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SOME TRANSFORMATIONS ON THE BILATERAL SERIES ${}_2\psi_2$

ZHIZHENG ZHANG AND ZEYA JIA

ABSTRACT. The purpose of this paper is to derive several new transformation formulas between bilateral and unilateral basic hypergeometric series. From these transformations, some q-series identities are obtained.

1. Introduction. Let a and q be complex numbers, with $|q| < 1$ unless otherwise stated. Then

$$
\begin{aligned}
&(a)_0 = (a;q)_0 = 1, \qquad (a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad \text{for } n \in N,\\
&(a_1,a_2,\ldots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n,\\
&(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j),\\
&(a_1,a_2,\ldots,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty.
\end{aligned}
$$

In addition, we also use the following notation:

$$
(a;q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}. \tag{1}
$$

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Basic hypergeometric series and bilateral basic hypergeometric series are defined by

$$
{}_r\phi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix} ;q,z\right)
= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,
$$

and

$$
{}_r\psi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix} ;q,z\right)
= \sum_{n=-\infty}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,a_2,\ldots,b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{s-r} z^n,
$$

respectively. For more details, especially their convergence and the definitions on well-poised and very-well-poised series, see [**6**].

One of the most important summation theorems for basic hypergeometric series is the Heines q-analog of the Gauss summation theorem:

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\ q;\ \frac{c}{ab}\right] = \frac{(c/a,\ c/b\ ;q)_\infty}{(c,\ c/ab\ ;q)_\infty}, \tag{2}
$$

where $|c/ab| < 1$. See [**6,** (1.5.1)].

Another one of the most important summation theorems for basic hypergeometric series is Ramanujan's sum for ${}_1\psi_1$:

$$
{}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}\ q;\ z\right] = \frac{(q,\ b/a,\ az,\ q/az;q)_\infty}{(b,\ q/a,\ z,\ b/az\ ;q)_\infty}, \tag{3}
$$

where $|b/a| < |z| < 1$. See [**6,** (5.2.1)].

In [**1**], Bailey gave the following summation formula:

$$
{}_2\psi_2\left(\begin{matrix} a,b \\ c,bq \end{matrix} ;q,\frac{q}{a}\right) = \frac{(q,q,bq/a,c/b;q)_\infty}{(q/a,bq,q/b,c;q)_\infty}, \tag{4}
$$

where $\max(|q/a|,|c|) < 1$.

In the theory of basic hypergeometric series, it is known that bilateral series are important objects. Many authors showed some relations between the bilateral series and the unilateral basic hypergeometric series. For example, in **[1]**–**[5]** and **[7]**–**[12]**, some transformation formulas had been obtained for the bilateral ${}_2\psi_2$ series. The purpose of this paper is to derive several new transformation formulas between bilateral and unilateral basic hypergeometric series. From these transformations, some q-series identities are obtained.

2. Several transformations between ${}_2\psi_2$ and ${}_2\phi_1$.

First of all, we give a result to be used later.

Lemma 2.1. *Let* $|c/ab| < 1$. *Then*

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & cq \end{matrix} ; q, \frac{c}{ab}\right] = \frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty}\left\{\frac{ab(1+c) - c(a+b)}{ab-c}\right\}. \tag{5}$$

Proof. We have

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b \\ & cq \end{matrix} ; q, \frac{c}{ab}\right] &= \sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,cq;q)_k}\left(\frac{c}{ab}\right)^k \\
&= (1-c)\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k(1-cq^k)}\left(\frac{c}{ab}\right)^k \\
&= (1-c)\sum_{k=0}^{\infty}\frac{(a,b;q)_k(1-cq^k+cq^k)}{(q,c;q)_k(1-cq^k)}\left(\frac{c}{ab}\right)^k \\
&= (1-c)\,{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ; q, \frac{c}{ab}\right] + c\,{}_2\phi_1\left[\begin{matrix} a, & b \\ & cq \end{matrix} ; q, \frac{cq}{ab}\right] \\
(*) \quad &= (1-c)\frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty} + c\frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty} \\
&= \frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty}\left\{\frac{(a-c)(b-c)}{(ab-c)} + c\right\} \\
&= \frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty}\left\{\frac{ab(1+c)-c(a+b)}{ab-c}\right\},
\end{aligned}$$

where $(*)$ follows from the q-Gaussian summation (2). The proof of the lemma is complete. □

Theorem 2.2. *Let* $|d/(acq)| < 1$*,* $|b/a| < 1$ *and* $|bq| < 1$*. Then*

$$
{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix} ; q, \frac{d}{acq}\right] = \frac{(q, b/a, d/c, d/a; q)_\infty}{(acq-d)(b, q/a, d/ac, d; q)_\infty}
\Big\{ a(cq-d)\,{}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c \end{matrix} ; q, qb/a\right]
+(a-1)cd\,{}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c \end{matrix} ; q, b/a\right]\Big\}. \tag{6}
$$

Proof. Let $a \to aq^n$, $b \to q/b$ and $c \to q^{1+n}$ in q-Gaussian summation formula (2). Then

$$
{}_2\phi_1\left[\begin{matrix} aq^n, & q/b \\ & q^{1+n} \end{matrix} ; q, \frac{b}{a}\right] = \frac{(q/a, bq^n; q)_\infty}{(q^{1+n}, b/a; q)_\infty}. \tag{7}
$$

