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EXPLICIT CONSTRUCTIONS FOR GENUS 3 JACOBIANS

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ABSTRACT. Given a genus 3 canonical curve $X = \{F = 0\}$ we derive a set of equations for an open affine set of the Jacobian $J(X)$. The law group on the Jacobian is also explicitly constructed and, as an application, a set of equations for Kummer's variety $K(X)$ is obtained.

1. Introduction. The aim of this note is to construct, in the spirit of [**4**], a set of explicit equations for an affine open set of the Jacobian variety of a canonical genus three curve X, defined by a quartic equation $F = 0$. We develop in detail the case when X has a hyper-flex point, although the construction is valid for any genus 3 non-hyperelliptic curve. By a *hyper-flex*, we understand a point $p \in X$ such that its tangent line intersects X of order 4 at p. Our ground field is $\mathbf{C}$, but the results seem to be valid in any algebraically closed field of characteristic different from 2 or 3.

The idea is to work with a subset of degree 3 non-special effective divisors and associate it with a pencil of conics. Not only can an open set of JX be described, but also the multiplication by -1 and the law of the group (even when not explicitly described in terms of affine coordinates).

The construction comes from a standard argument involving resolution of locally free sheaves on plane non-singular curves. If $\mathcal{F}$ is an invertible sheaf on the non-singular quartic $X = \{F = 0\}$, then $\mathcal{F}$ is a Cohen-Macaulay $\mathcal{O}_{\mathbf{P}^2}$-module and, therefore, we have a minimal resolution:

$$0 \longrightarrow L_1 \xrightarrow{\alpha} L_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

with L_1 and L_0 of the same rank and $\det \alpha = \lambda F$, $\lambda \in \mathbf{C}$, just because the vanishing of this determinant corresponds to the geometrical locus

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where the map α is not surjective. In particular, the sum of the degrees of the diagonal entries of α must equal 4.

Following **[1]** (Proposition 3.5 and Example 3.7) we have:

Proposition 1.1. *Let $X = \{F = 0\}$ be a non-singular quartic on $\mathbf{P}^2$. Let D be a zero degree, non-principal divisor on X. Then $\mathcal{O}_X(D)$ admits a minimal resolution*:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-3)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^2}(-1)^{\oplus 2} \longrightarrow \mathcal{O}_X(D) \longrightarrow 0.$$

In this way, $\det \alpha = 0$ gives rise to a decomposition $F = AG - BH$ for conics (A, B, G, H). This expression is uniquely determined, as is the minimal resolution, by D up to the actions of $GL(2)$ on $\mathcal{O}_{\mathbf{P}^2}(-3)^{\oplus 2}$ and $\mathcal{O}_{\mathbf{P}^2}(-1)^{\oplus 2}$. In conclusion, there is a one-to-one correspondence between non-principal degree zero divisors on X and conics (A, B, C, D) satisfying $F = AG - BH$, modulo the action of $GL(2) \times GL(2)$ given by:

$$(A_0, B_0) \cdot \begin{pmatrix} A & B \\ H & G \end{pmatrix} = A_0^{-1} \begin{pmatrix} A & B \\ H & G \end{pmatrix} B_0.$$

In Section 2 we explicitly describe a quotient for this action in an open set of the space of ordered quadruples of conics. More specifically, we construct an affine variety $Z \subset \mathbf{P}^{17}$ that corresponds bijectively with an open subset of $\operatorname{Pic}^3(X)$. Z is described by means of all the possible decompositions $F = AG - BH$ for a certain class of conics A and B. We prove that this correspondence is in fact an isomorphism of algebraic varieties.

In Section 3 we apply this construction to describing multiplication by (-1) in JX through a simple geometric construction, proving that Z is invariant under multiplication by (-1) and explicitly writing the morphism:

$$Z \xrightarrow{-1} Z.$$

This description allows us to obtain equations for the open set $Z/\{\pm 1\}$ of the Kummer variety $K(X)$. We describe geometrically how to obtain the sum of two divisors $D - 3\infty$, $D' - 3\infty \in JX$.

All of the paper is quite elementary and only requires the basic definitions and facts from algebraic curve theory (references **[2, 3]** contain all the necessary material).

2. Explicit construction of genus 3 Jacobians. Let $X = \{F = 0\}$ be a plane genus 3 curve defined by the non-singular homogeneous quartic polynomial F. We assume, in order to simplify the discussion, that X has a hyper-flex point. By the end of the section we shall indicate how this hypothesis can be removed.

Fix, once and for all, a set of homogeneous coordinates in $\mathbf{P}^2$ in such a way that the only point of X at the infinity line $z = 0$ is $(0 : 1 : 0)$. This point will be denoted by ∞. Thus, ∞ is a hyper-flex point: $X.\{z = 0\} = 4\infty$.

Denote by $\mathrm{Div}^{3,+}(X)$ the set of degree 3 effective divisors on X, and define the following subset:

$$\begin{aligned}\mathrm{Div}_0^{3,+}(X) := \{D = p_1 + p_2 + p_3 \in \mathrm{Div}^{3,+}(X) \mid \\ h^1(X, \mathcal{O}(D)) = 0,\ \text{and } \infty \neq p_i\} \\ \cap \{D \in \mathrm{Div}^{3,+}(X) \mid p_i \neq \infty \\ \text{and } h^1(X, \mathcal{O}(p_i + p_j + \infty)) = 0 \quad \text{for all } i \neq j\}.\end{aligned}$$

Note that, as the canonical divisor K_X is cut out by the linear system of lines in $\mathbf{P}^2$, the above conditions say geometrically that ∞ is not in support of our divisors and neither the three points in the support, nor the two points and ∞ are collinear.

