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ASYMPTOTIC BEHAVIOR OF THE FINITE-TIME RUIN PROBABILITY WITH PAIRWISE QUASI-ASYMPTOTICALLY INDEPENDENT CLAIMS AND CONSTANT INTEREST FORCE

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ABSTRACT. This paper will obtain an asymptotic formula of the finite-time ruin probability in a generalized risk model with constant interest force, in which the claim sizes are pairwise quasi-asymptotically independent and their arrival process is an arbitrary counting process. In particular, when the claim inter-arrival times follow a certain dependence structure, the result obtained can also include an asymptotic formula for the infinite-time ruin probability.

1. Risk model. In this paper, we investigate the finite-time ruin probability in a generalized risk model with constant interest force, where the claim sizes $\{X_i, i \geq 1\}$ form a sequence of nonnegative, identically distributed, but not necessarily independent, random variables (r.v.s) with common distribution F, and the claim arrival process $\{N(t), t \geq 0\}$ is a general counting process, independent of $\{X_i, i \geq 1\}$. Hence, the aggregate claim amount up to time $t \geq 0$ is expressed as

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

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with $S(t) = 0$ if $N(t) = 0$. Assume that the total amount of premiums accumulated before time $t \geq 0$, denoted by $C(t)$, is a nonnegative and nondecreasing stochastic process with $C(0) = 0$ and $C(t) < \infty$ almost surely (a.s.) for every $0 \leq t < \infty$. Let $0 \leq r < \infty$ be the constant interest force and $0 \leq x < \infty$ the insurer's initial reserve. Thus, the total reserve up to time $0 \leq t < \infty$ of the insurance company, denoted by $U_r(t)$, satisfies

$$U_r(t) = xe^{rt} + \int_0^t e^{r(t-s)} C(ds) - \int_0^t e^{r(t-s)} S(ds). \tag{1.1}$$

Apparently, by the conditions on $C(t)$, one can easily know that, for any fixed $0 < t < \infty$,

$$0 \leq \widetilde{C}(t) = \int_0^t e^{-rs} C(ds) < \infty \quad \text{almost surely,} \tag{1.2}$$

where $\widetilde{C}(t)$ is the discounted value of premiums accumulated before time t. The ruin probability within a finite time T is defined by

$$\psi_r(x, T) = P(U_r(t) < 0 \quad \text{for some } 0 \leq t \leq T), \tag{1.3}$$

and the infinite-time ruin probability is

$$\psi_r(x, \infty) = P(U_r(t) < 0 \quad \text{for some } 0 \leq t < \infty). \tag{1.4}$$

For later use, let $\{\theta_i, i \geq 1\}$ denote the claim inter-arrival times, and then let $\tau_k = \sum_{i=1}^k \theta_i, k \geq 1$, denote the arrival times of successive claims, which can constitute a counting process

$$N(t) = \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k \leq t\}}, \quad t \geq 0, \tag{1.5}$$

where $\mathbf{1}_A$ is the indicator function of an event A. If $\{\theta_i, i \geq 1\}$ are independent and identically distributed (i.i.d.) r.v.s, then $\{N(t), t \geq 0\}$ is the renewal process and the risk model with $\{N(t), t \geq 0\}$ a renewal process which was studied by Tang [**18**], Chen and Ng [**2**], Hao and Tang [**7**] and some others. If $\{\theta_i, i \geq 1\}$ follows a certain dependence structure, then $\{N(t), t \geq 0\}$ is a quasi-renewal process and the risk model with $\{N(t), t \geq 0\}$ a quasi-renewal process was studied by Li et al. [**13**], Yang and Wang [**25**], Wang et al. [**23**], Liu et al. [**14**] and so on. However, for the model of this paper $\{N(t), t \geq 0\}$ is an arbitrary counting process, and thus no assumption is made on the dependence structure of $\{\theta_i, i \geq 1\}$, also see Wang **[22]**.

2. Introduction and main results. Throughout this paper, all limit relationships are taken as $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) = C < \infty$, write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $C \leq 1$, write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $b(x) \lesssim a(x)$, and write $a(x) = o(b(x))$ if $C = 0$.

As generally admitted, in the insurance industry how to model dangerous claim sizes is one of the main worries of practicing actuaries, and actually most practitioners choose the claim-size distribution from the heavy-tailed distribution class, one of which is the subexponential class. Say that a distribution V belongs to the subexponential class, denoted by $V \in \mathcal{S}$, if $\overline{V}(x) = 1 - V(x) > 0$ for all $x > 0$ and

$$\overline{V^{*2}}(x) \sim 2\overline{V}(x),$$

where V^{*2} denotes the 2-fold convolution of V. Clearly, if $V \in \mathcal{S}$ then V is long-tailed, denoted by $V \in \mathcal{L}$ and characterized by

$$\overline{V}(x + y) \sim \overline{V}(x) \quad \text{for all } y \neq 0.$$

Another important class of heavy-tailed distributions is the dominated variation class, say that a distribution V belongs to the dominated variation class, denoted by $V \in \mathcal{D}$, if

$$\overline{V}(xy) = O(\overline{V}(x)) \quad \text{for all } y > 0.$$

A slightly smaller class of $\mathcal{D}$ is the consistent variation class, say that a distribution V belongs to the consistent variation class, denoted by $V \in \mathcal{C}$, where

$$\lim_{y \searrow 1} \liminf_{x \to \infty} \overline{V}(xy)/\overline{V}(x) = 1,$$

or equivalently,

$$\lim_{y \nearrow 1} \limsup_{x \to \infty} \overline{V}(xy)/\overline{V}(x) = 1.$$

It is well known that the following inclusion relationships are valid and proper:

$$\mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

For a distribution V and $y > 0$, we set

$$J_V^+ = -\lim_{y\to\infty} \log \overline{V}_*(y)/\log y \quad \text{with} \quad \overline{V}_*(y) = \liminf_{x\to\infty} \overline{V}(xy)/\overline{V}(x)$$

and

$$J_V^- = -\lim_{y\to\infty} \log \overline{V}^*(y)/\log y \quad \text{with} \quad \overline{V}^*(y) = \limsup_{x\to\infty} \overline{V}(xy)/\overline{V}(x).$$

For more details of heavy-tailed distributions and their applications to insurance and finance, the readers are referred to Bingham et al. [**1**] and Embrechts et al. [**4**].