Rewrite the equation as

$$
\frac{(q, b/a; q)_\infty}{(q/a, b; q)_\infty} \sum_{k=0}^{\infty} \frac{(q/b; q)_k (a; q)_{n+k}}{(q; q)_k (q; q)_{n+k}} \left(\frac{b}{a}\right)^k = \frac{(a; q)_n}{(b; q)_n}. \tag{8}
$$

Multiplying both sides by $(c; q)_n/(d; q)_n(d/acq)^n$, we have the following result after summing over all integers n.

$$
{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix} ; q, \frac{d}{acq}\right]
= \frac{(q, b/a; q)_\infty}{(q/a, b; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(c; q)_n}{(d; q)_n} \left(\frac{d}{acq}\right)^n \sum_{k=0}^{\infty} \frac{(q/b; q)_k (a; q)_{n+k}}{(q; q)_k (q; q)_{n+k}} \left(\frac{b}{a}\right)^k. \tag{9}
$$

Replacing n by $n-k$, the summation on the right-hand side of the above equation equals

$$
\sum_{k=0}^{\infty} \frac{(q/b; q)_k (c; q)_{-k}}{(q; q)_k (d; q)_{-k}} \left(\frac{bcq}{d}\right)^k \sum_{n=-\infty}^{\infty} \frac{(a, cq^{-k}; q)_n}{(q, dq^{-k}; q)_n} \left(\frac{d}{aqc}\right)^n.
$$

Hence,

$$
{}_2\psi_2\left[\begin{matrix} a, & c, \\ b, & d, \end{matrix} ; \; q, \frac{d}{aqc}\right]
= \frac{(q, b/a; q)_\infty}{(q/a, b; q)_\infty} \sum_{k=0}^{\infty} \frac{(q/b; q)_k (c; q)_{-k}}{(q; q)_k (d; q)_{-k}} \left(\frac{bcq}{d}\right)^k
{}_2\phi_1\left[\begin{matrix} a, & cq^{-k} \\ & dq^{-k} \end{matrix} ; \; q, d/aqc\right]. \tag{10}
$$

With the help of (5), the second sum of the above formula equals

$$
{}_2\phi_1\left[\begin{matrix} a, & cq^{-k} \\ & dq^{-k} \end{matrix} ; q, d/aqc\right]
= \frac{(dq^{-k}/a, d/c; q)_\infty}{(dq^{-k}, d/ac; q)_\infty} \left\{ \frac{acq(1 + dq^{-1-k}) - d(a + cq^{-k})}{acq - d} \right\}. \tag{11}
$$

Substituting (11) into (10) and then using (1), we get

$$
\begin{aligned}
{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix} ; \; q, \frac{d}{aqc}\right] &= \frac{(q, b/a; q)_\infty}{(q/a, b; q)_\infty} \sum_{k=0}^{\infty} \frac{(q/b; q)_k (c; q)_{-k}}{(q; q)_k (d; q)_{-k}} \left(\frac{bqc}{d}\right)^k \\
&\quad \times \frac{(dq^{-k}/a, d/c; q)_\infty}{(dq^{-k}, d/ac; q)_\infty} \left\{ \frac{acq(1 + dq^{-1-k}) - d(a + cq^{-k})}{acq - d} \right\} \\
&= \frac{(q, b/a, d/c, d/a; q)_\infty}{(q/a, b, d/ac, d; q)_\infty} \\
&\quad \times \left\{ \frac{a(cq - d)}{(acq - d)} \sum_{k=0}^{\infty} \frac{(c; q)_{-k} (q/b; q)_k}{(d/a; q)_{-k} (q; q)_k} \left(\frac{bcq}{d}\right)^k \right. \\
&\quad \left. + \frac{(a - 1)dc}{(acq - d)} \sum_{k=0}^{\infty} \frac{(c; q)_{-k} (q/b; q)_k}{(d/a; q)_{-k} (q; q)_k} \left(\frac{bc}{d}\right)^k \right\} \\
&= \frac{(q, b/a, d/c, d/a; q)_\infty}{(q/a, b, d/ac, d; q)_\infty} \\
&\quad \times \left\{ \frac{a(cq - d)}{(acq - d)} \sum_{k=0}^{\infty} \frac{(q/b, aq/d; q)_k}{(q, q/c; q)_k} \left(\frac{bq}{a}\right)^k \right. \\
&\quad \left. + \frac{(a - 1)dc}{(acq - d)} \sum_{k=0}^{\infty} \frac{(q/b, aq/d; q)_k}{(q, q/c; q)_k} \left(\frac{b}{a}\right)^k \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(q, b/a, d/c, d/a; q)_\infty}{(acq-d)(b, q/a, d/ac, d; q)_\infty} \\
&\quad \times \Bigg\{ a(cq-d)\, {}_2\phi_1 \left[\begin{matrix} q/b, & qa/d \\ & q/c \end{matrix} ; \; q, qb/a \right] \\
&\quad + (a-1)cd \; {}_2\phi_1 \left[\begin{matrix} q/b, & qa/d \\ & q/c \end{matrix} ; q, b/a \right] \Bigg\}.
\end{aligned}
$$

The proof of the theorem is completed. □

This theorem includes several known and new q-series identities. We tabulate the main cases as follows.

