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4-DISSECTIONS AND 8-DISSECTIONS FOR SOME INFINITE PRODUCTS

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ABSTRACT. In this paper, we establish 4- and 8-dissections for some infinite products. In particular, we generalize Hirschhorn's formulas for 8-dissections of a continued fraction of Gordon and its reciprocal. Our results also imply a theorem on the periodicity of signs of the coefficients of an infinite product given by Chan and Yesilyurt.

1. Introduction and main results. The aim of this paper is to establish 4- and 8-dissections for some infinite products which implies some results discovered by Hirschhorn [**6**] and Chan and Yesilyurt [**4**].

Throughout this paper, let q be a non-zero complex number of modulus less than 1. We use the standard notation

$$[z;q]_\infty = (z;q)_\infty (q/z;q)_\infty, \quad z \neq 0,$$

where

$$(a;q)_\infty = \sum_{n=0}^{\infty} (1-aq^n),$$

and, as usual,

$$[z_1, z_2, \ldots, z_n; q]_\infty = \prod_{i=1}^{n} [z_i; q]_\infty .$$

It is easy to verify that

$$[z, q/z; q^2]_\infty = [z;q]_\infty, \tag{1.1}$$

$$[-z, z; q]_\infty = [z^2; q^2]_\infty, \tag{1.2}$$

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$$[-1;q]_\infty = \frac{2}{[q;q^2]_\infty} \tag{1.3}$$

and

$$[z^{-1};q]_\infty = -z^{-1}[z;q]_\infty. \tag{1.4}$$

The m-dissection of the power series

$$P(q) = \sum_{n=0}^{\infty} a_n q^n$$

is the presentation of $P(q)$ as

$$P(q) = P_0(q) + P_1(q) + \cdots + P_{m-1}(q), \tag{1.5}$$

where

$$P_k(q) = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}.$$

Gordon's continued fraction is given by

$$G(q) = 1 + q + \frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^6}{1+q^7+}\cdots = \frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}.$$

This identity was established by Gordon [**6**]. Hirschhorn [**8**] discovered the 8-dissections of $G(q)$ and its reciprocal, thereby demonstrating periodicity of the sign of the coefficients in the expansions of $G(q)$ and its reciprocal, and in particular that certain coefficients are zero, a phenomenon first observed and shown by Richmond and Szekeres [**12**]. Alladi and Gordon [**1**], Andrews and Bressoud [**3**], Hirschhorn [**7**] and Chan and Yesilyurt [**4**] generalized these themes. Recently, the authors [**16**] proved Hirschhorn's results by an iterative method.

Hirschhorn's results on the 8-dissections of $G(q)$ and $G^{-1}(q)$ can be stated as follows.

Theorem 1.1 (Hirschhorn [**8**]). *We have*

$$\begin{aligned} G(q) = {} & \frac{[-q^{24},-q^{32};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty} + \frac{q[-q^{16},-q^{24};q^{64}]_\infty}{[q^8,q^8;q^{32}]_\infty[q^{32};q^{64}]_\infty} \\ & + \frac{q^2[-q^{16},-q^{24};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty} - \frac{q^{12}[-q^8,-1;q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty} \end{aligned}$$

$$-\frac{q^5[-q^8,-q^{16};q^{64}]_\infty}{[q^8,q^8;q^{32}]_\infty[q^{32};q^{64}]_\infty}-\frac{q^6[-q^8,-q^{16};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty}, \tag{1.6}$$

and

$$\begin{aligned} G^{-1}(q) = {} & \frac{[-q^{16},-q^{24};q^{64}]_\infty}{[q^8,q^8;q^{32}]_\infty[q^{32};q^{64}]_\infty}-\frac{q[-q^{16},-q^{24};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty} \\ & +\frac{q^3[-q^8,-q^{32};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty}-\frac{q^4[-q^8,-q^{16};q^{64}]_\infty}{[q^8,q^8;q^{32}]_\infty[q^{32};q^{64}]_\infty} \\ & +\frac{q^5[-q^8,-q^{16};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty}-\frac{q^7[-1,-q^{24};q^{64}]_\infty}{[q^8,q^{16};q^{32}]_\infty[q^{32};q^{64}]_\infty}. \end{aligned} \tag{1.7}$$

