ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 5, 2014

ON THE STRUCTURE OF SPLIT INVOLUTIVE LIE ALGEBRAS

ANTONIO J. CALDERÓN MARTÍN AND JOSÉ M. SÁNCHEZ DELGADO

ABSTRACT. We study the structure of arbitrary split involutive Lie algebras. We show that any of such algebras L is of the form $L = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the involutive abelian Lie subalgebra H and any I_j a well described involutive ideal of L satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of L is characterized and it is shown that L is the direct sum of the family of its minimal involutive ideals, each one being a simple split involutive Lie algebra.

1. Introduction and previous definitions.

1.1. Throughout this paper, involutive Lie algebras L of arbitrary dimension and over an arbitrary field $\mathbb{K}$ are considered. It is worthwhile to mention that, unless otherwise stated, there is not any restriction on $\dim L_\alpha$, the products $[L_\alpha, (L_\alpha)^*]$, or $\{k \in \mathbb{K} : k\alpha \in \Lambda\}$, where L_α denotes the root space associated to the root α, and Λ the set of nonzero roots of L.

In Section 2, we develop techniques of connections of roots in the framework of split involutive Lie algebras so as to show that L is of the form $L = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the involutive abelian Lie subalgebra H and any I_j a well-described involutive ideal of L satisfying $[I_j, I_k] = 0$ if $j \neq k$.

In Section 3, and under certain conditions, the simplicity of L is characterized, and it is shown that L is the direct sum of the family of its minimal involutive ideals, each one being a simple split involutive Lie algebra.

2010 AMS *Mathematics subject classification.* Primary 17B65, 17B20, 17B05.
Keywords and phrases. Infinite dimensional Lie algebras, involutive Lie algebras, split Lie algebras, roots, root spaces.
Received by the editors on May 3, 2011, and in revised form on June 11, 2012.
DOI:10.1216/RMJ-2014-44-5-1445

1.2. Let L be a Lie algebra over the base field $\mathbb{K}$, and let $- : \mathbb{K} \to \mathbb{K}$ be an involutive automorphism, (we say $-$ is a conjugation on $\mathbb{K}$). An *involution* on L is a conjugate-linear map, $* : L \to L$, $(x \mapsto x^*)$, such that $(x^*)^* = x$ and $[x, y]^* = [y^*, x^*]$ for any $x, y \in L$. A Lie algebra endowed with an involution is an *involutive Lie algebra*. An *involutive subset* of an involutive algebra is a subset globally invariant by the involution. We say that L is *simple* if the product is nonzero and its only ideals are $\{0\}$ and L. From now on, $(L, *)$ denotes an involutive Lie algebra.

1.3. Let us introduce the class of split algebras in the framework of involutive Lie algebras. Denote by H a maximal involutive abelian subalgebra of L. For a linear functional commuting with the involution

$$\alpha : (H, *) \longrightarrow (\mathbb{K}, -),$$

that is, $\alpha(h^*) = \overline{\alpha(h)}$ for any $h \in H$, we define the *root space* of L, (with respect to H), associated to α as the subspace

$$L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)v_\alpha \quad \text{for any } h \in H\}.$$

The elements $\alpha : (H, *) \to (\mathbb{K}, -)$ satisfying $L_\alpha \neq 0$ are called *roots* of L with respect to H, and we denote $\Lambda := \{\alpha : (H, *) \to (\mathbb{K}, -) : L_\alpha \neq 0\}$.

We say that L is a *split involutive Lie algebra*, with respect to H, if

$$L = H \oplus \left(\bigoplus_{\alpha \in \Lambda} L_\alpha \right).$$

We also say that Λ is the *root system* of L. Observe that, taking into account $H^* = H$, the root space associated to the zero root L_0 is contained in H.

As examples of split involutive Lie algebras, we have the L^*-algebras **[2, 4, 5]** and the involutive Lie algebras with a Cartan decomposition considered in **[1]**.

Lemma 1.1. *For any $\alpha, \beta \in \Lambda \cup \{0\}$ the following assertions hold.*

(i) *If $[L_\alpha, L_\beta] \neq 0$ then $\alpha + \beta \in \Lambda \cup \{0\}$ and $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.*
(ii) $(L_\alpha)^* = L_{-\alpha}$.

Proof.

(i) It is an immediate consequence of the Jacobi identity.