(a) Taking $b \to q$ in Theorem 2.2, we obtain Lemma 2.1.

(b) Taking $c \to b$ in Theorem 2.2, by applying the Cauchy identity

$$
\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty},
$$

we can obtain Ramanujan's ${}_1\psi_1$ summation (3).

(c) Taking $d \to aq$ (or $d \to cq$) in Theorem 2.2, we obtain that

$$
{}_2\psi_2 \left[\begin{matrix} c, & a \\ b, & aq \end{matrix} ; \; q, \frac{1}{c} \right] = a \frac{(q, b/a, aq/c, q; q)_\infty}{(b, q/a, q/c, aq; q)_\infty}, \tag{12}
$$

where $|1/c| < 1$ and $|bq| < 1$, which is different from (4). Here we give another proof of (12). Since

$$
\begin{aligned}
{}_2\psi_2 \left[\begin{matrix} c, & a \\ b, & aq \end{matrix} ; \; q, \frac{1}{c} \right] &= \sum_{n=-\infty}^{\infty} \frac{(1-a)}{(1-aq^n)} \frac{(c;q)_n}{(b;q)_n} \left(\frac{1}{c}\right)^n \\
&= (1-a) \sum_{n=-\infty}^{\infty} \frac{(c;q)_n}{(b;q)_n} \frac{(1-aq^n+aq^n)}{(1-aq^n)} \left(\frac{1}{c}\right)^n \\
&= (1-a) \sum_{n=-\infty}^{\infty} \frac{(c;q)_n}{(b;q)_n} \left(\frac{1}{c}\right)^n \\
&\quad + (1-a) \sum_{n=-\infty}^{\infty} \frac{(c;q)_n}{(b;q)_n} \frac{aq^n}{(1-aq^n)} \left(\frac{1}{c}\right)^n
\end{aligned}
$$

$$
\begin{aligned}
&= (1-a)\sum_{n=-\infty}^{\infty}\frac{(c;q)_n}{(b;q)_n}\left(\frac{1}{c}\right)^n \\
&\quad + a\sum_{n=-\infty}^{\infty}\frac{(a,c;q)_n}{(aq,b;q)_n}\left(\frac{q}{c}\right)^n \\
&= (1-a)\,{}_1\psi_1\left[\begin{matrix} c \\ b \end{matrix}\;;\;q,\frac{1}{c}\right] \\
&\quad + a\,{}_2\psi_2\left[\begin{matrix} c, & a \\ b, & aq \end{matrix}\;;q,\frac{q}{c}\right] \\
(*)\quad &= 0 + a\frac{(q,b/a,aq/c,q;q)_\infty}{(b,q/a,q/c,aq;q)_\infty} \\
&= a\frac{(q,b/a,aq/c,q;q)_\infty}{(b,q/a,q/c,aq;q)_\infty},
\end{aligned}
$$

where $(*)$ follows from (3) and (4), the proof is completed.

(d) Taking $d \to q$ in Theorem 2.2, we have

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & c \\ & b \end{matrix}\;;\;q,\frac{1}{ac}\right] &= \frac{(q/c,q/a;q)_\infty}{(ac-1)(b,q/ac;q)_\infty} \\
&\times\left\{a(c-1)\,{}_2\phi_1\left[\begin{matrix} q/b, & a \\ & q/c \end{matrix}\;;\;q,bq/a\right]\right. \\
&\qquad \left. +(a-1)c\,{}_2\phi_1\left[\begin{matrix} q/b, & a \\ & q/c \end{matrix}\;;q,b/a\right]\right\},
\end{aligned}
$$

where $|b/a| < 1$ and $|1/ac| < 1$.

(e) Taking $b \to aq$ in Theorem 2.2, we have

$$
\begin{aligned}
(13)\quad {}_2\psi_2\left[\begin{matrix} a, & c \\ aq, & d \end{matrix}\;;\;q,\frac{d}{acq}\right] &= \frac{(q,q,d/c,d/a;q)_\infty}{(acq-d)(aq,q/a,d/ac,d;q)_\infty} \\
&\times\left\{a(cq-d)\,{}_2\phi_1\left[\begin{matrix} 1/a, & aq/d \\ & q/c \end{matrix}\;;\;q,q^2\right]\right. \\
&\qquad \left. + (a-1)cd\,{}_2\phi_1\left[\begin{matrix} 1/a, & aq/d \\ & q/c \end{matrix}\;;\;q,q\right]\right\},
\end{aligned}
$$

where $|aq| < 1$ and $|d/acq| < 1$.