For the risk model with constant interest force and i.i.d heavy-tailed claims, there are many results on the asymptotic behavior for the finite-time ruin probability $\psi_r(x,T)$, $0 < T \le \infty$, see Klüppelberg and Stadtmüller [**10**], Kalashnikov and Konstantinides [**9**], Konstantinides et al. [**12**], Tang [**18, 19**], Wang [**22**], Hao and Tang [**7**], among others. In particular, Wang [**22**] showed that, in a risk model, the claim sizes $\{X_i, i \ge 1\}$ are i.i.d. and nonnegative r.v.s with common distribution $F \in \mathcal{L} \cap \mathcal{D}$, and their arrival process $\{N(t), t \ge 0\}$ is a general counting process satisfying $EN(T) > 0$ and $E(1+\delta)^{N(T)} < \infty$ for any fixed $0 < T < \infty$ and some $\delta = \delta(T) > 0$, and $\{X_i, i \ge 1\}$, $\{N(t), t \ge 0\}$ and $\{C(t), t \ge 0\}$ are mutually independent. Then, for fixed $0 < T < \infty$,

$$\psi_r(x,T) \sim \int_0^T \overline{F}(xe^{rt})\, dEN(t). \tag{2.1}$$

But, in most practical situations the independence assumption on claim sizes is unrealistic. Recently, more and more researchers have paid attention to a risk model with dependent claim sizes and/or dependent inter-arrival times, see Chen and Ng [**2**], Kong and Zong [**11**], Li et al. [**13**], Yang and Wang [**25**], Wang et al. [**23**], Gao et al. [**5**], Liu et al. [**13**], and so on. Therein, Wang et al. [**23**] have introduced a new dependence structure among r.v.s, which is as follows:

Definition 2.1. Say that the r.v.s $\{\xi_n : n \ge 1\}$ are widely upper orthant dependent (WUOD), if there exists a finite positive real number sequence $\{g_U(n) : n \ge 1\}$ such that, for each $n \ge 1$ and for all

$x_i \in (-\infty, \infty)$, $1 \le i \le n$,

$$P\Big(\bigcap_{i=1}^{n}\{\xi_i > x_i\}\Big) \le g_U(n)\prod_{i=1}^{n} P(\xi_i > x_i). \tag{2.2}$$

If the inequality (2.2) is changed into

$$P\Big(\bigcap_{i=1}^{n}\{\xi_i \le x_i\}\Big) \le g_L(n)\prod_{i=1}^{n} P(\xi_i \le x_i),$$

where $\{g_L(n) : n \ge 1\}$ is another finite positive real number sequence, then say that $\{\xi_n : n \ge 1\}$ are widely lower orthant dependent (WLOD).

Clearly, if $\{\xi_i, i \ge 1\}$ are WLOD, then $\{-\xi_i, i \ge 1\}$ are WUOD. From the definitions of WLOD and WUOD r.v.s, Wang et al. **[23]** gave the following proposition.

Proposition 2.1.

(i) *Assume that* $\{\xi_i, i \ge 1\}$ *are* WLOD (WUOD) r.v.*s. If* $\{f_i(\cdot), i \ge 1\}$ *are nondecreasing, then* $\{f_i(\xi_i), i \ge 1\}$ *are also* WLOD (WUOD); *if* $\{f_i(\cdot), i \ge 1\}$ *are nonincreasing, then* $\{f_i(\xi_i), i \ge 1\}$ *are* WUOD (WLOD).

(ii) *If* $\{\xi_i, i \ge 1\}$ *are nonnegative and* WUOD r.v.*s, then for each* $n \ge 1$,

$$E\prod_{i=1}^{n}\xi_i \le g_U(n)\prod_{i=1}^{n} E\xi_i.$$

In particular, if $\{\xi_i, i \ge 1\}$ *are* WUOD, *then for each* $n \ge 1$ *and any* $s > 0$,

$$E\exp\Big\{s\sum_{i=1}^{n}\xi_i\Big\} \le g_U(n)\prod_{i=1}^{n} E\exp\{s\xi_i\}.$$

For some properties and examples of WUOD and WLOD r.v.s, we refer the readers to Wang et al. **[23]** and Wang and Cheng **[24]**. For a nonstandard renewal risk model with WUOD claim sizes and WLOD inter-arrival times, Wang et al. **[23]** assumed that $\{X_i, i \ge 1\}$, $\{N(t), t \ge 0\}$ and $\{C(t), t \ge 0\}$ are mutually independent, and proved that if $F \in \mathcal{L} \cap \mathcal{D}$, then relation (2.1) holds uniformly for time T varying in a finite interval.

In this paper, we will consider the dependence structure among claim sizes from the WUOD structure to a more general dependence structure, whose definition is given below.

Definition 2.2. Say that the r.v.s $\{\xi_i, i \geq 1\}$ with distributions V_i, $i \geq 1$, respectively, are pairwise quasi-asymptotically independent if

$$P(\xi_i > x, \xi_j > x) = o(\overline{V_i}(x) + \overline{V_j}(x)) \quad \text{for } i \neq j,\ i, j \geq 1. \tag{2.3}$$

Remark 2.1. The term "pairwise quasi-asymptotic independence" is borrowed from Resnick [**17**] and Chen and Yuen [**3**]. Clearly, if $\{\xi_i, i \geq 1\}$ are identically distributed, relation (2.3) is equivalent to

$$P(\xi_i > x, \xi_j > x) = o(P(X_i > x)) \quad \text{for } i \neq j,\ i, j \geq 1,$$

that is to say, $\{\xi_i, i \geq 1\}$ are pairwise asymptotically independent or bivariate upper tail independent, see Zhang et al. [**27**], Gao and Wang [**6**], and the references therein. We also remark that the pairwise quasi-asymptotically independent r.v.s can cover not only common negatively dependent r.v.s but also some positively dependent r.v.s.

Inspired by the above results, in this paper we aim at establishing an asymptotic formula (2.1) for the finite-time ruin probability $\psi_r(x, T)$, $0 < T \leq \infty$, in a generalized risk model with constant interest force, where the claim sizes are pairwise quasi-asymptotic independent and their arrival process is an arbitrary counting process. In our main results, we will discuss two cases: one that is the premium process $\{C(t); t \geq 0\}$ is independent of $\{X_i, i \geq 1\}$ and $\{N(t), t \geq 0\}$, and the other is that $\{C(t); t \geq 0\}$ is not necessarily independent of $\{X_i, i \geq 1\}$ or $\{N(t), t \geq 0\}$.

The following are our main results, the first one is concerned with the finite-time ruin probability for any fixed $0 < T < \infty$.

