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ON A LOGARITHMIC HARDY-BLOCH TYPE SPACE

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ABSTRACT. In this paper, given $0 < p < \infty$, we define a logarithmic Hardy-Bloch type space

$$BH_{p,L} = \Big\{f(z) \in H(D) : ||f||_{p,L} = \sup_{z\in D}(1-|z|)\log\frac{e}{1-|z|}M_p(|z|,f') < \infty\Big\}.$$

Then we mainly study the relation between $BH_{p,L}$ and three classical spaces: Hardy space, Dirichlet type space and VMOA. We also obtain some estimates on the growth of $f \in BH_{p,L}$.

1. Introduction. Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane C, and let $H(D)$ denote the set of all analytic functions on D. For $0 \le r < 1$, $f(z) \in H(D)$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p}, \quad 0<p<\infty,$$
$$M_\infty(r,f) = \max_{0\le\theta\le 2\pi}|f(re^{i\theta})|.$$

For $0 < p \le \infty$, the Hardy space H^p consists of those functions $f \in H(D)$, for which

$$\|f\|_{H_p} = \sup_{0\le r<1} M_p(r,f) < \infty.$$

The Dirichlet type space $\mathscr{D}^p_{p-1}$ consists of those functions $f \in H(D)$,

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for which

$$\int_D (1-|z|)^{p-1}|f'(z)|^p dA(z) < \infty,$$

where $dA(z) = 1/\pi\, dx\, dy$ denotes the norm Lebesgue area measure on D. Hence, $f \in \mathscr{D}^p_{p-1}$ if and only if

$$(1) \quad \int_D (1-|z|)^{p-1}|f'(z)|^p dA(z) = 2\pi \int_0^1 r(1-r)^{p-1} M_p^p(r, f')\, dr < \infty.$$

The $\mathscr{D}^p_{p-1}$ is closely related to H^p. A classical result of Littlewood and Paley **[9]** (see also **[10]**) asserts that

$$H^p \subset \mathscr{D}^p_{p-1}, \quad 2 \le p < \infty.$$

On the other hand, see, e.g., **[11, 14]**, we have

$$\mathscr{D}^p_{p-1} \subset H^p, \quad 0 < p \le 2.$$

These inclusions are strict if $p \neq 2$.

Given $0 < p \le \infty$ and $0 \le \alpha < \infty$, we write $BH_{p,\alpha}$ and $BH_{p,L}$ for the spaces of those $f(z) \in H(D)$, such that

$$\begin{aligned} BH_{p,\alpha} = \Big\{ & f(z) \in H(D) : ||f||_{p,\alpha} \\ & = \sup_{z\in D} (1-|z|)^{\alpha} M_p(|z|, f') < \infty \Big\}, \\ BH_{p,L} = \Big\{ & f(z) \in H(D) : ||f||_{p,L} \\ & = \sup_{z\in D} (1-|z|) \log \frac{e}{1-|z|} M_p(|z|, f') < \infty \Big\}. \end{aligned}$$

It is easy to prove that $BH_{p,\alpha}$ and $BH_{p,L}$ are complete under the norms

$$\begin{aligned} ||f||_{p_\alpha} &= ||f||_{p,\alpha} + |f(0)|, \\ ||f||_{p_L} &= ||f||_{p,L} + |f(0)|. \end{aligned}$$

When $p \ge 1$, the two spaces above are Banach spaces.

For the two spaces above, it is evident that $BH_{p,L} \supseteq BH_{q,L}$ for $0 < p < q \le \infty$. With the terminology just introduced, we have $BH_{\infty,\alpha} = \mathscr{B}_\alpha$ and $BH_{\infty,L} = \mathscr{B}_L$, where $\mathscr{B}_\alpha$ and $\mathscr{B}_L$ denote the α-Bloch space and the logarithmic Bloch space, respectively, the properties on these two spaces are abundant. More information about $\mathscr{B}_\alpha$ and $\mathscr{B}_L$ can be found in [**12, 15, 16, 17, 18, 20, 21**].

When $0 \leq \alpha < 1$, the following result is due to Hardy and Littlewood, see [**8**].