(ii) For any $h \in H$ and $v_\alpha \in L_\alpha$, we have $[h, v_\alpha]^* = (\alpha(h)v_\alpha)^* = \overline{\alpha(h)}v_\alpha^*$ .From here $[h^*, v_\alpha^*] = -\overline{\alpha(h)}v_\alpha^* = -\alpha(h^*)v_\alpha^*$. The facts $H^* = H$ and $*^2 = *$ conclude the proof. □

A subset Λ_0 of Λ is called a *root subsystem* if $\alpha \in \Lambda_0$ implies $-\alpha \in \Lambda_0$ and if $\alpha, \beta \in \Lambda_0$, $\alpha + \beta \in \Lambda$, then necessarily $\alpha + \beta \in \Lambda_0$. For a root subsystem Λ_0 of Λ, we define $H_{\Lambda_0} := \mathrm{span}_{\mathbb{K}}\{[L_\alpha, (L_\alpha)^*] : \alpha \in \Lambda_0\}$ and $V_{\Lambda_0} := \bigoplus_{\alpha \in \Lambda_0} L_\alpha$. It is straightforward to verify that $L_{\Lambda_0} := H_{\Lambda_0} \oplus V_{\Lambda_0}$ is an involutive Lie subalgebra of L that we call the involutive Lie subalgebra *associated* to the root subsystem Λ_0.

2. Connections of roots. Decompositions.

In the following, L denotes a split involutive Lie algebra with $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ the corresponding root spaces decomposition. We begin by developing connections of roots techniques in this setting.

Definition 2.1. Let α and β be two nonzero roots. We say that α is *connected* to β if there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$$

is a family of nonzero roots, $\alpha_1 = \alpha$ and $\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \in \pm\beta$. We also say that $\{\alpha_1, \ldots, \alpha_n\}$ is a *connection* from α to β.

The next result shows the connection relation is of equivalence.

Proposition 2.2. *The relation $\sim$ in Λ defined by $\alpha \sim \beta$ if and only if α is connected to β is of equivalence.*

Proof. $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$.

Let us see the symmetric character of $\sim$. If $\alpha \sim \beta$, there exists a connection

$$\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n\} \subset \Lambda$$

from α to β. Then

$$\{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\alpha_2+\dots+\alpha_{n-1}, \alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n\} \subset \Lambda, \tag{1}$$

$\alpha_1 = \alpha$ and $\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n \in \{\beta, -\beta\}$. Hence, we can distinguish two possibilities. In the first one,

$$\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n, = \beta \tag{2}$$

and in the second one,

$$\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n = -\beta. \tag{3}$$

Suppose we have the first one. By the symmetry of Λ, we can consider the set of nonzero roots

$$\{\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n, -\alpha_n, -\alpha_{n-1}, \dots, -\alpha_3, -\alpha_2\} \subset \Lambda.$$

By equation (1), this family of elements in Λ clearly satisfies

$$\begin{gathered}
\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n \in \Lambda, \\
\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n-\alpha_n = \alpha_1+\alpha_2+\dots+\alpha_{n-1} \in \Lambda, \\
\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n-\alpha_n-\alpha_{n-1} = \alpha_1+\alpha_2+\dots+\alpha_{n-2} \in \Lambda, \\
\vdots \\
\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n-\alpha_n-\alpha_{n-1}\dots-\alpha_3-\alpha_2 = \alpha_1 \in \Lambda,
\end{gathered}$$

since also $\alpha_1+\alpha_2+\dots+\alpha_{n-1}+\alpha_n = \beta$ (by equation (2)), and $\alpha_1 = \alpha$. From here, β is connected to α, that is, $\beta \sim \alpha$.

Suppose now we are in the second possibility given by equation (3). In this case, we have as above that $\{-\alpha_1-\alpha_2-\dots-\alpha_{n-1}-\alpha_n, \alpha_n, \alpha_{n-1}, \dots, \alpha_2\}$ is a connection from β to α and $\sim$ is symmetric.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, and write $\{\alpha_1, \dots, \alpha_n\}$ for a connection from α to β and $\{\beta_1, \dots, \beta_m\}$ for a connection from β to γ. If $m > 1$, then $\{\alpha_1, \dots, \alpha_n, \beta_2, \dots, \beta_m\}$ is a connection from α to γ in case $\alpha_1+\dots+\alpha_n = \beta$, and $\{\alpha_1, \dots, \alpha_n, -\beta_2, \dots, -\beta_m\}$ in the case $\alpha_1+\dots+\alpha_n = -\beta$. If $m = 1$, then $\gamma \in \pm\beta$, and so $\{\alpha_1, \dots, \alpha_n\}$ is a connection from α to γ. Therefore, $\alpha \sim \gamma$ and $\sim$ is of equivalence. $\square$

We denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \beta \sim \alpha\}.$$

Let us observe that $\{\alpha\}$ is a connection from α to $-\alpha$. So $-\alpha \in \Lambda_\alpha$.