Theorem 2.1. *Consider the generalized risk model introduced in Section* 1 *with $r \geq 0$, where the claim sizes $\{X_i, i \geq 1\}$ are pairwise quasi-asymptotically independent* r.v.*s with common distribution $F \in \mathcal{C}$, and for any fixed $0 < T < \infty$ such that $EN(T) > 0$, there exists some*

$p > J_F^+$ *such that* $E(N(T))^{p+1} < \infty$. *Then relation* (2.1) *holds for the fixed* $0 < T < \infty$, *if one of the following conditions is valid*:

(i) *the premium process* $\{C(t), t \geq 0\}$ *is independent of* $\{X_i, i \geq 1\}$ *and* $\{N(t), t \geq 0\}$;

(ii) *the discounted value of premiums accumulated by time* T, *defined in* (1.2), *satisfies that*

$$P(\widetilde{C}(T) > x) = o(\overline{F}(x)).$$

Evidently, for any fixed $0 < T < \infty$, the condition that $EN(T) > 0$ is equivalent to $P(N(T) > 0) = P(\tau_1 \leq T) > 0$, and the condition that $E(N(T))^{p+1} < \infty$ for some $p > J_F^+$, is more relaxed than that of Wang **[22]**, namely, $E(1+\delta)^{N(T)} < \infty$ for some $\delta > 0$. In the next main result, we extend the set for T to an infinite set $(0, \infty]$.

Theorem 2.2. *Under the conditions of Theorem* 2.1 *with* $r > 0$ *and* $J_F^- > 0$, *we further assume that the claim inter-arrival times* $\{\theta_i, i \geq 1\}$ *are* WLOD r.v.*s such that*

$$\lim_{n \to \infty} g_L(n) e^{-\epsilon_0 n} = 0 \quad \text{for some } \epsilon_0 > 0, \tag{2.4}$$

and that the total discounted amount of premiums is finite, that is,

$$0 \leq \widetilde{C} = \int_0^\infty e^{-rs} C(ds) < \infty \quad \text{almost surely.} \tag{2.5}$$

Then relation (2.1) *still holds for all* $0 < T \leq \infty$, *if one of the following conditions is valid*:

(i) *the premium process* $\{C(t), t \geq 0\}$ *is independent of* $\{X_i, i \geq 1\}$ *and* $\{N(t), t \geq 0\}$;

(ii) *the total discounted amount of premiums satisfies*

$$P(\widetilde{C} > x) = o(\overline{F}(x)).$$

Remark 2.2. In Theorems 2.1 and 2.2 above, condition (i) has been considered by Wang **[22]**, Yang and Wang **[25]**, Wang et al. **[23]**, Liu et al. **[14]** and many others; while condition (ii), which does not require independence between the premium process and the claim process, allows for a more realistic case that the premium rate varies as a deterministic or stochastic function of the insurer's current reserve, as that considered by Petersen [**16**], Michaud [**15**], Jasiulewicz [**8**], Tang [**18**] and Gao et al. [**5**].

Applying Theorem 2.1, we now put forward a special case when $r = 0$.

Corollary 2.1. *Consider the above risk model with* $r = 0$*, if the other conditions of Theorem* 2.1 *are true, then for fixed* $0 < T < \infty$ *and any* $\alpha > 0$*,*

$$\psi_r(x, T) \sim \overline{F}(x) EN(T) \sim \alpha^{-1} \int_x^{x+\alpha EN(T)} \overline{F}(y)\, dy. \tag{2.6}$$

The remaining part of the paper is divided into two parts. In Section 3 we present some lemmas that are crucial to proving our main results in Section 4.

3. Lemmas. In order to prove the main results, we need the following lemmas, among which the first lemma is a combination of [**1,** Proposition 2.2.1] and [**20,** Lemma 3.5].

Lemma 3.1. *If a distribution* $V \in \mathcal{C}$*, then*:

(i) *for any* $p > J_V^+$*, there are positive constants* C *and* D *such that*

$$\frac{\overline{V}(y)}{\overline{V}(x)} \le C\,(x/y)^p \tag{3.1}$$

holds for all $x \ge y \ge D$*; and, for any* $\widehat{p} < J_V^-$*, there are positive constants* $\widehat{C}$ *and* $\widehat{D}$ *such that*

$$\frac{\overline{V}(y)}{\overline{V}(x)} \ge \widehat{C}\,(x/y)^{\hat{p}} \tag{3.2}$$

holds for all $x \ge y \ge \widehat{D}$;

(ii) *for any* $p > J_V^+$*, the following holds*:

$$x^{-p} = o(\overline{V}(x)). \tag{3.3}$$

The following second lemma will play an important role in proving the main results and is also of its own merit and close to the spirit of **[21,** Proposition 5.1].

Lemma 3.2. *If* $\{\xi_i, 1 \le i \le n\}$ *are* n *real-valued and pairwise quasi-asymptotically independent* r.v.*s with distributions* $V_i \in \mathcal{C}$, $1 \le i \le n$, *respectively, such that*

$$V_i(-x) = o(\overline{V_i}(x)), \quad 1 \le i \le n, \tag{3.4}$$

then for any fixed $0 < a \le b < \infty$,

$$P\biggl(\sum_{i=1}^{n} c_i\xi_i > x\biggr) \sim \sum_{i=1}^{n} P(c_i\xi_i > x) \tag{3.5}$$

holds uniformly for all $\underline{c_n} = (c_1, c_2, \dots, c_n) \in [a, b]^n$, *that is,*

$$\lim_{x\to\infty} \sup_{\underline{c_n}\in[a,b]^n} \left| \frac{P\Bigl(\sum_{i=1}^{n} c_i\xi_i > x\Bigr)}{\sum_{i=1}^{n} P(c_i\xi_i > x)} - 1 \right| = 0.$$

Proof. Relation (3.5) is clear if $n = 1$. Hence, we assume $n \ge 2$. On the one hand, for an arbitrarily fixed $v \in (1/2, 1)$,

$$\begin{aligned} P\biggl(\sum_{i=1}^{n} c_i\xi_i > x\biggr) &\le P\biggl(\bigcup_{i=1}^{n}\{c_i\xi_i > vx\}\biggr) \\ &\quad + P\biggl(\sum_{i=1}^{n} c_i\xi_i > x,\ \bigcap_{i=1}^{n}\{c_i\xi_i \le vx\}\biggr) \\ &= I_1 + I_2. \end{aligned} \tag{3.6}$$