As a corollary of Theorem 1.2, Hirschhorn **[8]** discovered that

Corollary 1.2. *Let*

$$G(q)=\sum_{n=0}^\infty a_nq^n \quad \textit{and} \quad G^{-1}(q)=\sum_{n=0}^\infty b_nq^n.$$

We have, for $n\ge 0$,

$$\begin{array}{rrrr} a_{8n}>0, & a_{8n+1}>0, & a_{8n+2}>0, & a_{8n+3}=0, \\ a_{8n+12}<0, & a_{8n+5}<0, & a_{8n+6}<0, & a_{8n+7}=0, \\ b_{8n}>0, & b_{8n+1}<0, & b_{8n+2}=0, & b_{8n+3}>0, \\ a_{8n+4}<0, & a_{8n+5}>0, & a_{8n+6}=0, & a_{8n+7}<0. \end{array}$$

Recently, Chan and Yesilyurt **[4]** generalized Corollary 1.2. They established the following theorem.

Theorem 1.3. *Suppose m is divisible by 8 and $\gcd(m,r)=1$. Let*

$$\frac{[q^{3r};q^m]_\infty}{[q^r;q^m]_\infty}=\sum_{n=0}^\infty c_nq^n, \quad 3r<m, \tag{1.8}$$

and

$$\frac{[q^{3r-m};q^m]_\infty}{[q^r;q^m]_\infty}=\sum_{n=0}^\infty d_nq^n, \quad m/3<r<m/2. \tag{1.9}$$

We have, for $n \geq 0$,

$$
\begin{aligned}
&c_{8n} > 0, \qquad c_{8n+r} > 0, \qquad c_{8n+2r} > 0,\\
&c_{8n+m-2r} < 0, \qquad\qquad c_{8n+m-3r} < 0,\\
&c_{8n+m+4r} < 0 \quad \text{if } m > 8r\\
&c_{8n+2m-4r} < 0 \quad \text{if } m < 8r\\
&c_n = 0 \qquad\qquad \text{if } n \equiv 3r \pmod 4;\\
&d_{8n} > 0, \qquad a_{8n+m-r} > 0, \quad a_{8n+r} > 0,\\
&d_{8n+4r-m} < 0, \qquad\qquad a_{8n+3r-m} < 0,\\
&d_{8n+5r-m} < 0 \quad \text{if } 3m > 8r,\\
&d_{8n+2m-3r} < 0 \quad \text{if } 3m < 8r,\\
&d_n = 0 \qquad\qquad \text{if } n \equiv 2r \pmod 4.
\end{aligned}
$$

In this paper, we establish 4- and 8-dissections of some infinite products, which can be stated as follows.

Theorem 1.4. *Let* $m > 0$ *be divisible by* 4, *let* $u > 0$ *and* $r > 0$ *be odd integers and* $u - r \equiv 2 \pmod 4$. *We have*

$$\frac{[q^u; q^m]_\infty}{[q^r; q^m]_\infty} = A_0(q^4) + qA_1(q^4) + q^2A_2(q^4) + q^3A_3(q^4), \tag{1.10}$$

where $A_0(q)$, $A_1(q)$, $A_2(q)$ *and* $A_3(q)$ *are defined by*

$$A_0(q) = \frac{[q^{(u+r)/4}; q^{m/2}]_\infty [q^{(m-u+3r)/4}, q^{(m+u+r)/4}; q^m]_\infty}{[q^{m/4}; q^{m/2}]_\infty [q^{m/2}, q^r; q^m]_\infty}, \tag{1.11}$$

$$A_2(q) = q^{(r-1)/2} \frac{[q^{(u+r)/4}; q^{m/2}]_\infty [q^{(m-u-r)/4}, q^{(m+u-3r)/4}; q^m]_\infty}{[q^{m/4}; q^{m/2}]_\infty [q^{m/2}, q^r; q^m]_\infty}, \tag{1.12}$$

$$A_i(q) = q^{(3r-i)/4} \frac{[q^{(m-u-r)/4}; q^{m/2}]_\infty [q^{(u-3r)/4}, q^{(2m-u-r)/4}; q^m]_\infty}{[q^{m/4}; q^{m/2}]_\infty [q^{m/2}, q^r; q^m]_\infty}, \tag{1.13}$$