Proposition 2.3. *Let $\alpha \in \Lambda$. Then the following assertions hold*:

(i) *Λ_α is a root subsystem.*
(ii) *If $\gamma \in \Lambda$ satisfies that $\gamma \notin \Lambda_\alpha$, then $[L_\beta, L_\gamma] = 0$ and $[[L_\beta, (L_\beta)^*], L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$.*

Proof.

(i) Given $\beta \in \Lambda_\alpha$, there exists a connection $\{\alpha_1, \dots, \alpha_n\}$ from α to β. It is clear that $\{\alpha_1, \dots, \alpha_n\}$ also connects α to $-\beta$, and therefore $-\beta \in \Lambda_\alpha$. Given $\beta, \delta \in \Lambda_\alpha$ such that $\beta + \delta \in \Lambda$, there exists a connection $\{\alpha_1, \dots, \alpha_n\}$ from α to β. Hence, $\{\alpha_1, \dots, \alpha_n, \delta\}$ is a connection from α to $\beta + \delta$ in the case $\alpha_1 + \cdots + \alpha_n = \beta$ and $\{\alpha_1, \dots, \alpha_n, -\delta\}$ in the case $\alpha_1 + \cdots + \alpha_n = -\beta$. So $\beta + \delta \in \Lambda_\alpha$.
(ii) Let us suppose that there exists $\beta \in \Lambda_\alpha$ such that $[L_\beta, L_\gamma] \neq 0$ with $\gamma \notin \Lambda_\alpha$. Then $\beta + \gamma \in \Lambda$ and we have as in (i) that α is connected to $\beta + \gamma$. Since Λ_α is a root subsystem, then $\gamma \in \Lambda_\alpha$, a contradiction. Therefore, $[L_\beta, L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. As $-\beta \in \Lambda_\alpha$ for any $\beta \in \Lambda_\alpha$, we also have that $[(L_\beta)^*, L_\gamma] = [L_{-\beta}, L_\gamma] = 0$. By applying the Jacobi identity, we obtain $[[L_\beta, (L_\beta)^*], L_\gamma] = 0$. ☐

Theorem 2.4. *The following assertions hold*:

(i) *For any $\alpha \in \Lambda$, the involutive Lie subalgebra*

$$L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$$

of L associated to the root subsystem Λ_α is an (*involutive*) *ideal of L.*
(ii) *If L is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$.*

Proof.

(i) We have by Proposition 2.3 that $[L_\beta, L_\gamma] = 0$ and that $[[L_\beta, (L_\beta)^*], L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. As we also have Λ_α is a root subsystem, we get

$$[L_{\Lambda_\alpha}, L] = \Bigg[\bigoplus_{\beta\in\Lambda_\alpha}[L_\beta,(L_\beta)^*] \oplus \bigoplus_{\beta\in\Lambda_\alpha} L_\beta,$$
$$H \oplus \Bigg(\bigoplus_{\beta\in\Lambda_\alpha} L_\beta\Bigg) \oplus \Bigg(\bigoplus_{\gamma\notin\Lambda_\alpha} L_\gamma\Bigg)\Bigg] \subset L_{\Lambda_\alpha}.$$

(ii) The simplicity of L implies $L_{\Lambda_\alpha} = L$, and therefore $\Lambda_\alpha = \Lambda$. □

Theorem 2.5. *For a vector space complement* $\mathcal{U}$ *of* $\mathrm{span}_{\mathbb{K}}\{[L_\alpha,(L_\alpha)^*] : \alpha\in\Lambda\}$ *in* H, *we have*

$$L = \mathcal{U} + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ *is one of the involutive ideals* L_{Λ_α} *of* L *described in Theorem* 2.4 (i), *satisfying* $[I_{[\alpha]}, I_{[\beta]}] = 0$ *if* $[\alpha]\neq[\beta]$.

Proof. By Proposition 2.2, we can consider the quotient set $\Lambda/\sim := \{[\alpha] : \alpha\in\Lambda\}$. Let us denote by $I_{[\alpha]} := L_{\Lambda_\alpha}$. We have $I_{[\alpha]}$ is well defined and, by Theorem 2.4 (i) an involutive ideal of L. Therefore,

$$L = \mathcal{U} + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

By applying Proposition 2.3 (ii) we also obtain $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha]\neq[\beta]$. □

Let us denote by $\mathcal{Z}(L)$ the center of L.