For I_1, by $V_i \in \mathcal{C}$, $1 \le i \le n$, and the arbitrariness of $v \in (1/2, 1)$, we have

$$\begin{aligned} &\lim_{v\nearrow 1} \limsup_{x\to\infty} \sup_{\underline{c_n}\in[a,b]^n} \frac{I_1}{\sum_{i=1}^{n} P(c_i\xi_i > x)} \\ &\quad \le \lim_{v\nearrow 1} \limsup_{x\to\infty} \sup_{\underline{c_n}\in[a,b]^n} \frac{\sum_{i=1}^{n} \overline{V_i}(vx/c_i)}{\sum_{i=1}^{n} \overline{V_i}(x/c_i)} = 1. \end{aligned} \tag{3.7}$$

For I_2, it follows that

$$I_2 = P\left(\sum_{i=1}^{n} c_i\xi_i > x,\ \frac{x}{n} < \max_{1\le k\le n} c_k\xi_k \le vx\right) \tag{3.8}$$
$$\le \sum_{k=1}^{n} P\left(\sum_{i=1,i\neq k}^{n} c_i\xi_i > (1-v)x,\ c_k\xi_k > \frac{x}{n}\right)$$
$$\le \sum_{k=1}^{n}\sum_{i=1,i\neq k}^{n} P\left(c_i\xi_i > \frac{(1-v)x}{n-1},\ c_k\xi_k > \frac{x}{n}\right).$$

Since $1/n > (1-v)/(n-1)$, we obtain from (2.3), (3.8) and $V_i \in \mathcal{C} \subset \mathcal{D}$, $1 \le i \le n$, that, for all $\underline{c_n} \in [a,b]^n$,

$$\lim_{x\to\infty} \frac{I_2}{\sum_{i=1}^{n} P(c_i\xi_i > x)} \le \lim_{x\to\infty} \sum_{k=1}^{n}\sum_{i=1,i\neq k}^{n} \tag{3.9}$$
$$\cdot \frac{P\left(c_i\xi_i > [(1-v)x]/(n-1),\ c_k\xi_k > [(1-v)x]/(n-1)\right)}{P(c_i\xi_i > x) + P(c_k\xi_k > x)}$$
$$\le \lim_{x\to\infty} \sum_{k=1}^{n}\sum_{i=1,i\neq k}^{n}$$
$$\cdot \frac{P(\xi_i > (1-v)/(n-1)(x/b),\ \xi_k > (1-v)/(n-1)(x/b))}{\overline{V_i}([1-v]/[n-1](x/b)) + \overline{V_k}([1-v]/[n-1](x/b))}$$
$$\cdot \frac{\overline{V_i}([1-v]/[n-1](x/b)) + \overline{V_k}([1-v]/[n-1](x/b))}{\overline{V_i}(x/a) + \overline{V_k}(x/a)} = 0.$$

Thus, substituting (3.7) and (3.9) into (3.6) yields that

$$P\left(\sum_{i=1}^{n} c_i\xi_i > x\right) \lesssim \sum_{i=1}^{n} P(c_i\xi_i > x) \tag{3.10}$$

holds uniformly for all $\underline{c_n} \in [a,b]^n$.

On the other hand, for an arbitrarily fixed $w > 1$,

$$
\begin{aligned}
(3.11)\qquad P\Big(\sum_{i=1}^{n} c_i\xi_i > x\Big) &\geq P\Big(\sum_{i=1}^{n} c_i\xi_i > x, \max_{1\leq k\leq n} c_k\xi_k > wx\Big) \\
&\geq \sum_{k=1}^{n} P\Big(\sum_{i=1}^{n} c_i\xi_i > x, c_k\xi_k > wx\Big) \\
&\quad - \sum_{1\leq i<j\leq n} P(c_i\xi_i > wx, c_j\xi_j > wx) \\
&= I_3 - I_4.
\end{aligned}
$$

For I_3, it holds that

$$
\begin{aligned}
(3.12)\qquad I_3 &\geq \sum_{k=1}^{n} P\Big(c_k\xi_k > wx, \sum_{i=1,i\neq k}^{n} c_i\xi_i > (1-w)x\Big) \\
&\geq \sum_{k=1}^{n} P(c_k\xi_k > wx) - \sum_{k=1}^{n}\sum_{i=1,i\neq k}^{n} P\Big(c_i\xi_i < \frac{(1-w)x}{n-1}\Big) \\
&= \sum_{k=1}^{n} P(c_k\xi_k > wx) - I_5.
\end{aligned}
$$

By (3.4) and $V_i \in \mathcal{C} \subset \mathcal{D}$, $1 \leq i \leq n$, we have that, for all $\underline{c_n} \in [a,b]^n$,

$$
\begin{aligned}
\limsup_{x\to\infty} \frac{I_5}{\sum_{i=1}^{n} P(c_i\xi_i > x)} &\leq \limsup_{x\to\infty} \sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{P(c_i\xi_i < (1-w)x/n-1)}{P(c_i\xi_i > x)} \\
&\leq \limsup_{x\to\infty} \sum_{k=1}^{n}\sum_{i=1,i\neq k}^{n} \frac{P(\xi_i < (1-w)/(n-1)(x/b))}{P(\xi_i > (w-1)/(n-1)(x/b))} \\
&\quad \cdot \frac{P(\xi_i > (w-1)/(n-1)(x/b))}{P(\xi_i > x/a)} = 0.
\end{aligned}
$$

This, along with (3.12), leads to

$$
(3.13)\qquad \liminf_{x\to\infty} \frac{I_3}{\sum_{i=1}^{n} P(c_i\xi_i > x)} \geq \liminf_{x\to\infty} \frac{\sum_{k=1}^{n} P(c_k\xi_k > wx)}{\sum_{k=1}^{n} P(c_k\xi_k > x)}.
$$

For I_4, by (2.3) we see that, for all $\underline{c_n} \in [a,b]^n$,

$$\begin{aligned}\limsup_{x\to\infty}&\frac{I_4}{\sum_{i=1}^n P(c_i\xi_i > x)}\\ &\le \limsup_{x\to\infty}\sum_{1\le i<j\le n}\frac{P(c_i\xi_i > wx, c_j\xi_j > wx)}{P(c_i\xi_i > wx)+P(c_j\xi_j > wx)}\\ &\le \limsup_{x\to\infty}\sum_{1\le i<j\le n}\frac{P(\xi_i > wx/b, \xi_j > wx/b)}{\overline{V_i}(wx/b)+\overline{V_j}(wx/b)}\\ &\cdot\frac{\overline{V_i}(wx/b)+\overline{V_j}(wx/b)}{\overline{V_i}(wx/a)+\overline{V_j}(wx/a)}\\ &= 0.\end{aligned} \tag{3.14}$$

From (3.11) to (3.14), it follows that

$$\begin{aligned}\lim_{w\searrow 1}\liminf_{x\to\infty}\inf_{\underline{c_n}\in[a,b]^n}&\frac{P(\sum_{i=1}^n c_i\xi_i > x)}{\sum_{i=1}^n P(c_i\xi_i > x)}\\ &\ge \lim_{w\searrow 1}\liminf_{x\to\infty}\inf_{\underline{c_n}\in[a,b]^n}\frac{\sum_{i=1}^n \overline{V_i}(wx/c_i)}{\sum_{i=1}^n \overline{V_i}(x/c_i)} = 1,\end{aligned}$$

which implies that

$$P\left(\sum_{i=1}^n c_i\xi_i > x\right) \gtrsim \sum_{i=1}^n P(c_i\xi_i > x) \tag{3.15}$$

holds uniformly for all $\underline{c_n} \in [a,b]^n$.