$$\text{if } r \equiv -i \pmod 4 \text{ for } i = 1,\ 3,\ \text{then}$$

(1.14)
$$A_i(q) = q^{(r-i)/4} \frac{[q^{(m-u-r)/4}; q^{m/2}]_\infty [q^{(u+r)/4}, q^{(2m-u+3r)/4}; q^m]_\infty}{[q^{m/4}; q^{m/2}]_\infty [q^{m/2}, q^r; q^m]_\infty},$$
$$\text{if } r \equiv i \pmod 4 \text{ for } i = 1,\ 3.$$

Theorem 1.5. *Let m and r be positive integers and $r \equiv 1 \pmod 2$, $m \equiv 0 \pmod 8$. We have*

$$\frac{[q^{3r}; q^m]_\infty}{[q^r; q^m]_\infty} = \sum_{i=0}^{7} q^i B_i(q^8), \tag{1.15}$$

where $B_i(q)$ are given by

$$B_0(q) = \frac{[-q^{3m/8}, -q^{3m/8+r}; q^m]_\infty}{[q^{m/8}, q^{m/4}; q^{m/2}]_\infty [q^{m/2}; q^m]_\infty}, \tag{1.16}$$

$$B_4(q) = -q^{m/8+(r-1)/2} \frac{[-q^{m/8-r}, -q^{m/8}; q^m]_\infty}{[q^{m/8}, q^{m/4}; q^{m/2}]_\infty [q^{m/2}; q^m]_\infty}, \tag{1.17}$$

$$B_{2i}(q) = q^{(r-i)/4} \frac{[q^{-3m/8}, -q^{3m/8-r}; q^m]_\infty}{[q^{m/8}, q^{m/4}; q^{m/2}]_\infty [q^{m/2}; q^m]_\infty}, \tag{1.18}$$
$$\textit{if } r \equiv i \pmod 4 \text{ for } i = 1,\ 3,\ \textit{then}$$

$$B_{2i}(q) = -q^{(m-2r-2i)/8} \frac{[q^{-m/8}, -q^{m/8+r}; q^m]_\infty}{[q^{m/8}, q^{m/4}; q^{m/2}]_\infty [q^{m/2}; q^m]_\infty}, \tag{1.19}$$
$$\textit{if } r \equiv -i \pmod 4 \text{ for } i = 1,\ 3,\ \textit{then}$$

$$B_i(q) = B_{4+i}(q) = 0, \quad \textit{if } r \equiv -i \pmod 4 \text{ for } i = 1,\ 3,\ \textit{then} \tag{1.20}$$

$$B_i(q) = q^{(r-i)/8} \frac{[-q^{m/4}, -q^{m/2-r}; q^m]_\infty}{[q^{m/8}, q^{m/8}; q^{m/2}]_\infty [q^{m/2}; q^m]_\infty}, \tag{1.21}$$
$$\textit{if } r \equiv i \pmod 8 \text{ for } i = 1,\ 3,\ 5,\ 7,\ \textit{then}$$

$$B_i(q) = -q^{(m-3r-i)/8} \frac{[-q^{m/4}, -q^r; q^m]_\infty}{[q^{m/8}, q^{m/8}; q^{m/2}]_\infty [q^{m/2}; q^m]_\infty}, \tag{1.22}$$
$$\textit{if } r \equiv i + 4 \pmod 8 \text{ for } i = 1,\ 3,\ 5,\ 7.$$

Theorem 1.6. *Let m, r be positive integers and $r \equiv 1 \pmod 2$, $m \equiv 0 \pmod 8$. We have*

$$\frac{[q^{3r-m}; q^m]_\infty}{[q^r; q^m]_\infty} = \sum_{i=0}^{7} q^i C_i(q^8), \tag{1.23}$$

where $C_i(q)$ are given by

$$C_0(q)=\frac{[-q^{m/4},-q^{m-r};q^m]_\infty}{[q^{m/2};q^m]_\infty[q^{m/8};q^{m/4}]_\infty}, \tag{1.24}$$

$$C_4(q)=-q^{(r-1)/2-(m/8)}\frac{[-q^{m/2-r},-q^{m/4};q^m]_\infty}{[q^{m/2};q^m]_\infty[q^{m/8};q^{m/4}]_\infty}, \tag{1.25}$$