Corollary 2.6. *If* $\mathcal{Z}(L) = 0$ *and* $[L, L] = L$, *then* L *is the direct sum of the involutive ideals given in Theorem* 2.4,

$$L = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Proof. From $[L, L] = L$, it is clear that $L = \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. The direct character of the sum now follows from the facts $[I_{[\alpha]}, I_{[\beta]}] = 0$, if $[\alpha]\neq[\beta]$, and $\mathcal{Z}(L) = 0$. □

3. The simple components. In this section, we study if any of the components in the decomposition given in Corollary 2.6 is simple. Under certain conditions, we give an affirmative answer. From now on, $\mathrm{char}\,(\mathbb{K}) = 0$.

Lemma 3.1. *Let $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$ be a split Lie algebra. If I is an ideal of L then $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha))$.*

Proof. We can see $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$ as a weight module with respect to the split Lie algebra L, with maximal abelian subalgebra H, in the natural way. The character of the ideal of I gives us that I is a submodule of L. It is well known that a submodule of a weight module is again a weight module. From here, I is a weight module respect to L, (and H), and so $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha))$. □

Lemma 3.2. *Let L be a split Lie algebra with $\mathcal{Z}(L) = 0$. Then there is no nonzero ideal of L contained in H.*

Proof. Suppose there exists a nonzero ideal I of L such that $I \subset H$. We have $[I, H] \subset [H, H] = 0$. We also have that the fact $[I, \bigoplus_{\alpha\in\Lambda} L_\alpha] \subset I \subset H$ implies $\alpha(I) = 0$ for any $\alpha \in \Lambda$, and so $[I, \bigoplus_{\alpha\in\Lambda} L_\alpha] = 0$. From here, $I \subset \mathcal{Z}(L) = 0$, a contradiction. □

Definition 3.3. We say that a split Lie algebra L is *root-multiplicative* if $\alpha, \beta \in \Lambda$ are such that $\alpha + \beta \in \Lambda$. Then $[L_\alpha, L_\beta] \neq 0$.

As examples of root-multiplicative split involutive Lie algebras we have the semisimple separable L^*-algebras and the semisimple locally finite involutive split Lie algebras over a field of characteristic zero. Indeed, as we can take a locally finite involutive split subalgebra dense in any L^*-algebra [**2, 4, 5**], it is enough to consider a semisimple locally finite involutive split Lie algebra $\mathcal{L}$, but it is well known that, in any of such algebras, if $\alpha, \beta, \alpha + \beta \in \Lambda$, then $[\mathcal{L}_\alpha, \mathcal{L}_\beta] = \mathcal{L}_{\alpha+\beta}$, (see [**6,** Proposition I.7 (v) and Theorem III.19]), and so $\mathcal{L}$ is a root-multiplicative involutive split Lie algebra.

Following the terminology of the theory of graduations on Lie algebras, we say that an involutive split Lie algebra L is of *maximal length* if $\dim L_\alpha = 1$ for any $\alpha \in \Lambda$. Observe that if L is of maximal length, then Lemma 3.1 lets us assert that, given any nonzero ideal I of L, then

$$I = (I \cap H) \oplus \left(\bigoplus_{\alpha \in \Lambda_I} L_\alpha \right) \quad \text{where } \Lambda_I \subset \Lambda. \tag{4}$$

As examples of involutive Lie algebras of maximal length we have the involutive Lie algebras considered in [**1**]. Let us see a couple of more detailed examples.

Example 3.4. Consider the (non simple) Lie algebra $L = \mathfrak{gl}(I, \mathbb{C})$ of all of the $I \times I$-matrices with only finitely many nonzero entries, I being an arbitrary non-empty set. Now define the involution

$$\begin{aligned} * : \mathfrak{gl}(I, \mathbb{C}) &\longrightarrow \mathfrak{gl}(I, \mathbb{C}) \\ (x_{jk})^*_{j,k \in I} &= (\overline{x}_{kj})_{j,k \in I}, \end{aligned}$$

which makes of $\mathfrak{gl}(I, \mathbb{C})$ an involutive Lie algebra.

