_3x_4x_6) \oplus (x_2x_3x_4x_6) \\ &\oplus (x_3x_4x_5x_6) \longrightarrow (x_1x_4x_6) \oplus (x_2x_4x_6) \oplus (x_3x_4x_5) \oplus (x_3x_4x_6) \\ &\longrightarrow 0.\end{aligned}$$

Then $(0, 0, 1, 1, 1, 1)$ corresponding to the monomial $x_3x_4x_5x_6$ is an extremal multigraded shift of $\mathbb{F}$. However, if M is Cohen-Macaulay, then the extremal multigraded shifts appear only in the last module of the minimal multigraded free resolution; otherwise, there exists a free direct summand in the i-th syzygy module of M for some $i < \operatorname{proj\,dim} M$. But by a result of Dutta this cannot happen; see [**5,** Corollary 1.2].

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