$$C_2(q)=C_6(q)=0, \tag{1.26}$$

$$C_i(q)=q^{(m-r-i)/8}\frac{[-q^{m/8},-q^{r-m/8};q^m]_\infty}{[q^{m/8},q^{m/4};q^{m/2}]_\infty[q^{m/2};q^m]_\infty}, \tag{1.27}$$

if $r\equiv -i \pmod 8$ for $i=1,\ 3,\ 5,\ 7,$ *then*

$$C_i(q)=-q^{(3r-m-i)/8}\frac{[-q^{5m/8-r},-q^{3m/8};q^m]_\infty}{[q^{m/8},q^{m/4};q^{m/2}]_\infty[q^{m/2};q^m]_\infty}, \tag{1.28}$$

if $r\equiv 4-i \pmod 8$ for $i=1,\ 3,\ 5,\ 7,$ *then*

$$C_i(q)=q^{(r-i)/8}\frac{[-q^{m/8},-q^{m/8+r};q^m]_\infty}{[q^{m/8},q^{m/4};q^{m/2}]_\infty[q^{m/2};q^m]_\infty}, \tag{1.29}$$

if $r\equiv i \pmod 8$ for $i=1,\ 3,\ 5,\ 7,$ *then*

$$C_i(q)=-q^{(5r-m-i)/8}\frac{[-q^{3m/8-r},-q^{3m/8};q^m]_\infty}{[q^{m/8},q^{m/4};q^{m/2}]_\infty[q^{m/2};q^m]_\infty}, \tag{1.30}$$

if $r\equiv 4+i \pmod 8$ for $i=1,\ 3,\ 5,\ 7.$

Taking $r=1$ and $m=8$ in Theorem 1.5, we deduce the formula (1.6). Also, setting $r=3$ and $m=8$ in Theorem 1.6, we obtain the formula (1.7). At last, it follows from Theorems 1.5 and 1.6 that Theorem 1.3 holds.

2. Two lemmas. In order to prove our main results, we need the following two lemmas which are proved using the following identity (see **[5, 14, 15]**)

$$\left[\chi\lambda,\frac{\chi}{\lambda},\mu\nu,\frac{\mu}{\nu};q\right]_\infty-\left[\chi\nu,\frac{\chi}{\nu},\lambda\mu,\frac{\mu}{\lambda};q\right]_\infty=\frac{\mu}{\lambda}\left[\chi\mu,\frac{\chi}{\mu},\lambda\nu,\frac{\lambda}{\nu};q\right]_\infty, \tag{2.1}$$

where χ, λ, μ and ν are non-zero complex numbers.

Lemma 2.1. *Let $m > 0$ be an even integer and $r > 0$, $u > 0$ odd integers. We have*

$$\frac{[q^u;q^m]_\infty}{[q^r;q^m]_\infty} = \alpha(q^2) + q\beta(q^2), \tag{2.2}$$

where $\alpha(q^2)$ and $\beta(q^2)$ are defined by

$$\alpha(q^2) = \frac{[q^{u+r}, q^{m+r-u}; q^{2m}]_\infty}{[q^m, q^{2r}; q^{2m}]_\infty}, \tag{2.3}$$

$$\beta(q^2) = q^{r-1}\frac{[q^{u-r}, q^{m-u-r}; q^{2m}]_\infty}{[q^m, q^{2r}; q^{2m}]_\infty}. \tag{2.4}$$

Proof. In view of (1.2) and (2.2), we see that

$$\begin{aligned}\alpha(q^2) &= \frac{1}{2}\left(\frac{[q^u;q^m]_\infty}{[q^r;q^m]_\infty} + \frac{[-q^u;q^m]_\infty}{[-q^r;q^m]_\infty}\right)\\ &= \frac{1}{2}\frac{[q^u,-q^r;q^m]_\infty + [-q^u,q^r;q^m]_\infty}{[q^{2r};q^m]_\infty}. \end{aligned} \tag{2.5}$$

Setting $\chi \to q^{(3u-r)/4}$, $\lambda \to q^{(u+r)/4}$, $\mu \to -q^{(u+r)/4}$, $\nu \to q^{(u-3r)/4}$ and $q \to q^m$ in (2.1), then dividing $[q^{u-r};q^{2m}]_\infty$ on both sides and utilizing (1.1), (1.2) and (1.3), we deduce that

$$[q^u,-q^r;q^m]_\infty - 2\frac{[q^{u+r}, q^{m-u+r}; q^{2m}]_\infty}{[q^m;q^{2m}]_\infty} = -[-q^u,q^r;q^m]. \tag{2.6}$$

Employing (2.5) and (2.6), we easily arrive at (2.3).