Theorem A. *Let $0 \leq \alpha < 1$ and $1 \leq p \leq \infty$. Then*

$$BH_{p,\alpha} = \Lambda^p_{1-\alpha} = \left\{ f \in H^p : \omega_p(t,f) = O(t^{1-\alpha}), \quad \text{as } t \to 0 \right\},$$

where

$$\omega_p(t,f) = \sup_{0<|h|\leq t} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(e^{i(\theta+h)}\right) - f\left(e^{i\theta}\right) \right|^p d\theta \right)^{1/p}, \quad t > 0, \ \text{if } 1 \leq p < \infty,$$

$$\omega_\infty(t,f) = \sup_{0<|h|\leq t} \left(\operatorname{ess\,sup} \left| f\left(e^{i(\theta+h)}\right) - f\left(e^{i\theta}\right) \right| \right), \quad t > 0.$$

Thus, $BH_{p,\alpha}$ is a mean Lipschitz space. On this basis, Blasco [**2**] *and Girela and Máquez* [**5**] *extend the result.*

When $\alpha = 1$, the following result is known (see [**7**] and [**6**]).

Theorem B. *Let $f \in H(D)$, if $0 < p < \infty$ and $f \in BH_{p,1}$. Then*

$$M_p(r,f) = O\left(\left(\log \frac{1}{1-r} \right)^\beta \right),$$

where

(i) $\beta = 1/p$, *for* $0 < p < 2$,
(ii) $\beta = 1/2$ *for* $2 \leq p < \infty$.

Our main goal in this paper is to show that, if $f \in BH_{p,L}$, whose rate of growth $M_p(r,f')$ $(1 < p < \infty)$ is between the ones for the functions in $BH_{p,\alpha}$ $(0 < \alpha < 1)$ and $BH_{p,1}$, then $f \in (H^p \bigcap \mathscr{D}^p_{p-1})$, see Theorem 2.1. We also characterize that $BH_{p,L} \subset (H^p \bigcap \mathscr{D}^p_{p-1})$ for $p \in (1,\infty)$ is strict. However, the containment is not true for $0 < p \leq 1$. We also note that $BH_{p,L} \not\subset VMOA$ for every $0 < p < \infty$. As for the growth of $f \in BH_{p,L}$, we give a sharp estimate.

Throughout this paper, C, which may change from one occurrence to the next, denotes a positive and finite constant only dependent on p and α.

2. Proof of main results.

Theorem 2.1. *Suppose* $1 < p < \infty$. *Then* $BH_{p,L} \subset (H^p \bigcap \mathscr{D}^p_{p-1})$.

To complete the proof, we need the following three lemmas.

Lemma 2.1. [**6**] *If* $2 < p < \infty$, *then there is a constant* C_p *depending only on* p *such that*

$$||f||_{H_p} \le C_p\bigg(|f(0)| + \bigg(\int_0^1 (1-r)M_p^2(r,f')\,dr\bigg)^{1/2}\bigg),$$
$$\text{for all } f \in H(D).$$

Lemma 2.2. [**7**] *If* $0 < p \le 2$, *then there is a constant* C_p *depending only on* p *such that*

$$||f||^p_{H^p} \le C_p\bigg(|f(0)|^p + \int_D (1-|z|)^{p-1}|f'(z)|^p dA(z)\bigg),$$
$$\text{for every } f \in \mathscr{D}^p_{p-1}.$$

Lemma 2.3. *If* $0 < \alpha,\ \beta < \infty$, $x \in (0,e]$, *then* $f(x) = x^\alpha(\log e/x)^\beta$ *increases on* $(0, e^{1-\beta/\alpha}]$, *decreases on* $[e^{1-\beta/\alpha}, e]$.

The proof of this lemma is easy; we omit the details here.