Theorem 2.3. *Let $|b/(aq)| < 1$, $|b| < 1$ and $|d/(ac)| < 1$. Then*

$$\begin{aligned}
&a(q-b)(acq-d)\frac{(b,q/a,d/ac,d;q)_\infty}{(q,b/a,d/c,d/a;q)_\infty}{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d\end{matrix} ; q, \frac{d}{ac}\right] \\
&= cd(aq-b)(a-1){}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c\end{matrix} ; q, b/aq\right] \\
&\quad + (a(aq-b)(cq-d)-qcd(a-1)^2){}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c\end{matrix} ; q, b/a\right] \\
&\quad - aq(a-1)(cq-d){}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c\end{matrix} ; q, bq/a\right].
\end{aligned} \tag{14}$$

Proof. Let $a \to aq^n$, $b \to q/b$ and $c \to q^n$ in (5):

$$\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} aq^n, & q/b \\ & q^{n+1}\end{matrix} ; q, \frac{b}{aq}\right] \\
&\qquad = \frac{(q/a, bq^n;q)_\infty}{(q^{n+1}, b/a;q)_\infty}\left\{\frac{aq(1+q^n)-q(abq^{n-1}+1)}{aq-b}\right\},
\end{aligned} \tag{15}$$

where $|b/(aq)| < 1$. Rewrite the equation as

$$\begin{aligned}
&(aq-b)\frac{(q,b/a;q)_\infty}{(b,q/a;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_{n+k}(q/b;q)_k}{(q;q)_{n+k}(q;q)_k}\left(\frac{b}{aq}\right)^k \\
&\qquad = (a-1)q\frac{(a;q)_n}{(b;q)_n} + a(q-b)\frac{(a;q)_n}{(b;q)_n}q^n.
\end{aligned} \tag{16}$$

Multiplying both sides by $(c;q)_n/(d;q)_n z^n$, we have the following result after summing over all integers n.

$$\begin{aligned}
&q(a-1){}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d\end{matrix} ; q, z\right] + a(q-b){}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d\end{matrix} ; q, zq\right] \\
&= (aq-b)\frac{(q,b/a;q)_\infty}{(b,q/a;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(c;q)_n}{(d;q)_n} \\
&\quad \times z^n\sum_{k=0}^{\infty}\frac{(a;q)_{n+k}(q/b;q)_k}{(q;q)_{n+k}(q;q)_k}\left(\frac{b}{aq}\right)^k \\
&= (aq-b)\frac{(q,b/a;q)_\infty}{(b,q/a;q)_\infty}\sum_{k=0}^{\infty}\frac{(c;q)_{-k}(q/b;q)_k}{(d;q)_{-k}(q;q)_k}\left(\frac{b}{aqz}\right)^k
\end{aligned} \tag{17}$$

$$\times {}_2\phi_1\left[\begin{matrix} a, & cq^{-k} \\ & dq^{-k} \end{matrix};\ q, z\right].$$

Taking $z \to d/(aqc)$ in (17), we have

$$\begin{aligned}
&q(a-1){}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix};\ q, \frac{d}{acq}\right] + a(q-b){}_2\psi_2\left[\begin{matrix} a, & c \\ b, & d \end{matrix};\ q, \frac{d}{ac}\right] \\
&\quad = (aq-b)\frac{(q, b/a; q)_\infty}{(b, q/a; q)_\infty}\sum_{k=0}^{\infty}\frac{(c;q)_{-k}(q/b;q)_k}{(d;q)_{-k}(q;q)_k}\left(\frac{bc}{d}\right)^k \\
&\qquad \times {}_2\phi_1\left[\begin{matrix} a, & cq^{-k} \\ & dq^{-k} \end{matrix};\ q, \frac{d}{acq}\right] \\
(*) &\quad = (aq-b)\frac{(q, b/a; q)_\infty}{(b, q/a; q)_\infty}\sum_{k=0}^{\infty}\frac{(c;q)_{-k}(q/b;q)_k}{(d;q)_{-k}(q;q)_k}\left(\frac{bc}{d}\right)^k \\
&\qquad \times \frac{((d/a)q^{-k}, d/c; q)_\infty}{(dq^{-k}, d/ac; q)_\infty}\cdot\frac{ac(q+dq^{-k}) - d(a+cq^{-k})}{acq-d} \\
&\quad = \frac{a(cq-d)(aq-b)}{(acq-d)}\frac{(q, b/a, d/c, d/a; q)_\infty}{(b, q/a, d/ac, d; q)_\infty} \\
&\qquad \times \sum_{k=0}^{\infty}\frac{(q/b, aq/d; q)_k}{(q, q/c; q)_k}\left(\frac{b}{a}\right)^k \\
&\qquad + \frac{cd(aq-b)(a-1)}{(acq-d)}\frac{(q, b/a, d/c, d/a; q)_\infty}{(b, q/a, d/ac, d; q)_\infty} \\
&\qquad \times \sum_{k=0}^{\infty}\frac{(q/b, aq/d; q)_k}{(q, q/c; q)_k}\left(\frac{b}{aq}\right)^k \\
&\quad = \frac{(aq-b)(q, b/a, d/c, d/a; q)_\infty}{(acq-d)(b, q/a, d/ac, d; q)_\infty} \\
&\qquad \times \left\{a(qc-d){}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c \end{matrix};\ q, \frac{b}{a}\right]\right. \\
&\qquad\qquad \left. + (a-1)cd{}_2\phi_1\left[\begin{matrix} q/b, & qa/d \\ & q/c \end{matrix};\ q, \frac{b}{aq}\right]\right\},
\end{aligned}$$

where $(*)$ follows by applying (11) in the second sum. Substituting (6) into the above equation, the proof of the theorem can be obtained. $\square$

This theorem includes the following several corollaries:

(a) Taking $d \to aq$ in Theorem 2.3, the theorem becomes Bailey's summation (4).