On the other hand, by (1.1) and (1.2), we have

$$\begin{aligned}\beta(q^2) &= \frac{1}{2q}\left(\frac{[q^u;q^m]_\infty}{[q^r;q^m]_\infty} - \frac{[-q^u;q^m]_\infty}{[-q^r;q^m]_\infty}\right)\\ &= \frac{1}{2q}\frac{[q^u,-q^r;q^m]_\infty - [-q^u,q^r;q^m]_\infty}{[q^{2r};q^m]_\infty}. \end{aligned} \tag{2.7}$$

Taking $\chi \to q^{(3u+r)/4}$, $\lambda \to q^{(u-r)/4}$, $\mu \to q^{(u+3r)/4}$, $\nu \to -q^{(u-r)/4}$ and $q \to q^m$ in (2.1), then dividing $[q^{u+r};q^{2m}]_\infty$ on both sides and employing (1.1), (1.2) and (1.3), we see that

$$[q^u,-q^r;q^m]_\infty - [-q^u,q^r;q^m]_\infty = 2q^r\frac{[q^{u-r}, q^{m-u-r}; q^{2m}]_\infty}{[q^m;q^{2m}]_\infty}. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain (2.4), and this completes the proof. □

Lemma 2.2. *Let m be a positive even integer and r, u positive odd integers. We have*

$$[q^r, q^u; q^m]_\infty = \alpha(q^2) + q\beta(q^2), \tag{2.9}$$

where $\alpha(q^2)$ and $\beta(q^2)$ are defined by

$$\alpha(q^2) = \frac{[-q^{u+r}, -q^{m-u+r}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty}, \tag{2.10}$$

$$\beta(q^2) = -q^{r-1}\frac{[-q^{u-r}, -q^{m-u-r}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty}. \tag{2.11}$$

Proof. It is easy to see that

$$\alpha(q^2) = \frac{1}{2}\left([q^u, q^r; q^m]_\infty + [-q^u, -q^r; q^m]_\infty\right). \tag{2.12}$$

Setting $\chi \to q^{(u+r)/2}$, $\lambda \to q^{(u-r)/2}$, $\mu \to -q^{(u-r)/2}$, $\nu \to i$ and $q \to q^m$ in (2.1), then dividing $[-q^{u-r}; q^{2m}]_\infty$ on both sides and using (1.1), (1.2) and (1.3), we see that

$$[q^u, q^r; q^m]_\infty - 2\frac{[-q^{u+r}, -q^{m-u+r}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty} = -[-q^u, -q^r; q^m]. \tag{2.13}$$

It follows from (2.12) and (2.13) that (2.10) holds.

Similarly, we obtain:

$$\beta(q^2) = \frac{1}{2q}\left([q^u, q^r; q^m]_\infty - [-q^u, -q^r; q^m]_\infty\right). \tag{2.14}$$

Putting $\chi \to q^{(u+r)/2}$, $\lambda \to q^{(u-r)/2}$, $\mu \to -q^{(u+r)/2}$, $\nu \to i$ and $q \to q^m$ in (2.1), then dividing $[-q^{u+r}; q^{2m}]_\infty$ on both sides and employing (1.1), (1.2) and (1.3), we deduce that

$$[q^u, q^r; q^m]_\infty - [-q^u, -q^r; q^m]_\infty = -2q^r\frac{[-q^{u-r}, -q^{m-u-r}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty}. \tag{2.15}$$