Consequently, we will complete the proof of this lemma by combining (3.10) and (3.15). □

Remark 3.3. Clearly, if $\{\xi_i, 1 \le i \le n\}$ are nonnegative and pairwise quasi-asymptotically independent, then condition (3.4) is satisfied naturally, and it can be canceled for Lemma 3.2.

Lemma 3.3. *Consider the counting process* $\{N(t), t \ge 0\}$ *in* (1.5) *with* WLOD *inter-arrival times* $\{\theta_i, i \ge 1\}$ *satisfying* (2.4). *Then, for any fixed* $T > 0$ *and any* $p > 0$,

$$E(N(T))^p < \infty. \tag{3.16}$$

Proof. Note that, by the Markov inequality and Proposition 2.1,

$$E(N(T))^p \le \sum_{n=1}^{\infty} n^p P(\tau_n \le T) \le e^T \sum_{n=1}^{\infty} n^p E(e^{-\tau_n}) \le e^T \sum_{n=1}^{\infty} g_L(n) n^p \exp\{n \log(Ee^{-\theta_1})\}. \tag{3.17}$$

Applying (2.4) and setting $\epsilon_0 = -\log(Ee^{-\theta_1}) - c$ for some $c > 0$, we can find some $n_0 \ge 0$ such that, for all $n \ge n_0$,

$$g_L(n) \le e^{-cn} \exp\{-n \log(Ee^{-\theta_1})\},$$

which, along with (3.17), leads to

$$E(N(T))^p \le e^T \left(\sum_{n=1}^{n_0-1} g_L(n) n^p (Ee^{-\theta_1})^n + \sum_{n \ge n_0} n^p e^{-cn} \right) < \infty.$$

This ends the proof. $\square$

The lemma below is due to [**22,** Lemma 3.5].

Lemma 3.4. *For the generalized risk model introduced in Section* 1 *with* $EN(T) > 0$, *for any fixed* $0 < T < \infty$, *we have*

$$\sum_{i=1}^{\infty} P(X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le T\}} > x) = \int_0^T \overline{F}(xe^{rt})\, dEN(t).$$

4. Proofs of main results.

Proof of Theorem 2.1. We now proceed to prove Theorem 2.1, and the ideas are inspired by the proof of [**22,** Theorem 2.2]. Recalling the surplus process (1.1), we attain its discounted value as

$$\widetilde{U}_r(t) = e^{-rt} U_r(t) = x + \widetilde{C}(t) - \sum_{i=1}^{N(t)} X_i e^{-r\tau_i}, \quad t \ge 0, \tag{4.1}$$

where $\widetilde{C}(t)$ is defined by (1.2). By the definition (1.3) of the finite-time ruin probability, we have

$$\begin{aligned}\psi_r(x,T) &= P(\widetilde{U}_r(t) < 0 \text{ for some } 0 < t \le T)\\ &= P\Big(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} > x + \widetilde{C}(t) \text{ for some } 0 < t \le T\Big).\end{aligned}$$

Then, it follows that

$$\psi_r(x,T) \le P\Big(\sum_{i=1}^{N(T)} X_i e^{-r\tau_i} > x\Big) \tag{4.2}$$

and

$$\begin{aligned}\psi_r(x,T) &= P\Big(\bigcup_{0<t\le T}\Big\{\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} > x + \widetilde{C}(t)\Big\}\Big)\\ &\ge P\Big(\sum_{i=1}^{N(T)} X_i e^{-r\tau_i} > x + \widetilde{C}(T)\Big).\end{aligned} \tag{4.3}$$

According to the conditions of Theorem 2.1, we conclude that, for any given $\varepsilon > 0$ and any fixed $T > 0$, there exists a positive integer $m_0 = m_0(T,\varepsilon) > 1$ such that

$$\frac{Ce^{rTp} E(N(T))^{p+1} \mathbf{1}_{\{N(T)>m_0\}}}{EN(T)} \le \varepsilon, \tag{4.4}$$

where $C > 0$ and $p > J_F^+$ are two constants as (similar to?)that in (3.1).

On the one hand, we deal with (4.2) to show the upper bound of $\psi_r(x,T)$. Let m_0 be fixed as above. It is easy to see that

$$\begin{aligned}&P\Big(\sum_{i=1}^{N(T)} X_i e^{-r\tau_i} > x\Big)\\ &\quad= \Big(\sum_{n=1}^{m_0} + \sum_{n=m_0+1}^{\infty}\Big) P\Big(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(T) = n\Big) = H_1 + H_2.\end{aligned} \tag{4.5}$$

First consider H_1. Let $G(t_1, t_2, \ldots, t_{n+1})$ be the joint distribution of the random vector $(\tau_1, \tau_2, \ldots, \tau_{n+1})$, where $n = 1, 2, \ldots, m_0$. Thus, by Lemma 3.2 and the independence of $\{X_i, i \geq 1\}$ and $\{N(t), t \geq 0\}$, we derive that there exists an $x_1 = x_1(T, \varepsilon)$ such that, for all $x > x_1$,