(b) Taking $d \to q$ in Theorem 2.3, we have

$$\begin{aligned} &a(q-b)(ac-1)\frac{(b,q/ac;q)_\infty}{(q/c,b/a;q)_\infty}{}_2\phi_1\left[\begin{matrix} a, & c \\ & b \end{matrix};\, q, \frac{q}{ac}\right] \\ &\qquad = c(aq-b)(a-1){}_2\phi_1\left[\begin{matrix} a, & q/b \\ & q/c \end{matrix};\, q, \frac{b}{aq}\right] \\ &\qquad\quad + (a(aq-b)(c-1)-cq(a-1)^2){}_2\phi_1\left[\begin{matrix} a, & q/b \\ & q/c \end{matrix};\, q, b/a\right] \\ &\qquad\quad - aq(a-1)(c-1){}_2\phi_1\left[\begin{matrix} a, & q/b \\ & q/c \end{matrix};\, q, \frac{bq}{a}\right] \end{aligned}$$

where $|q/(ac)| < 1$ and $|b/(aq)| < 1$.

(c) Taking $b \to aq$ in Theorem 2.3, we have

$$\begin{aligned} &{}_2\psi_2\left[\begin{matrix} a, & c \\ aq, & d \end{matrix};\, q, \frac{d}{ac}\right] = \frac{(q,q,d/c,d/a;q)_\infty}{(acq-d)(aq,q/a,d/ac,d;q)_\infty} \\ &\qquad \left\{(cq-d){}_2\phi_1\left[\begin{matrix} 1/a, & aq/d \\ & q/c \end{matrix};\, q, q^2\right]\right. \\ &\qquad\qquad \left. + \frac{(a-1)cd}{a}{}_2\phi_1\left[\begin{matrix} 1/a, & aq/d \\ & q/c \end{matrix};\, q, q\right]\right\} \end{aligned}$$

where $|d/(ac)| < 1$ and $|a| < 1$.

(d) Taking $c \to 1$ in Theorem 2.3, we have

$$\begin{aligned} &a(q-b)(aq-d)\frac{(b,q/a;q)_\infty}{(q,b/a;q)_\infty}{}_2\phi_1\left[\begin{matrix} q/b, & q/d \\ & q/a \end{matrix};\, q, b\right] \\ &\qquad = d(aq-b)(a-1){}_2\phi_1\left[\begin{matrix} q/b, & aq/d \\ & q \end{matrix};\, q, \frac{b}{aq}\right] \\ &\qquad\quad + (a(aq-b)(q-d)-qd(a-1)^2){}_2\phi_1\left[\begin{matrix} q/a, & aq/d \\ & q \end{matrix};\, q, \frac{b}{a}\right] \\ &\qquad\quad - aq(a-1)(q-d){}_2\phi_1\left[\begin{matrix} q/b, & aq/d \\ & q \end{matrix};\, q, \frac{bq}{a}\right] \end{aligned}$$

where $|d/a| < 1$, $|b| < 1$ and $|b/(aq)| < 1$.

3. Several transformations between ${}_2\psi_2$ and ${}_3\phi_2$ series. In **[12]**, we gave a transformation between ${}_2\psi_2$ and ${}_3\phi_2$ series. Here we discuss further results.

Theorem 3.1. *Let $|b/(aq)| < 1$ and $|q/a| < 1$. Then*

$$\begin{aligned}
&\frac{(q/a,-q/a,b,cq/a;q)_\infty}{(aq-b)(b/a;q)_\infty(q^2;q^2)_\infty}\left\{q(a-1)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a\end{matrix}\;;\;q,-\frac{q}{a}\right]\right.\\
&\qquad\left.+a(q-b)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a\end{matrix}\;;\;q,-\frac{q^2}{a}\right]\right\}\\
&=\left(cq,\frac{cq^2}{a^2};q^2\right)_\infty {}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2/c \\ & q, & q^2/c\end{matrix}\;;\;q^2,(b^2)/(a^2q^2)\right]\\
&\quad+\frac{(b-q)}{q(1-q)(c-q)}\left(c,\frac{cq}{a^2};q^2\right)_\infty\\
&\quad\times{}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & a^2q/c \\ & q^3, & q^3/c\end{matrix}\;;\;q^2,\frac{b^2}{a^2q^2}\right].
\end{aligned}\tag{18}$$

Proof. Taking $d\to(cq)/a$, $z\to -q/a$ in (17), we get that

$$\begin{aligned}
&q(a-1)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a\end{matrix}\;;\;q,-\frac{q}{a}\right]+a(q-b)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a\end{matrix}\;;\;q,-\frac{q^2}{a}\right]\\
&=(aq-b)\frac{(q,b/a;q)_\infty}{(b,q/a;q)_\infty}\sum_{k=0}^{\infty}\frac{(c;q)_{-k}(q/b;q)_k}{(cq/a;q)_{-k}(q;q)_k}\left(-\frac{b}{q^2}\right)^k\\
&\quad{}_2\phi_1\left[\begin{matrix} a, & cq^{-k} \\ & cq^{1-k}/a\end{matrix}\;;\;q,-\frac{q}{a}\right].
\end{aligned}\tag{19}$$