Employing (2.14) and (2.15), we derive (2.11), and this proof is complete. □

3. Proof of Theorem 1.4. In this section, we provide a proof of Theorem 1.4 by employing Lemma 2.1.

Proof. Taking $u \to (m+r-u)/2$ in (2.2), we have

$$\frac{[q^{(m+r-u)/2};q^m]_\infty}{[q^r;q^m]_\infty} = \frac{[q^{(m-u+3r)/2},q^{(m+u+r)/2};q^{2m}]_\infty + q^r[q^{(m-u-r)/2},q^{(m+u-3r)/2};q^{2m}]_\infty}{[q^m,q^{2r};q^{2m}]_\infty}. \tag{3.1}$$

The congruence $u-r\equiv 2 \pmod 4$ implies that $u+r\equiv 0 \pmod 4$ and $u-3r\equiv 0 \pmod 4$. It follows from (1.10), (2.2) and (3.1) that

$$\begin{aligned} A_0(q^4)+q^2A_2(q^4) &= \frac{[q^{u+r};q^{2m}]_\infty}{[q^m;q^{2m}]_\infty}\frac{[q^{m+r-u};q^{2m}]_\infty}{[q^{2r};q^{2m}]_\infty} \\ &= \frac{[q^{u+r};q^{2m}]_\infty}{[q^m;q^{2m}]_\infty[q^{2m},q^{4r};q^{4m}]_\infty}\left([q^{m-u+3r},q^{m+u+r};q^{4m}]_\infty\right. \\ &\qquad \left.+q^{2r}[q^{m-u-r},q^{m+u-3r};q^{4m}]_\infty\right), \end{aligned}$$

which yields (1.11) and (1.12).

Also, taking $u\to(u-r)/2$ in (2.2), we have

$$\begin{aligned} \frac{[q^{(u-r)/2};q^m]_\infty}{[q^r;q^m]_\infty} &= \frac{1}{[q^m,q^{2r};q^{2m}]_\infty}\left([q^{(u+r)/2},q^{(2m-u+3r)/2};q^{2m}]_\infty\right. \\ &\qquad \left.+q^r[q^{(u-3r)/2},q^{(2m-u-r)/2};q^{2m}]_\infty\right). \end{aligned} \tag{3.2}$$

It follows from (1.10), (2.2) and (3.2) that

$$\begin{aligned} qA_1(q^4)+q^3A_3(q^4) &= q^r\frac{[q^{m-u-r};q^{2m}]_\infty}{[q^m;q^{2m}]_\infty}\frac{[q^{u-r};q^{2m}]_\infty}{[q^{2r};q^{2m}]_\infty} \\ &= q^r\frac{[q^{m-u-r};q^{2m}]_\infty}{[q^m;q^{2m}]_\infty[q^{2m},q^{4r};q^{4m}]_\infty}[q^{u+r},q^{2m-u+3r};q^{4m}]_\infty \\ &\quad + q^{3r}\frac{[q^{m-u-r};q^{2m}]_\infty}{[q^m;q^{2m}]_\infty[q^{2m},q^{4r};q^{4m}]_\infty}[q^{u-3r},q^{2m-u-r};q^{4m}]_\infty, \end{aligned} \tag{3.3}$$

which implies (1.13) and (1.14). This completes the proof. $\square$

4. Proofs of Theorems 1.5 and 1.6. In this section, we present proofs of Theorems 1.5 and 1.6 by means of Lemma 2.2.

Proof. In Theorem 1.4, taking $u = 3r$, we can obtain the 4-dissection of $[q^{3r}; q^m]/[q^r; q^m]_\infty$. Setting $u \to m/2 - r$, $r \to m/4 + r$ and $q \to q^m$ in (2.9), we have

$$\begin{aligned}(4.1)\quad & [q^{m/2-r}, q^{m/4+r}; q^m]_\infty \\ &= \frac{[-q^{3m/4}, -q^{3m/4+2r}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty} \\ &\quad - \frac{q^{m/4+r}[-q^{m/4-2r}, -q^{m/4}, q^{2m}]_\infty}{[q^m; q^{2m}]_\infty}.\end{aligned}$$

Setting $u = 3r$ in (1.11) and employing (1.15) and (4.1), we see that

$$\begin{aligned}B_0(q^8) + q^4 B_4(q^8) &= \frac{[q^{2m-4r}, q^{m+4r}; q^{4m}]_\infty}{[q^m, q^{2m}; q^{4m}]_\infty} \\ &= \frac{[-q^{3m}, -q^{3m+8r}; q^{8m}]_\infty}{[q^m, q^{2m}; q^{4m}]_\infty [q^{4m}; q^{8m}]_\infty} \\ &\quad - \frac{q^{m+4r}[-q^{m-8r}, -q^m; q^{8m}]_\infty}{[q^m, q^{2m}; q^{4m}]_\infty [q^{4m}; q^{8m}]_\infty},\end{aligned}$$

which yields (1.16) and (1.21).