Proof of Theorem 2.1. Take $f \in BH_{p,L}$ and assume, without loss of generality, that $f(0) = 0$. When $2 < p < \infty$, for $0 < r < 1$, set $f_r(z) = f(rz)$. Applying Lemma 2.1 to f_r, we obtain that

$$\begin{aligned} M_p^2(r,f) &\le C\int_0^1 r^2(1-s)M_p^2(rs,f')\,ds \\ &\le C\int_0^1 \frac{1-s}{(1-rs)^2(\log e/(1-rs))^2}\,ds. \end{aligned}$$

Since $rs < s$, Lemma 2.3 implies

$$M_p^2(r,f) \le C\int_0^1 \frac{1}{(1-s)(\log e/(1-s))^2}\,ds < \infty.$$

When $1 < p \le 2$, using Lemma 2.2 yields

$$\begin{aligned} M_p^p(r,f) &\le C\int_D r^p(1-|w|)^{p-1}|f'(rw)|^p dA(w) \\ &\le C\int_0^1 \frac{(1-s)^{p-1}}{(1-rs)^p(\log e/(1-rs))^p}\,ds \\ &\le C\int_0^1 \frac{1}{(1-s)(\log e/(1-s))^p}\,ds. \end{aligned}$$

Hence $f \in H^p$. The assertion that $f \in \mathscr{D}_{p-1}^p$ can be easily obtained from (1) finishes the proof. $\square$

This theorem is not true for $0 < p \le 1$ and $p = \infty$. Indeed, when $p = \infty$, the space $BH_{\infty,L} = \mathscr{B}_L$. We take $f(z) = \log\log e/(1-z)$. Then

$$M_\infty(r,f') = O\left(\frac{1}{(1-r)\log e/(1-r)}\right),$$

that is, $f \in \mathscr{B}_L$, but $f \notin H^\infty$. In the case $0 < p \le 1$, we only prove the following result.

Theorem 2.2. *Suppose* $0 < p \le 1$. *Let*

$$f(z) = \frac{1}{(1-z)^{1/p}\log e/(1-z)}, \quad z \in D.$$

Then $f \in BH_{p,L}$, *but* $f \notin H^p$.

The following three lemmas are needed in the proof of Theorem 2.2.

Lemma 2.4. *If* $a > 1$ *and* $0 < r < 1$, *then there exist two constants*

$$C_1 = \frac{2}{a-1}\left(1 - \frac{1}{(1+\pi)^{a-1}}\right)$$

and

$$C_2 = \frac{4\pi^a}{a-1},$$

such that

$$C_1\,(1-r)^{1-a} \le \int_0^{2\pi} \left|\rho e^{i\theta} - r\right|^{-a} dt \le C_2\,(1-r)^{1-a}, \quad \rho = \frac{1}{2}(1+r).$$

The proof is similar to [**3,** Lemma in Chapter 4.6], so we omit the details.

Lemma 2.5. *For $0 < \alpha$, $\beta < \infty$ and $z \in D$, let*

$$f(z) = \frac{(1-|z|)^{\alpha}\,(\log e/(1-|z|))^{\beta}}{(|1-z|)^{\alpha}\,(\log e/|1-z|)^{\beta}}.$$

Then $f(z) \le 1$ in the case $\beta \le \alpha \log e/2$, and $f(z) \le (e^{\alpha-\beta}(6\beta)^{\beta})/2^{\alpha}\alpha^{\beta}$ in the case $\beta > \alpha \log e/2$.

Proof. For $\beta \le \alpha \log e/2$, by Lemma 2.3, we have

$$(1-|z|)^{\alpha}\left(\log \frac{e}{1-|z|}\right)^{\beta} \le (|1-z|)^{\alpha}\left(\log \frac{e}{|1-z|}\right)^{\beta}, \quad \text{for all } z \in D,$$

which implies $f(z) \le 1$.

When $\beta > \alpha \log e/2$, let $z \in D_1 = \{z \in D, |1-z| < e^{1-(\beta/\alpha)}\}$, we deduce that $|f(z)| \le 1$. On the other hand, the condition $z \in D\backslash D_1$ implies that

$$f(z) \le \frac{e^{\alpha-\beta}(\beta/\alpha)^{\beta}}{2^{\alpha}\,(\log e/2)^{\beta}} \le \frac{6^{\beta} e^{\alpha-\beta}(\beta/\alpha)^{\beta}}{2^{\alpha}} \le \frac{e^{\alpha-\beta}(6\beta)^{\beta}}{2^{\alpha}\alpha^{\beta}}.$$

This finishes the proof. $\square$

The following lemma may be found in [**8**].