$$
\begin{aligned}
H_1 &= \sum_{n=1}^{m_0} \int_{\{0<t_1\le t_2\le\cdots\le t_n\le T, t_{n+1}>T\}} \\
&\quad \cdot P\left(\sum_{i=1}^{n} X_i e^{-rt_i} > x\right) dG(t_1, t_2, \ldots, t_{n+1}) \\
&\le (1+\varepsilon) \sum_{n=1}^{m_0} \sum_{i=1}^{n} \int_{\{0<t_1\le t_2\le\cdots\le t_n\le T, t_{n+1}>T\}} \\
&\quad \cdot P(X_i e^{-rt_i} > x)\, dG(t_1, t_2, \ldots, t_{n+1}) \\
&= (1+\varepsilon) \sum_{n=1}^{m_0} \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x, N(T) = n) \\
&\le (1+\varepsilon) \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} P(X_i e^{-r\tau_i} > x, N(T) = n) \\
&= (1+\varepsilon) \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i} \mathbf{1}_{(\tau_i \le T)} > x) \\
&= (1+\varepsilon) \int_0^T \overline{F}(xe^{rt})\, dEN(t),
\end{aligned}
\tag{4.6}
$$

where, in the last step, we used Lemma 3.4. Next consider H_2, which is divided into two parts as:

$$
\begin{aligned}
H_2 &= \left(\sum_{m_0<n<x/D} + \sum_{n\ge x/D}\right) P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(T) = n\right) \\
&= H_{21} + H_{22},
\end{aligned}
\tag{4.7}
$$

where the constant D is the same one as in (3.1). For H_{21}, for the fixed $T > 0$, it follows from (3.1) and (4.4) that, for all $x \geq D$,

$$
H_{21} \le \sum_{m_0<n<x/D} P\left(\sum_{i=1}^{n} X_i > x\right) P(N(T) = n) \tag{4.8}
$$

$$\begin{aligned}&\le \sum_{m_0<n<x/D} n\overline{F}(x/n)P(N(T)=n)\\&\le C\overline{F}(x)\sum_{m_0<n<x/D} n^{p+1}P(N(T)=n)\\&\le C\overline{F}(x)E(N(T))^{p+1}\mathbf{1}_{\{N(T)>m_0\}}\\&\le \varepsilon\overline{F}(x)\frac{EN(T)}{e^{rTp}}\\&\le C\varepsilon\int_0^T \overline{F}(xe^{rt})\,dEN(t).\end{aligned}$$

For H_{22}, by (3.3) and (4.4), there exists an $x_2 = x_2(\varepsilon)$ such that, for all $x > \max\{D, x_2\}$,

$$\begin{aligned}H_{22}&\le P(N(T)\ge x/D)\\&\le (x/D)^{-p-1}E(N(T))^{p+1}\mathbf{1}_{\{N(T)>m_0\}}\\&\le \varepsilon\,\overline{F}(x)E(N(T))^{p+1}\mathbf{1}_{\{N(T)>m_0\}}\\&\le \varepsilon^2\int_0^T \overline{F}(xe^{rt})\,dEN(t).\end{aligned}\tag{4.9}$$

Therefore, we obtain from (4.5)-(4.9) that, for all $x>x_3=\max\{D,x_1,x_2\}$,

$$\psi_r(x,T)\le (1+\varepsilon+C\varepsilon+\varepsilon^2)\int_0^T \overline{F}(xe^{rt})\,dEN(t).\tag{4.10}$$

On the other hand, we now deal with (4.3) to estimate the lower bound of $\psi_r(x,T)$. Under Theorem 2.1 (i), by conditioning on $\widetilde{C}(T)$ defined in (1.2) and along similar lines to the derivation of H_1, there exists an $x_4 = x_4(T,\varepsilon)$ such that, for all $x > x_4$,

(4.11)

$$\begin{aligned}\psi_r(x,T)\ge&\sum_{n=1}^{m_0}\int_{\{0<t_1\le t_2\le\cdots\le t_n\le T,t_{n+1}>T\}}\int_0^\infty P\Big(\sum_{i=1}^n X_ie^{-rt_i}>x+y\Big)\\&\cdot P(\widetilde{C}(T)\in dy)\,dG(t_1,t_2,\ldots,t_{n+1})\\\ge&(1-\varepsilon)\sum_{n=1}^{m_0}\sum_{i=1}^{n}\int_{\{0<t_1\le t_2\le\cdots\le t_n\le T,t_{n+1}>T\}}\int_0^\infty\end{aligned}$$

$$
\begin{aligned}
&\cdot P(X_i > xe^{rt_i} + ye^{rT})P(\widetilde{C}(T) \in dy)\, dG(t_1, t_2, \ldots, t_{n+1}) \\
&\geq (1-\varepsilon)^2 \sum_{n=1}^{m_0} \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x, N(T) = n) \\
&= (1-\varepsilon)^2 \left(\sum_{n=1}^{\infty} - \sum_{n=m_0+1}^{\infty} \right) \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x, N(T) = n) \\
&= (1-\varepsilon)^2 \int_0^T \overline{F}(xe^{rt})\, dEN(t) - (1-\varepsilon)^2 H_3,
\end{aligned}
$$

where m_0 is defined in (4.4), and the third step is from the dominated convergence theorem and $F \in \mathcal{C} \subset \mathcal{L}$. For H_3, we obtain from (3.1) and (4.4) that, for all $x \geq D$,

$$
\begin{aligned}
H_3 &\leq \sum_{n=m_0+1}^{\infty} \sum_{i=1}^{n} P(X_i > x, N(T) = n) \\
&= \overline{F}(x) \sum_{n=m_0+1}^{\infty} nP(N(T) = n) \\
&= \overline{F}(x) EN(T) \mathbf{1}_{\{N(T) > m_0\}} \\
&\leq \varepsilon \overline{F}(x) \frac{EN(T)}{Ce^{rTp}} \leq \varepsilon \int_0^T \overline{F}(xe^{rt})\, dEN(t),
\end{aligned}
$$

which, together with (4.11), implies that, for all $x \geq x_5 = \max\{x_4, D\}$,

$$
\begin{aligned}
\psi_r(x, T) &\geq (1-\varepsilon)^2 \int_0^T \overline{F}(xe^{rt})\, dEN(t) \\
&\quad - (1-\varepsilon)^2 \varepsilon \int_0^T \overline{F}(xe^{rt})\, dEN(t) \\
&= (1-\varepsilon)^3 \int_0^T \overline{F}(xe^{rt})\, dEN(t).
\end{aligned} \tag{4.12}
$$

Therefore, we conclude from (4.10) and (4.12) that, for all $x \geq \max\{x_3, x_5\}$,

(4.13)

$$
(1-\varepsilon)^3 \int_0^T \overline{F}(xe^{rt})\, dEN(t) \leq \psi_r(x, T)
$$

$$\leq (1+\varepsilon+C\varepsilon+\varepsilon^2)\int_0^T \overline{F}(xe^{rt})\,dEN(t).$$