In the second sum on the right side of above equation, applying the Bailey-Daum summation formula [**6,** (1.8.1)], we get:

$${}_2\phi_1\left[\begin{matrix} a, & b \\ & aq/b\end{matrix}\;;\;q,-\frac{q}{b}\right]=\frac{(-q;q)_\infty(aq,aq^2/b^2;\;q^2)_\infty}{(aq/b,-q/b;\;q)_\infty}.\tag{20}$$

We obtain that

$$\begin{aligned}
&q(a-1)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a\end{matrix}\;;\;q,-\frac{q}{a}\right]+a(q-b)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a\end{matrix}\;;\;q,-\frac{q^2}{a}\right]\\
&=(aq-b)\frac{(q,b/a,-q;q)_\infty}{(b,q/a;q)_\infty}\sum_{k=0}^{\infty}\frac{(q/b;q)_k(c;q)_{-k}}{(q;q)_k(cq/a;q)_{-k}}
\end{aligned}\tag{21}$$

$$\times \left(-\frac{b}{q^2}\right)^k \frac{(cq^{1-k}, cq^{2-k}/a^2; q^2)_\infty}{(cq^{1-k}/a, -q/a; q)_\infty}.$$

With the help of (1), after some simplification, we have

(22)
$$q(a-1)_2\psi_2\begin{bmatrix} a, & c & ; q, -\dfrac{q}{a} \\ b, & cq/a & \end{bmatrix} + a(q-b)_2\psi_2\begin{bmatrix} a, & c & ;\ q, -\dfrac{q^2}{a} \\ b, & cq/a & \end{bmatrix}$$
$$= (aq-b)\frac{(q, b/a, -q; q)_\infty}{(b, q/a, -q/a, cq/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(q/b; q)_k}{(q, q/c; q)_k}$$
$$\left(\frac{b}{cq}\right)^k q^{\binom{k}{2}} \left(cq^{1-k}, \frac{cq^{2-k}}{a^2}; q^2\right)_\infty.$$

Since

$$\left(\frac{cq^{1-k}}{a};\ q\right)_\infty = \left(\frac{cq^{1-k}}{a};\ q\right)_k \left(\frac{cq}{a};\ q\right)_\infty,$$
$$\left(\frac{cq^{1-k}}{a};\ q\right)_k = (-1)^k \left(\frac{c}{a}\right)^k q^{(-k^2+k)/2} \left(\frac{a}{c};\ q\right)_k,$$
$$(cq^{1-2k};\ q^2)_\infty = (-1)^k c^k q^{-k^2} \left(\frac{q}{c};\ q^2\right)_k (cq;\ q^2)_\infty,$$
$$(cq^{-2k};\ q^2)_\infty = (-1)^k c^k q^{-k^2-k} \left(\frac{q^2}{c};\ q^2\right)_k (c;\ q^2)_\infty,$$
$$\left(\frac{cq^{2-2k}}{a^2};\ q^2\right)_\infty = (-1)^k \left(\frac{c^k}{a^{2k}}\right) q^{-k^2+k} \left(\frac{a^2}{c};\ q^2\right)_k \left(\frac{q^2c}{a^2};\ q^2\right)_\infty,$$
$$\left(\frac{cq^{1-2k}}{a^2};\ q^2\right)_\infty = (-1)^k \left(\frac{c^k}{a^{2k}}\right) q^{-k^2} \left(\frac{a^2q}{c};\ q^2\right)_k \left(\frac{qc}{a^2};\ q^2\right)_\infty,$$

Hence, we have

(23)
$$q(a-1)_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a \end{matrix}\ ;\ q, -\frac{q}{a}\right]$$
$$+ a(q-b)_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a \end{matrix}\ ;\ q, -\frac{q^2}{a}\right]$$
$$= (aq-b)\frac{(q, b/a, -q; q)_\infty}{(b, q/a, -q/a, cq/a; q)_\infty}$$

$$\times\left[\left(cq, \frac{cq^2}{a^2};\ q^2\right)_\infty \sum_{k=0}^{\infty} \frac{(q/b;q)_{2k}}{(q,q/c;q)_{2k}}\right.$$
$$\times\left(\frac{b}{aq}\right)^{2k}\left(\frac{q}{c}, \frac{a^2}{c};\ q^2\right)_k$$
$$+\left(c, \frac{cq}{a^2};\ q^2\right)_\infty \sum_{k=0}^{\infty} \frac{(q/b;q)_{2k+1}}{(q,q/c;q)_{2k+1}}$$
$$\left.\times\left(\frac{b}{cq}\right)\left(\frac{b}{aq}\right)^{2k}\left(\frac{q^2}{c}, \frac{a^2q}{c}; q^2\right)_k\right]$$
$$= \frac{(aq-b)(b/a;q)_\infty(q^2,cq,cq^2/a^2;q^2)_\infty}{(q/a,-q/a,b,cq/a;q)_\infty}$$
$${}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2/c \\ & q, & q^2/c \end{matrix};\ q^2, \frac{b^2}{a^2q^2}\right]$$
$$+\frac{(aq-b)(b-q)}{q(1-q)(c-q)}\frac{(b/a;q)_\infty(q^2,c,cq/a^2;q^2)_\infty}{(q/a,-q/a,b,cq/a;q)_\infty}$$
$$\times {}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & a^2q/c \\ & q^3, & q^3/c \end{matrix};\ q^2, \frac{b^2}{a^2q^2}\right].$$

The proof of the theorem has been completed. □

This theorem has the following corollaries:

(a) Taking $b \to q$ in Theorem 3.1, we obtain the Bailey-Daum summation formula [**6,** (1.8.1)].