Lemma 2.6. *Suppose that $0 < p < \infty$ and $f \in H^p$. Then*

$$\int_0^1 M_{\infty}^p(r,f)\,dr \le \pi ||f||_{H^p}^p.$$

Proof of Theorem 2.2. Take the function

$$g_c(z) = \frac{z^c}{(1-z)\,(\log 1/(1-z))^c}.$$

Exercise 3 [**3,** Chapter I] says that $g_c \in H^1$ $(c > 1)$. However, set $z = re^{i\theta}$,

$$\int_0^1 M_\infty^1(r, g_c)\,dr \geq \int_0^1 \frac{r^c}{(1-r)\,(\log 1/(1-r))^c}\,dr = \infty,$$
$$\text{for every } 0 \leq c \leq 1,$$

which, by Lemma 2.6, implies $g_c(z) \notin H^1$ for $0 \leq c \leq 1$.

For $r = |z| \geq 1/2$, there exists a constant $C > 0$, such that $|\log 1/(1-z)| \geq C$. Therefore,

$$\begin{aligned}\frac{|f(z)|^p}{|g_1(z)|} &= \left|\frac{1/(1-z)\log(e/(1-z))^p}{z/(1-z)\log 1/(1-z)}\right| \\ &\geq 2\frac{|\log 1/(1-z)|}{|(\log e/(1-z))^p|} \\ &\geq 2\frac{|\log 1/(1-z)||\log e/(1-z)|^{1-p}}{1+|\log 1/(1-z)|} \\ &\geq \frac{2C}{1+C}\left(\log\frac{e}{2}\right)^{1-p}.\end{aligned}$$

It follows that $f \notin H^p$. On the other hand,

$$f'(z) = \frac{1/p}{(1-z)^{1+(1/p)}\log e/(1-z)} - \frac{1}{(1-z)^{1+(1/p)}\,(\log e/(1-z))^2}.$$

We have

$$\begin{aligned}|f'(z)|^p &\leq \frac{1}{|1-z|^{1+p}(\log|e/(1-z)|)^p}\left(\frac{1}{p} + \frac{1}{\log|e/(1-z)|}\right)^p \\ &\leq \left(\frac{1}{p} + \frac{1}{\log e/2}\right)^p \frac{1}{|1-z|^{1+p}(\log|e/(1-z)|)^p}.\end{aligned}$$

Using Lemmas 2.4 and 2.5, we obtain that

$$\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{1+p}(\log|e/(1-re^{i\theta})|)^p}\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{p/2}(\log|e/(1-re^{i\theta})|)^p|1-re^{i\theta}|^{1+(p/2)}}\,d\theta$$
$$\le \frac{C}{(1-r)^{p/2}\left(\log e/(1-r)\right)^p}\int_0^{2\pi}\frac{1}{|1-re^{i\theta}|^{1+(p/2)}}\,d\theta$$
$$= \frac{C}{(1-r)^{p}\left(\log e/(1-r)\right)^p}.$$

This shows that $f(z) \in BH_{p,L}$, and the proof is complete. $\square$

We also note that $BH_{p,L} \subset (H^p \bigcap \mathscr{D}^p_{p-1})$ for every $p \in (1,\infty)$ is proper. Indeed, we have the following theorem.

Theorem 2.3. *Given p with $1 < p < \infty$, there exists a function f which belongs to $(H^p \bigcap \mathscr{D}^p_{p-1})\backslash BH_{p,L}$.*

Proof. Let

$$f(z) = \frac{1}{(1-z)^{1/p}\left(\log(2e^{2\sqrt{p}})/(1-z)\right)^{1/\sqrt{p}}}, \quad z \in D.$$

Then

$$\log\frac{2e^{2\sqrt{p}}}{1-|z|} \sim \log\frac{1}{1-|z|} \quad \text{as } |z| \to 1^-.$$

Take g_c in the proof of Theorem 2.2 with $c = \sqrt{p}$. Then

$$M_p^p(r,f) \sim M_1(r, g_{\sqrt{p}}), \quad r = |z| \to 1^-,$$

which implies $f \in H^p$ for every $1 < p < \infty$.

As for proving $f \in \mathscr{D}^p_{p-1}$ $(1 < p < \infty)$, it can be deduced by directly calculating

$$\int_0^1 r(1-r)^{p-1}M_p^p(r,f')\,dr < \infty.$$

Thus, we find $f \in H^p \bigcap \mathscr{D}^p_{p-1}$.