Under Theorem 2.1 (ii), we see from $F \in \mathcal{C}$ that, for each $t_i > 0$, $i = 1, 2, \ldots, n$, appearing in (4.6),

$$\lim_{l\searrow 0}\liminf_{x\to\infty}\frac{\overline{F}((1+l)xe^{rt_i})}{\overline{F}(xe^{rt_i})} = 1,$$

which means that there exist an $l_0 > 0$ and an $x_6 = x_6(\varepsilon, t_i)$ such that, for all $x \geq x_6$,

$$\overline{F}((1+l_0)xe^{rt_i}) \geq (1-\varepsilon)\overline{F}(xe^{rt_i}). \tag{4.14}$$

It follows from (4.3) that, for $l_0 > 0$ as above,

$$\begin{aligned}\psi_r(x,T) &\geq P\Big(\sum_{i=1}^{N(T)} X_i e^{-r\tau_i} > x + \widetilde{C}(T), \widetilde{C}(T) \leq l_0 x\Big)\\ &\geq P\Big(\sum_{i=1}^{N(T)} X_i e^{-r\tau_i} > (1+l_0)x, \widetilde{C}(T) \leq l_0 x\Big)\\ &\geq P\Big(\sum_{i=1}^{N(T)} X_i e^{-r\tau_i} > (1+l_0)x\Big) - P(\widetilde{C}(T) > l_0 x)\\ &= H_4 - H_5.\end{aligned} \tag{4.15}$$

For H_4, arguing as (4.11) and H_3 and using (4.14) can yield that, for all $x \geq \max\{x_5, x_6\}$,

$$\begin{aligned}H_4 \geq{}& \sum_{n=1}^{m_0}\int_{\{0<t_1\leq t_2\leq\cdots.\leq t_n\leq T, t_{n+1}>T\}}\\ &\cdot P\Big(\sum_{i=1}^{n} X_i e^{-rt_i} > (1+l_0)x\Big)\, dG(t_1, t_2, \ldots, t_{n+1})\\ \geq{}& (1-\varepsilon)\sum_{n=1}^{m_0}\sum_{i=1}^{n}\int_{\{0<t_1\leq t_2\leq\cdots\leq t_n\leq T, t_{n+1}>T\}}\\ &\cdot P(X_i > (1+l_0)xe^{rt_i})dG(t_1, t_2, \ldots, t_{n+1})\end{aligned} \tag{4.16}$$

$$\geq (1-\varepsilon)^2 \sum_{n=1}^{m_0} \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x, N(T) = n)$$
$$\geq (1-\varepsilon)^3 \int_0^T \overline{F}(xe^{rt})\, dEN(t).$$

For H_5, by Theorem 2.1 (ii) and $F \in \mathcal{C} \subset \mathcal{D}$, we get

$$\limsup_{x\to\infty} \frac{H_5}{\overline{F}(x)} = \limsup_{x\to\infty} \frac{P(\widetilde{C}(T) > l_0 x)}{\overline{F}(l_0 x)} \frac{\overline{F}(l_0 x)}{\overline{F}(x)} = 0,$$

from which we derive that there exists an $x_7 = x_7(\varepsilon)$ such that, for all $x \geq \max\{x_7, D\}$,

$$H_5 \leq \varepsilon \overline{F}(x) \leq C_0 \varepsilon \int_0^T \overline{F}(xe^{rt})\, dEN(t), \tag{4.17}$$

where $C_0 = Ce^{rTp}/EN(T)$. Substituting (4.16) and (4.17) into (4.15), we have that, for all $x \geq x_8 = \max\{x_5, x_6, x_7\}$,

$$\psi_r(x, T) \geq ((1-\varepsilon)^3 - C_0\varepsilon) \int_0^T \overline{F}(xe^{rt})\, dEN(t). \tag{4.18}$$

Hence, for all $x \geq \max\{x_3, x_8\}$, it holds that

$$\begin{aligned} \big((1-\varepsilon)^3 - C_0\varepsilon\big) \int_0^T \overline{F}(xe^{rt})\, dEN(t) &\leq \psi_r(x, T) \\ &\leq (1 + \varepsilon + C\varepsilon + \varepsilon^2) \\ &\quad \times \int_0^T \overline{F}(xe^{rt})\, dEN(t). \end{aligned} \tag{4.19}$$

As a result, by using (4.13) and (4.19), and taking into account the arbitrariness of $\varepsilon > 0$, we prove that, for any fixed $0 < T < \infty$, relation (2.1) holds under either of conditions (i) and (ii) of Theorem 2.1. □

Proof of Theorem 2.2. When $0 < T < \infty$, we know from (2.4) and Lemma 3.3 that Theorem 2.2 is a special case of Theorem 2.1, and then it follows immediately from the proof of Theorem 2.1. Hence, in the rest, we only need to prove the case when $T = \infty$. By (1.4) and (4.1), we have

$$\psi_r(x, \infty) = P(\widetilde{U}_r(t) < 0 \text{ for some } 0 \leq t < \infty).$$

Thus, we see that

$$P\Big(\sum_{i=1}^{\infty} X_i e^{-r\tau_i} > x+\widetilde{C}\Big) \le \psi_r(x,\infty) \le P\Big(\sum_{i=1}^{\infty} X_i e^{-r\tau_i} > x\Big), \tag{4.20}$$

where $\widetilde{C}$ is defined by (2.5). According to [**26,** Remark 2], we find that, if $F \in \mathcal{C}$ with $J_F^- > 0$ and $r > 0$, then

$$P\Big(\sum_{i=1}^{\infty} X_i e^{-r\tau_i} > x\Big) \sim \sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i} > x\right) = \int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t). \tag{4.21}$$

This and (4.20) show that, for any given $\varepsilon > 0$, there exists an $x_9 = x_9(\varepsilon)$ such that, for all $x \ge x_9$,

$$\psi_r(x,\infty) \le (1+\varepsilon)\int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t). \tag{4.22}$$

Subsequently, we will analyze the lower bound of $\psi_r(x,\infty)$. For any fixed p and $\widehat{p}$, $p > J_F^+$, $\widehat{p} < J_F^-$, we apply (3.1) and (3.2) to derive that, for all $x \ge \max\{D, \widehat{D}\}$ and all $0 < T < \infty$,

$$\frac{\int_T^{\infty}\overline{F}(xe^{rt})\, dEN(t)}{\int_0^{\infty}\overline{F}(xe^{rt})\, dEN(t)} = \frac{\int_T^{\infty}\overline{F}(xe^{rt})/\overline{F}(x)\, dEN(t)}{\int_0^{\infty}\overline{F}(xe^{rt})/\overline{F}(x)\, dEN(t)} \le \frac{C}{\widehat{C}}\cdot\frac{\int_T^{\infty} e^{-r\hat{p}t}dEN(t)}{\int_0^{\infty} e^{-rpt}dEN(t)}. \tag{4.23}$$