(b) Since the power of a is two on the right-hand sides of equation (18), we can get the following symmetric transformations:

$$\frac{(cq/a;q)_\infty}{(aq-b)(b/a;q)_\infty}\left\{(a-1)q\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a \end{matrix};\ q, -\frac{q}{a}\right]\right.$$
$$\left.+a(q-b)\,{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a \end{matrix};\ q, -\frac{q^2}{a}\right]\right\}$$
$$= \frac{(-cq/a;q)_\infty}{(aq+b)(-b/a;q)_\infty}\{(a+1)q$$
$$\times\, {}_2\psi_2\left[\begin{matrix} -a, & c \\ b, & -cq/a \end{matrix};\ q, q/a\right]$$

$$+a(q-b){}_2\psi_2\left[\begin{matrix}-a, & c \\ b, & -cq/a\end{matrix}\,;\,q,\frac{q^2}{a}\right]\Bigg\}.$$

If we take $b \to q$ in the above equation, then

$$\begin{aligned}\frac{(cq/a;q)_\infty}{(b/a;q)_\infty}{}_2\phi_1&\left[\begin{matrix}a, & c \\ & cq/a\end{matrix}\,;\,q,-\frac{q}{a}\right]\\ &= \frac{(-cq/a;q)_\infty}{(-b/a;q)_\infty}{}_2\phi_1\left[\begin{matrix}-a, & c \\ & -cq/a\end{matrix}\,;\,q,\frac{q}{a}\right].\end{aligned}$$

This transformation is a special case of the iterate of Heine's ${}_2\phi_1$ transformation (see [**6,** equation (III.2)])

$$(24)\qquad {}_2\phi_1\left[\begin{matrix}a, & b \\ & c\end{matrix}\,;\,q,z\right] = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left[\begin{matrix}abz/c, & b \\ & bz\end{matrix}\,;q,c/b\right].$$

(c) Taking $c \to 0$, the right side of (18) equals

$$(25)\qquad \begin{aligned}&\frac{(aq-b)(b/a;q)_\infty(q^2;q^2)_\infty}{b(q/a,-q/a,b;q)_\infty}{}_2\phi_1\left[\begin{matrix}q/b, & q^2/b \\ & q\end{matrix}\,;\,q^2,\frac{b^2}{q^4}\right]\\ &\quad+\frac{(aq-b)(b/a;q)_\infty(q^2;q^2)_\infty}{b(q/a,-q/a,b;q)_\infty}\frac{(b-q)}{(1-q)(-q^2)}\\ &\qquad\times{}_2\phi_1\left[\begin{matrix}q^2/b, & q^3/b \\ & q^3\end{matrix}\,;\,q^2,\frac{b^2}{q^4}\right].\end{aligned}$$

Applying Lemma 2.1, we have

$$(26)\qquad {}_2\phi_1\left[\begin{matrix}q/b,q^2/b \\ q\end{matrix}\,;\,q^2,b^2/q^4\right] = \frac{(b,b/q;q^2)_\infty}{(q,b^2/q^2;q^2)_\infty}\left\{\frac{q(q^2-b)(q+1)}{(q^4-b^2)}\right\},$$

$$(27)\qquad {}_2\phi_1\left[\begin{matrix}q^2/b, & q^3/b \\ & q^3\end{matrix}\,;\,q^2,\frac{b^2}{q^4}\right] = \frac{(b,bq;q^2)_\infty}{(q^3,b^2/q^2;q^2)_\infty}\left\{\frac{q^2(q^2-b)(q+1)}{(q^4-b^2)}\right\}.$$

Hence, the right side of (18) is equal to

$$(28)\qquad \begin{aligned}&\frac{(aq-b)(b/a;q)_\infty(q^2;q^2)_\infty}{b(q/a,-q/a,b;q)_\infty}\frac{(b,b/q;q^2)_\infty}{(q,b^2/q^2;\ q^2)_\infty}\\ &\quad\times\left[\left\{\frac{q(q^2-b)(q+1)}{(q^4-b^2)}\right\}+\frac{1}{q}\left\{\frac{q^2(q^2-b)(q+1)}{(q^4-b^2)}\right\}\right]\end{aligned}$$

$$= 2\frac{(aq-b)(b/a;q)_\infty(q^2;q^2)_\infty}{b(q/a,-q/a,b;q)_\infty}\cdot\frac{(b,b/q;q^2)_\infty}{(q,b^2/q^2;q^2)_\infty}\frac{q(q+1)}{(q^2+b)}.$$

Then, from (18), we obtain the following special case of Ramanujan's ${}_1\psi_1$ summation:

$${}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}\,;\,q,-\frac{q^2}{a}\right]=\frac{(b/a,q,-1/q,-q^2;q)_\infty}{(b,q/a,-q^2/a,-b/q^2;q)_\infty}.$$