On the other hand, using Lemma 2.4 yields

$$\frac{1}{2\pi}\int_0^{2\pi}|f'(re^{i\theta})|^p d\theta$$
$$\ge \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-re^{i\theta}|^{1+p}|\log(2e^{2\sqrt{p}})/(1-re^{i\theta})|^{\sqrt{p}}}$$

$$\begin{aligned}
&\times \left(\frac{1}{p} - \frac{1}{\sqrt{p}|\log(2e^{2\sqrt{p}})/(1-re^{i\theta})|}\right)^p d\theta \\
&\geq \frac{\delta}{2\pi(2p)^p} \frac{1}{(\log(2e^{2\sqrt{p}})/(1-r))^{\sqrt{p}}} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{1+p}}\, d\theta \\
&\geq \frac{\delta}{2\pi(2p)^p} \frac{1}{(1-r)^p(\log(2e^{2\sqrt{p}})/(1-r))^{\sqrt{p}}},
\end{aligned}$$

where $\delta > 0$ is a constant. Therefore,

$$\begin{aligned}
(1-r)\log\frac{e}{1-r} M_p(r,f') \geq \frac{\delta^{1/p}}{2p(2\pi)^{\sqrt{p}}} \frac{\log e/1-r}{(\log(2e^{2\sqrt{p}})/(1-r))^{1/\sqrt{p}}} \\
\longrightarrow \infty, \quad r \to 1^-,
\end{aligned}$$

which implies $f \notin BH_{p,L}$, and this concludes the proof. $\square$

Remark 2.1. If we take

$$f(z) = \frac{1}{(1-z)^{1/p}\log(2e^{2\sqrt{p}})/(1-z)}, \quad z \in D.$$

Carefully checking the proofs of Theorems 2.2 and 2.3, we have

$$M_p(r,f') \approx \frac{1}{(1-r)\log e/(1-r)}.$$

In **[17]**, the second author showed that $\beta_L \subseteq VMOA$, the vanishing mean oscillation of analytic functions in D. So we want to know whether $BH_{p,L} \subseteq VMOA$ for $p \in (0,\infty)$. However, the answer is negative. For more information for $VMOA$ space, see **[1, 4, 19]**.

Theorem 2.4. *Suppose* $0 < p < \infty$, *then*

$$f(z) = \int_0^z \frac{1}{(1-t)^{1+1/p}\log e/(1-t)}\, dt \in BH_{p,L},$$

but $f(z) \notin VMOA$.

Proof. The proof for $f \in BH_{p,L}$ is similar to Theorem 2.2, we omit the details here.

It is well known that the space

$$VMOA \subset BMOA \subset \mathscr{B} \bigcap \Big(\bigcap_{0<p<\infty} H^p \Big).$$

We have

$$f'(r) = \frac{1}{(1-r)^{1+1/p} \log e/(1-r)},$$

and then

$$(1-r)f'(r) = \frac{1}{(1-r)^{1/p} \log e/(1-r)},$$

which tends to infinity when r tends to 1. Hence, $f \notin \mathscr{B}$, it follows that $f \notin VMOA$. □

Our next objective is to estimate the growth of $f \in BH_{p,L}$ $(0 < p < \infty)$. We begin with two lemmas.

Lemma 2.7. *Given p with $1 \le p < \infty$ and r with $0 < r < 1$, then*

(2)
$$\int_0^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)}\, ds \le 2p \left(1 + pe^{2/p}\right) \frac{1}{(1-r)^{1/p} \log e/(1-r)}.$$

Proof. Let $r_* = 1 - e^{1-2p}$. Using Lemma 2.3, we easily obtain

$$\begin{aligned}\int_0^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)}\, ds &\le \int_0^r \frac{1}{(1-s)^{1+1/2p}}\, ds \\ &\le 2pe^{(2p-1)/2p}\end{aligned}$$

if $r \le r_*$. When $r > r_*$, we have

$$\begin{aligned}&\int_0^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)}\, ds \\ &\quad = \int_0^{r_*} \frac{1}{(1-s)^{1+1/p} \log e/(1-s)}\, ds + \int_{r_*}^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)}\, ds \\ &\quad \le 2pe^{(2p-1)/2p} + \frac{1}{(1-r)^{1+1/2p} \log e/(1-r)} \int_0^r \frac{1}{(1-s)^{1+1/2p}}\, ds \\ &\quad \le 2pe^{(2p-1)/2p} + 2p \frac{1}{(1-r)^{1/p} \log e/(1-r)}.\end{aligned}$$