For any $(j, k) \in I \times I$, we denote by

$$\begin{aligned} E_{jk} : I \times I &\longrightarrow \mathbb{C} \\ (l, m) &\longmapsto \delta_{jl}\delta_{km} \end{aligned}$$

the *matrix units*. Then, by defining $H = \mathrm{span}_{\mathbb{C}}\{E_{jj} : j \in I\}$, we have that H is a maximal involutive abelian subalgebra of $\mathfrak{gl}(I, \mathbb{C})$. By also defining

$$\begin{aligned} \varepsilon_k : H &\longrightarrow \mathbb{C} \\ E_{jj} &\longmapsto \delta_{jk}, \end{aligned}$$

we can consider the set of nonzero roots, with respect to H, given by

$$\Lambda = \{\varepsilon_j - \varepsilon_k : k \in I,\ j \neq k\},$$

with associated root spaces given by $L_{\varepsilon_j - \varepsilon_k} = \mathbb{C}E_{jk}$. Then we have that $\mathfrak{gl}(I, \mathbb{C})$ is an involutive split Lie algebra of maximal length, with a symmetric root system and root-multiplicative.

Example 3.5. Consider now the (simple) Lie algebra $L = \mathfrak{sl}(I, \mathbb{C})$ of all of the $I \times I$-matrices in $\mathfrak{gl}(I, \mathbb{C})$ with trace zero. By defining the involution $*$ as in Example 3.4, we have that $\mathfrak{sl}(I, \mathbb{C})$ becomes an involutive split Lie algebra of maximal length, with a symmetric root system and root-multiplicative, by taking as maximal involutive abelian subalgebra

$$H := \{E_{jj} - E_{kk} : j,k \in I,\ j \neq k\}$$

and where the set of nonzero roots is $\Lambda := \{\varepsilon_j - \varepsilon_k : j,k \in I,\ j \neq k\}$ with $\varepsilon_k(\sum_{j\in I} x_j E_{jj}) = x_k$.

Definition 3.6. A Lie algebra $\mathcal{L}$ is called *perfect* if $\mathcal{Z}(\mathcal{L}) = 0$ and $[\mathcal{L},\mathcal{L}] = \mathcal{L}$.

Proposition 3.7. *Let L be a perfect split involutive Lie algebra of maximal length and root-multiplicative. If L has all its nonzero roots connected, then any ideal I of L satisfies $I^* = I$.*

Proof. Consider I a nonzero ideal of L. By Lemma 3.2 and equation (4) we can write $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda_I} L_\alpha)$ with $\Lambda_I \subset \Lambda$ and $\Lambda_I \neq \emptyset$. Consider any $\alpha_0 \in \Lambda_I$ since $L_{\alpha_0} \subset I$. Let us show that $(L_{\alpha_0})^* \subset I$. Since $\alpha_0 \neq 0$, and taking into account that the facts $L = [L,L]$ and Corollary 2.6 imply $H = \sum_{\beta\in\Lambda}[L_\beta,(L_\beta)^*]$, we have that there exists $\beta \in \Lambda$ satisfying $\alpha_0([L_\beta,(L_\beta)^*]) \neq 0$. The maximal length of L now gives us that

$$[[L_\beta,(L_\beta)^*],L_{\alpha_0}] = L_{\alpha_0}. \tag{5}$$

If $\beta \in \pm\alpha_0$, we have as consequences of $(L_{-\alpha_0})^* = L_{\alpha_0}$ and equation (5) that $(L_{\alpha_0})^* = [(L_{\alpha_0})^*,[L_\beta,(L_\beta)^*]] \subset I$. If $\beta \notin \pm\alpha_0$, as α_0 and β are connected, the root-multiplicativity and the maximal length of L give us a connection $\{\gamma_1,\dots,\gamma_r\}$ from α_0 to β such that $\gamma_1 = \alpha_0$, $\gamma_1+\gamma_2,\dots,\gamma_1+\gamma_2+\cdots+\gamma_r \in \Lambda$, $\gamma_1+\gamma_2+\cdots+\gamma_r \in \pm\beta$ and

$$[L_{\gamma_1},L_{\gamma_2}] = L_{\gamma_1+\gamma_2}, [[L_{\gamma_1},L_{\gamma_2}],L_{\gamma_3}] = L_{\gamma_1+\gamma_2+\gamma_3},\dots,$$
$$[[\dots[[L_{\gamma_1},L_{\gamma_2}],L_{\gamma_3}],\dots],L_{\gamma_r}] = L_{\varepsilon\beta},$$

with $\varepsilon \in \pm1$. From here, we deduce that either $L_\beta \subset I$ or $L_{-\beta} = (L_\beta)^* \subset I$. In both cases,

$$[L_\beta,(L_\beta)^*] \subset I \tag{6}$$

and, as by equation (5) we have $(L_{\alpha_0})^* = [(L_{\alpha_0})^*,[L_\beta,(L_\beta)^*]]$, then we get $(L_{\alpha_0})^* \subset I$. Hence, $(\bigoplus_{\alpha\in\Lambda_I} L_\alpha)^* = \bigoplus_{\alpha\in\Lambda_I} L_\alpha$. Finally, the fact