Setting $u \to m/2 - r$, $r \to m/4 - r$ and $q \to q^m$ in (2.9), we obtain

$$\begin{aligned}(4.2)\quad & [q^{m/2-r}, q^{m/4-r}; q^m]_\infty \\ &= \frac{[-q^{3m/4}, -q^{3m/4-2r}; q^{2m}]_\infty - q^{m/4-r}[-q^{m/4+2r}, -q^{m/4}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty}.\end{aligned}$$

Taking $u = 3r$ in (1.12) and using (1.15) and (4.2), we have

$$\begin{aligned}q^2 B_2(q^8) + q^6 B_6(q^8) &= q^{2r} \frac{[q^{2m-4r}, q^{m-4r}; q^{4m}]_\infty}{[q^m, q^{2m}; q^{4m}]_\infty} \\ &= \frac{q^{2r}[-q^{3m}, -q^{3m-8r}; q^{8m}]_\infty}{[q^m, q^{2m}; q^{4m}]_\infty [q^{4m}; q^{8m}]_\infty} - \frac{q^{m-2r}[-q^{m+8r}, -q^m; q^{8m}]_\infty}{[q^m, q^{2m}; q^{4m}]_\infty [q^{4m}; q^{8m}]_\infty},\end{aligned}$$

which yields (1.18) and (1.19).

Putting $u = 3r$ in (1.13), we see that if $r \equiv -i \pmod 4$ for $i = 1,\ 3$, then

$$A_i(q) = 0.$$

Therefore, we have, if $r \equiv 3 \pmod 4$, then

$$B_1(q) = B_5(q) = 0,$$

and, if $r \equiv 1 \pmod 4$, then

$$B_3(q) = B_7(q) = 0,$$

which imply that (1.20) holds.

Setting $u \to q^{m/4+r}$, $r \to q^{m/4-r}$ and $q \to q^m$ in (2.9), we see that

$$\begin{aligned} &[q^{m/4+r}, q^{m/4-r}; q^m]_\infty \\ &\qquad = \frac{[-q^{m/2}, -q^{m-2r}; q^{2m}]_\infty - q^{m/4-r}[-q^{2r}, -q^{m/2}; q^{2m}]_\infty}{[q^m; q^{2m}]_\infty}. \end{aligned} \tag{4.3}$$

Taking $u = 3r$ in (1.14) and employing (1.15) and (4.3), we have

$$\begin{aligned} q^i B_i(q^8) + q^{4+i} B_{4+i}(q^8) &= q^r \frac{[q^{m-4r}, q^{m+4r}; q^{4m}]_\infty}{[q^m, q^m; q^{4m}]_\infty} \\ &= \frac{q^r[-q^{2m}, -q^{4m-8r}; q^{8m}]_\infty - q^{m-3r}[-q^{8r}, -q^{2m}; q^{8m}]_\infty}{[q^m, q^m; q^{4m}]_\infty [q^{4m}; q^{8m}]_\infty}, \end{aligned} \tag{4.4}$$

where $i = 1,\ 3$ and $r \equiv i \pmod 4$. It follows from (4.4) that (1.21) and (1.22) hold.

To conclude this section, we turn toward proving Theorem 1.6. Using (1.1) and (1.4), we see that

$$\frac{[q^{3r-m}; q^m]_\infty}{[q^r; q^m]_\infty} = -q^{3r-m} \frac{[q^{m-3r}; q^m]_\infty}{[q^r; q^m]_\infty} = -q^{3r-m} \frac{[q^{3r}; q^m]_\infty}{[q^r; q^m]_\infty}. \tag{4.5}$$

Employing Theorem 1.5 and (4.5), we can easily deduce Theorem 1.6. The proof is complete. $\square$

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