Hence, (2) holds. □

Lemma 2.8. [8] *Any function $f \in H^p$ $(0 < p < \infty)$ can be expressed in the form $f(z) = f_1(z) + f_2(z)$, where f_1 and f_2 are nonvanishing H^p functions such that $\|f_n\|_{H^p} \le 2\|f\|_{H^p}$, $n = 1, 2$.*

Theorem 2.5. *If $f \in BH_{p,L}$, $(0 < p < \infty)$, then*

$$|f(z)| \le \frac{C\|f\|p_L}{(1-r)^{1/p} \log e/(1-r)}, \quad r = |z|,$$

where

$$C = \begin{cases} 4\pi p(p-1)^{(p-1)/p} \cdot \left(1 + p \cdot e^{1/2p}\right), & 1 < p < \infty, \\ 8(1 + e^{1/2}), & p = 1, \\ 2^{2+(3/p)-3p}/(1-p), & 0 < p < 1. \end{cases}$$

Proof. For r with $0 < r < 1$, we have

$$f(re^{i\theta}) = \int_0^r f'(se^{i\theta}) e^{i\theta} ds + f(0).$$

We set $\rho = (1+s)/2$, by Cauchy formula

$$f'(se^{i\theta}) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f'(\zeta)}{\zeta - se^{i\theta}} \, d\zeta = \frac{\rho}{2\pi} \int_0^{2\pi} \frac{f'(\rho e^{i(t+\theta)}) e^{i(t-\theta)}}{\rho e^{it} - s} \, dt.$$

Case I: $1 < p < \infty$. Let q be the conjugate index of $p : (1/p) + (1/q) = 1$, then, by Hölder's inequality, Lemmas 2.4 and 2.7, we give

$$\begin{aligned} |f(re^{i\theta})| &\le |f(0)| + \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \frac{|f'(\rho e^{i(t+\theta)}) e^{i(t-\theta)}|}{|\rho e^{it} - s|} \, dt \, ds \\ &\le |f(0)| + \left(\frac{1}{2\pi}\right)^{1/q} \cdot \left(\frac{4\pi^q}{q-1}\right)^{1/q} \\ &\quad \int_0^r M_p(\rho, f') \cdot \frac{1}{(1-s)^{1/p}} \, ds \\ &\le |f(0)| + 4\pi p(p-1)^{(p-1)/p} \left(1 + p \cdot e^{1/2p}\right) \\ &\quad \cdot \frac{\|f\|_{p,L}}{(1-r)^{1/p} \log e/(1-r)} \\ &\le 4\pi p(p-1)^{(p-1)/p} \left(1 + p \cdot e^{1/2p}\right) \frac{||f||_{p_L}}{(1-r)^{1/p} \log e/(1-r)}. \end{aligned}$$

Case II: $p = 1$. Using Lemma 2.7 again yields

$$\begin{aligned}|f(re^{i\theta})| &\le |f(0)| + \frac{1}{2\pi}\int_0^r \int_0^{2\pi} \frac{|f'(\rho e^{i(t+\theta)})e^{i(t-\theta)}|}{|\rho e^{it} - s|}\,dt\,ds \\ &\le |f(0)| + 4\int_0^r \frac{1}{(1-s)^2 \log e/(1-s)}\,ds \cdot ||f||_{p,L} \\ &\le 8(1+e^{1/2})\frac{||f||_{p_L}}{(1-r)\log e/(1-r)}.\end{aligned}$$

Case III: $0 < p < 1$. If $f'(z) \neq 0$ in $z \in D$, then the function $F(z) = (f'(z))^p$ is analytic, and $f \in BH_{p,L}$ gives

$$M_1(r, F) = \{M_p(r, f')\}^p \le \frac{||f||_{p,L}^p}{(1-r)^p\,(\log e/(1-r))^p}.$$