Theorem 3.2. *Let* $|b/(aq)|<1$ *and* $|q/a|<1$. *Then*

$$\begin{aligned}
&a(q-b)\frac{(q/a,-q/a,b,cq/a;q)_\infty}{(b/a;q)_\infty}{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a \end{matrix}\,;\,q,-\frac{q^2}{a}\right]\\
&= \left(q^2,cq,\frac{cq^2}{a^2};q^2\right)_\infty\\
&\quad\times\left\{(aq-b){}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2/c \\ & q, & q^2/c \end{matrix}\,;\,q^2,\frac{b^2}{a^2q^2}\right]\right.\\
&\qquad\left.-q(a-1)\,{}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2/c \\ & q, & q^2/c \end{matrix}\,;\,q^2,\frac{b^2}{a^2}\right]\right\}\\
&\quad+\frac{(b-q)(q^2,c,cq/a^2;q^2)_\infty}{(1-q)}\\
&\quad\times\left\{\frac{(aq-b)}{q(c-q)}{}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & a^2q/c \\ & q^3, & q^3/c \end{matrix}\,;\,q^2,\frac{b^2}{a^2q^2}\right]\right.\\
&\qquad\left.-\frac{q(a-1)}{(c-q)}{}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & a^2q/c \\ & q^3, & q^3/c \end{matrix}\,;q^2,b^2/a^2\right]\right\}.
\end{aligned}\tag{29}$$

Proof. Substituting the main result of **[12]**:

$$\begin{aligned}
{}_2\psi_2\left[\begin{matrix} a, & c \\ b, & cq/a \end{matrix}\,;\,q,-\frac{q}{a}\right] &= \frac{(b/a;q)_\infty(q^2,cq,cq^2/a^2;q^2)_\infty}{(q/a,-q/a,b,cq/a;q)_\infty}\\
&\quad\times{}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2/c \\ & q, & q^2/c \end{matrix}\,;\,q^2,b^2/a^2\right]\\
&\quad+\frac{(b/a;q)_\infty(q^2,cq,cq^2/a^2;q^2)_\infty}{(q/a,-q/a,b,cq/a;q)_\infty}\,\frac{(b-q)}{(1-q)(c-q)}\\
&\quad\times{}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & a^2q/c \\ & q^3, & q^3/c \end{matrix}\,;\,q^2,\frac{b^2}{a^2}\right]
\end{aligned}\tag{30}$$

into Theorem 3.1, the proof can be completed. ☐

This theorem includes the following special cases:

(a) Taking $c \to a$ in Theorem 3.2, we have

$$\begin{aligned}
&\frac{a(q-b)(q/a,-q/a,b,q;q)_\infty}{(b/a;q)_\infty(q^2;q^2)_\infty}{}_2\phi_1\left[\begin{matrix} a, & a \\ & b \end{matrix}\ ;\ q, -\frac{q^2}{a}\right] \\
&\quad = \left(aq,\frac{q^2}{a};q^2\right)_\infty \left\{(aq-b){}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a \\ & q, & q^2/a \end{matrix}\ ;\ q^2, \frac{b^2}{a^2q^2}\right]\right. \\
&\qquad \left. -q(a-1){}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a \\ & q, & q^2/a \end{matrix}\ ;\ q^2, \frac{b^2}{a^2}\right]\right\} \\
&\quad + \frac{(b-q)(a,q/a;q^2)_\infty}{(1-q)}\left\{\frac{(aq-b)}{q(a-q)}\right. \\
&\quad \times {}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & aq \\ & q^3, & q^3/a \end{matrix}\ ;\ q^2, \frac{b^2}{a^2q^2}\right] - \frac{q(a-1)}{(a-q)} \\
&\quad \left. \times {}_3\phi_2\left[\begin{matrix} q^2/b, & q^3/b, & aq \\ & q^3, & q^3/a \end{matrix}\ ;\ q^2, \frac{b^2}{a^2}\right]\right\},
\end{aligned}$$

where $|q^2/a| < 1$ and $|b/(aq)| < 1$.

(b) Taking $c \to 1$ in Theorem 3.2, we have

$$\begin{aligned}
a(q-b)\frac{(b,q/a;q)_\infty}{(b/a,q;q)_\infty}{}_2\phi_1\left[\begin{matrix} q/b, & a \\ & q/a \end{matrix}\ ;\ q - \frac{b}{qa}\right] \\
= (aq-b){}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2 \\ & q, & q^2 \end{matrix}\ ;\ q^2, \frac{b^2}{a^2q^2}\right] \\
- q(a-1){}_3\phi_2\left[\begin{matrix} q/b, & q^2/b, & a^2 \\ & q, & q^2 \end{matrix}\ ;\ q^2, \frac{b^2}{a^2}\right],
\end{aligned}$$

where $|b/aq| < 1$.

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Department of Mathematics, Luoyang Teachers' College, Luoyang 471022, P.R. China and College of Mathematics and Information Science, Henan Normal University, Xinxiang 450001, P.R. China
Email address: zhzhzhang-yang@163.com

College of Mathematics and Information Science, Henan Normal University, Xinxiang 450001, P.R. China
Email address: jiawei163jzy@163.com