By Proposition 2.1, it follows that

$$\begin{aligned}\int_0^{\infty} e^{-r\hat{p}t}dEN(t) &= \sum_{n=1}^{\infty}\int_0^{\infty} e^{-r\hat{p}t}dP(\tau_n \le t) = \sum_{n=1}^{\infty} E(e^{-r\hat{p}\tau_n})\\ &\le \sum_{n=1}^{\infty} g_L(n)\exp\{n\log(Ee^{-r\hat{p}\theta_1})\}.\end{aligned}$$

Setting $\epsilon_0 = -\log(Ee^{-r\hat{p}\theta_1}) - c$ in (2.4) for some $c > 0$ proves that, there exists some $n_0 \ge 0$ such that, for all $n \ge n_0$,

$$g_L(n) \le e^{-cn}\exp\{-n\log(Ee^{-r\hat{p}\theta_1})\}.$$

Hence, we get that

$$\int_0^{\infty} e^{-r\hat{p}t}dEN(t) \le \sum_{n=1}^{n_0-1} g_L(n)(Ee^{-r\hat{p}\theta_1})^n + \sum_{n\ge n_0} e^{-cn} < \infty. \tag{4.24}$$

Likewise, we also get

$$\int_0^\infty e^{-rpt} dEN(t) < \infty. \tag{4.25}$$

Consequently, combining (4.23), (4.24) and (4.25), the right-hand side of (4.23) tends to 0 as T tends ∞, which yields that, for the given $\varepsilon > 0$, there exists a $0 < T_0 < \infty$ such that

$$\int_{T_0}^\infty \overline{F}(xe^{rt})\, dEN(t) \le \varepsilon \int_0^\infty \overline{F}(xe^{rt})\, dEN(t) \tag{4.26}$$

holds for all $x \ge \max\{D, \widehat{D}\}$. Under Theorem 2.2 (i), by applying (4.20), conditioning on $\widetilde{C}$ and arguing as in (4.11) and (4.12), we obtain that there exists an $x_{10} = \max\{\widehat{D}, x_5\}$ such that, for all $x \ge x_{10}$ and for the fixed $0 < T_0 < \infty$ as that in (4.26),

$$\begin{aligned} \psi_r(x,\infty) &\ge P\left(\sum_{i=1}^{N(T_0)} X_i e^{-r\tau_i} > x + \widetilde{C} \right) \\ &\ge (1-\varepsilon)^3 \int_0^{T_0} \overline{F}(xe^{rt})\, dEN(t) \\ &= (1-\varepsilon)^3 \left(\int_0^\infty - \int_{T_0}^\infty \right) \overline{F}(xe^{rt})\, dEN(t) \\ &= (1-\varepsilon)^4 \int_0^\infty \overline{F}(xe^{rt})\, dEN(t), \end{aligned} \tag{4.27}$$

where, in the last step, we used the inequality (4.26). Thus, combining (4.22) and (4.27), we derive that, for all $x \ge \max\{x_9, x_{10}\}$,

$$(1-\varepsilon)^4 \int_0^\infty \overline{F}(xe^{rt})\, dEN(t) \le \psi_r(x,\infty) \le (1+\varepsilon) \int_0^\infty \overline{F}(xe^{rt})\, dEN(t). \tag{4.28}$$

Under Theorem 2.2 (ii), by using (4.20) and arguing as in the derivation of (4.15), we attain that for $l_0 > 0$ and $0 < T_0 < \infty$ as in (4.14) and (4.26), respectively,

$$\begin{aligned} \psi_r(x,\infty) &\ge P\left(\sum_{i=1}^{N(T_0)} X_i e^{-r\tau_i} > (1+l_0)x \right) - P(\widetilde{C} > l_0 x) \\ &= H_6 - H_7. \end{aligned} \tag{4.29}$$

For H_6, along the same lines of H_4 with T replaced by T_0 can imply that, for all $x \geq x_{11} = \max\{x_6, x_{10}\}$,

$$\begin{aligned} H_6 &\geq (1-\varepsilon)^3 \int_0^{T_0} \overline{F}(xe^{rt})\, dEN(t) \\ &\geq (1-\varepsilon)^4 \int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t), \end{aligned} \tag{4.30}$$

where the last step is from (4.26). For H_7, by a derivation similar to (4.17), Theorem 2.2 (ii) and $F \in \mathcal{C} \subset \mathcal{D}$, there exists an $x_{12} = x_{12}(\varepsilon)$ such that, for all $x \geq x_{12}$,

$$H_7 \leq \varepsilon \overline{F}(x) \leq C_0 \varepsilon \int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t), \tag{4.31}$$

where C_0 is the same as that in (4.17). So, from (4.29)–(4.31), it follows that, for all $x \geq \max\{x_{11}, x_{12}\}$,

$$\psi_r(x, \infty) \geq ((1-\varepsilon)^4 - C_0\varepsilon) \int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t).$$

This, along with (4.22), can show that, for all $x \geq \max\{x_9, x_{11}, x_{12}\}$,

$$\begin{aligned} ((1-\varepsilon)^4 - C_0\varepsilon) \int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t) &\leq \psi_r(x,\infty) \\ &\leq (1+\varepsilon) \int_0^{\infty} \overline{F}(xe^{rt})\, dEN(t). \end{aligned} \tag{4.32}$$

Therefore, combining (4.28) with (4.32) and using the arbitrariness of $\varepsilon > 0$, we prove that, for $T = \infty$, relation (2.1) holds under either of conditions (i) and (ii) of Theorem 2.2, and this ends the proof. □

Proof of Corollary 2.1. Clearly, by Theorem 2.1, we obtain that, for any fixed $0 < T < \infty$,

$$\psi_r(x, T) \sim \overline{F}(x) EN(T).$$

Note that, for fixed $0 < T < \infty$,

$$\begin{aligned} \overline{F}(x)EN(T) &\geq \alpha^{-1} \int_x^{x+\alpha EN(T)} \overline{F}(y)\, dy \\ &\geq \overline{F}(x + \alpha EN(T)) EN(T) \sim \overline{F}(x) EN(T), \end{aligned}$$

where the last step is due to $F \in \mathcal{C} \subset \mathcal{L}$. So, we get relation (2.6) immediately. $\square$

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