By the Cauchy formula, we find

$$|F(re^{i\theta})| = |\frac{\rho}{2\pi}\int_0^{2\pi} \frac{F(\rho e^{it})e^{it}}{\rho e^{it} - z}\,dt| \le \frac{2||f||_{p,L}^p}{(1-r)^{1+p}\,(\log e/(1-r))^p},$$

which implies that

$$|f'(re^{i\theta})| \le \frac{\sqrt[p]{2}||f||_{p,L}}{(1-r)^{1+1/p}\log e/(1-r)}.$$

Then

$$\begin{aligned}M_1(r, f') &= \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^{1-p}|f'(re^{i\theta})|^p d\theta \\ &\le \{M_\infty(r, f')\}^{1-p}\,\{M_p(r, f')\}^p \\ &\le \frac{2^{(1-p)/p}}{(1-r)^{1/p}\log e/(1-r)} \cdot ||f||_{p,L}^p.\end{aligned}$$

We deduce

$$\begin{aligned}|f(re^{i\theta})| &\le |f(0)| + \frac{1}{2\pi}\int_0^r \int_0^{2\pi} \frac{|f'(\rho e^{i(t+\theta)})e^{i(t-\theta)}|}{|\rho e^{it} - s|}\,dt\,ds \\ &\le |f(0)| + 2^{2/p}\frac{p}{1-p} \cdot \frac{||f||_{p,L}}{(1-r)^{1/p}\log e/(1-r)} \\ &\le \frac{2^{2/p}}{1-p} \cdot \frac{||f||_{p_L}}{(1-r)^{1/p}\log e/(1-r)}.\end{aligned}$$

If $f'(z)$ has zeros, we fix $R<1$ and use Lemma 2.8 to write

$$f'(Rz) = f_1'(z) + f_2'(z),$$

where f_1 and f_2 do not vanish and

$$\|f_n'\|_{H^p} \leq 2M_p(R,f') \leq \frac{2||f||_{p,L}}{(1-R)\log e/(1-R)}, \quad n=1,\ 2.$$

Since $f_n'(z) \neq 0$ $(n=1,\ 2)$, it follows that

$$\begin{aligned}|f'(R^2e^{i\theta})| &\leq |f_1'(Re^{i\theta})| + |f_2'(Re^{i\theta})| \\ &\leq \frac{2^{2+1/p}||f||_{p,L}}{(1-R)^{1+1/p}\log e/(1-R)}.\end{aligned}$$

Then

$$|f'(re^{i\theta})| \leq \frac{2^{3+2/p}||f||_{p,L}}{(1-r)^{1+1/p}\log e/(1-r)},$$

which implies that

$$|f(re^{i\theta})| \leq \frac{2^{2+3/p-3p}}{1-p} \cdot \frac{||f||_{p_L}}{(1-r)^{1/p}\log e/(1-r)}.$$

This completes the proof of Theorem 2.5. □

For $0<p<\infty$, we set

$$H_{p,L}^{\infty} = \left\{ f\in H(D),\ |f(z)| = \sup\left(\frac{1}{(1-r)^{1/p}\log e/(1-r)}\right) < \infty,\ |z|=r\right\}.$$

Then $BH_{p,L}$ and the classic Bloch space $\mathscr{B}_1$ are included in $H_{p,L}^{\infty}$. It turns out that neither $H_{p,L}^{\infty}$ nor $H_{p,L}^{\infty}$ is contained in H^p for every $1<p<\infty$. Indeed,

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad z\in D$$

is in $\mathscr{B}_1$, which implies that $f\in H_{p,L}^{\infty}$, but not in H^p. On the other hand, taking the function $f(z)$ in Theorem 2.4, we find that $f\in H^p$ but not $f\notin H_{p,L}^{\infty}$.

We set $\mathcal{U}$ here to be the class of all univalent functions in D. Then Prawitz, see [**13,** page 17], deduced the following theorem.

Theorem D. *Suppose that* $0 < p < \infty$. *If* $f \in \mathcal{U}$ *and* $\int_0^1 M_\infty^p(r,f)\,dr < \infty$, *then* $f \in H^p$. Using this result, we can give the following theorem.

Theorem 2.6. *Suppose that* $1 < p < \infty$ *and* $f \in \mathcal{U} \bigcap H^\infty_{p,L}$. *Then* $f \in H^p$.

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