$H = \sum_{\beta \in \Lambda}[L_\beta, (L_\beta)^*]$ and equation (6) give us

$$H \subset I. \tag{7}$$

As $H^* = H$, we get, in particular, $(I \cap H)^* = I \cap H$. From here, and taking into account $(\bigoplus_{\alpha \in \Lambda_I} L_\alpha)^* = \bigoplus_{\alpha \in \Lambda_I} L_\alpha$, equation (4) lets us conclude $I^* = I$. □

Theorem 3.8. *Let L be a perfect split involutive Lie algebra, root-multiplicative and of maximal length. Then L is simple if and only if it has all its nonzero roots connected.*

Proof. The first implication is Theorem 2.4 (ii). To prove the converse, write $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ and consider I a nonzero ideal of L. By equation (7), we have $H \subset I$. Given any $\alpha \in \Lambda$ and taking into account $\alpha \neq 0$ and the maximal length of L, we have $[H, L_\alpha] = L_\alpha$ and so $L_\alpha \subset I$. We conclude $I = L$, and therefore L is simple. □

Theorem 3.9. *Let L be a perfect split involutive Lie algebra, root-multiplicative and of maximal length. Then L is the direct sum of the family of its minimal ideals, each one a simple split involutive Lie algebra having all its nonzero roots connected.*

Proof. By Corollary 2.6, $L = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$ is the direct sum of the involutive ideals $I_{[\alpha]} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha} = (\sum_{\beta \in [\alpha]}[L_\beta, (L_\beta)^*)]) \oplus (\bigoplus_{\beta \in [\alpha]} L_\beta)$ having any $I_{[\alpha]}$ as its root system, Λ_α, with all of its roots connected. Taking into account that $\Lambda_\alpha = [\alpha]$ is a root subsystem, we have that Λ_α has all of its roots Λ_α-connected, (connected through roots in Λ_α). We also have that any of the $I_{[\alpha]}$ is root-multiplicative as a consequence of the root-multiplicativity of L. Clearly, $I_{[\alpha]}$ is of maximal length, and finally $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]}) = 0$, (where $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$), as a consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$, (Corollary 2.6), and $\mathcal{Z}(L) = 0$; and, also, $I_{[\alpha]} = [I_{[\alpha]}, I_{[\alpha]}]$, by the facts $L = [L, L]$ and $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. We can apply Theorem 3.8 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $L = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. □

Remark 1. Finally, we would like to know that our results in Proposition 3.7 and Theorems 3.8 and 3.9 are also of interest in the framework of complete Lie algebras, that is, those Lie algebras $\mathcal{L}$ satisfying that their center are zero and all of their derivations are inner. A class of split Lie algebras we are considering is an example of complete Lie algebras, see **[3]**, and so our results give us a description of the structure of a particular (involutive) class of complete Lie algebras.

Acknowledgments. We would like to thank the referee for his detailed reading of this work and for his suggestions which have improved the final version of the same.

REFERENCES

1. A.J. Calderón, *On involutive Lie algebras having a Cartan decomposition*, Bull. Austr. Math. Soc. **69** (2004), 191–202.

2. J.A. Cuenca, A. García and C. Martín, *Structure theory for L^*-algebras*, Math-Proc. Camb. Phil. Soc. **107** (1990), 361–365.

3. B. Hsie, *On complete Lie algebras*, Comm. Alg. **34** (2006), 3743-3750.

4. J.R. Schue, *Hilbert space methods in the theory of Lie algebras*, Trans. Amer. Math. Soc. **95** (1960), 69–80.

5. ———, *Cartan decompositions for L^*-algebras*, Trans. Amer. Math. Soc. **98** (1961), 334–349.

6. N. Stumme, *The structure of locally finite split Lie algebras*, J. Alg. **220** (1999), 664–693.

Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain
Email address: **ajesus.calderon@uca.es**

Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain
Email address: **txema.sanchez@uca.es**