Moreover, the condition $h^1(X, \mathcal{O}(D)) = 0$ allows us to identify $\mathrm{Div}_0^{3,+}(X)$ with a subset of $\mathrm{Pic}^3(X)$. Indeed, by Riemann-Roch, any divisor $D \in \mathrm{Div}_0^{3,+}(X)$ satisfies $h^0(X, \mathcal{O}(D)) = 1$.

The next theorem explains our basic construction. To any $D \in \mathrm{Div}_0^{3,+}(X)$ we associate the pencil of conics cutting X out in the divisor $D + \infty$. Our choice of coordinates gives an explicit form for a basis of this pencil. These conics are naturally related to the minimal resolution alluded to in Section 2.

Theorem 2.1. *There exists a bijection between the sets* $\mathrm{Div}_0^{3,+}(X)$ *and*

$$Z = \{(A, B, G, H) \in \left(H^0(\mathcal{O}_{\mathbf{P}^2}(2))\right)^{\oplus 4} \mid F = AG - BH\},$$

with A and B of the following particular form in affine coordinates:

$$A = a_{00} + a_{10}x + a_{01}y - x^2,$$
$$B = b_{00} + b_{10}x + b_{01}y - xy,$$

and H satisfying that its coefficient h_{20} corresponding to the monomial x^2 equals 0.

Proof. Start with a non-principal divisor $D' = p_1 + p_2 + p_3 - 3\infty$. Sections of $\mathcal{O}_X(-D')$ are identified with locally defined functions having zeroes on p_i, $i = 1, 2, 3$ and a pole of order at most 3 at ∞. If $h^1(X, \mathcal{O}_X(p_1 + p_2 + p_3 + \infty)) = 0$, then the space of conics C satisfying that $X.C \geq p_1 + p_2 + p_3 + \infty$ form a pencil (note that, if $D = p_1+p_2+p_3 \in \mathrm{Div}_0^{3,+}(X)$, this condition is automatically satisfied). In this way, if A and B are a basis of this pencil, the map:

$$(G, H) \longrightarrow \left.\frac{AG - BH}{L}\right|_X,$$

with $L = \{z = 0\}$ and G and H locally regular functions, is surjective and gives the first step of a resolution for $\mathcal{O}_X(-D')$. The kernel of this map corresponds to relations of the form $AG - BH|_X = 0$, and we obtain a resolution as prescribed by Proposition 1.1. The existence of conics G, H satisfying this condition is guaranteed by Noether's theorem.

As explained in the introduction, the correspondence assigning (A, B) to the divisor D' is in general not well defined, as these conics are not uniquely determined by the construction. In order to get a well-defined bijective map, we restrict ourselves to divisors in $\mathrm{Div}_0^{3,+}(X)$.

Given $D \in \mathrm{Div}_0^{3,+}(X)$, we construct the pair of conics A, B considering the pencil of conics C satisfying $C.X \geq D + \infty$. In this pencil, we fix a basis; namely, that formed by the only conic A in the system having as tangent line at ∞ the line $\{z = 0\}$ and the conic B in the system defined by the condition $B(1 : 0 : 0) = 0$.

Sometimes, in order to emphasize the dependence of A and B on D, we write A_D, B_D instead of A and B.

These conics can be constructed in an explicit way. For instance, if $p_i \neq p_j$, for $i \neq j$, write $p_i = (x_i, y_i, 1)$ (recall that $p_i \neq \infty$). Consider the matrix:

$$M_D = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}. \tag{1}$$

This matrix is invertible because $h^1(X, \mathcal{O}(D)) = 0$. Thus, the systems:

$$M_D \begin{pmatrix} a_{00} \\ a_{10} \\ a_{01} \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix},$$

and

$$M_D \begin{pmatrix} b_{00} \\ b_{10} \\ b_{01} \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ x_3 y_3 \end{pmatrix},$$

have unique solutions. The conics having the previous solutions as coefficients are precisely A and B.

This construction made, we have the following properties:

Lemma 2.2. *For any* $D \in \mathrm{Div}_0^{3,+}(X)$, *the conics* A *and* B *described above satisfy*:

a) A *is irreducible,*
b) $a_{01} \neq 0$,
c) $A.B = D + \infty$,
d) G, $H \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ *exist such that* $F = AG - BH$.

The proof is straightforward. As remarked before, part d) follows from an application of Noether's theorem.

The conics G and H are not uniquely determined by A and B; however, if $\mathcal{I}$ denotes the ideal sheaf associated with the intersection of A and B, and we consider the Koszul complex:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2} \xrightarrow{(A,B)} \mathcal{O}_{\mathbf{P}^2}(2) \oplus \mathcal{O}_{\mathbf{P}^2}(2) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(4) \otimes \mathcal{I} \longrightarrow 0,$$

which is exact and induces an exact sequence in global sections. We see that all the possible G and H in the previous relation belong to pencils:

$$G + \lambda B \quad \text{and} \quad H + \lambda A. \tag{2}$$

Thus, if we fix the unique value of λ such that the conic $H + \lambda A$ has coefficient $h_{20} = 0$, then we have defined uniquely the conics G and H in the relation $F = AG - BH$.

This completes the first part of the proof, namely, to assign to each $D \in \mathrm{Div}_0^{+,3}$ the four conics (A, B, G, H).

Now, we make the converse construction. Assume the conics:

$$A = a_{00} + a_{10}x + a_{01}y - x^2,$$
$$B = b_{00} + b_{10}x + b_{01}y - xy,$$

satisfy the equation

$$F = AG - BH$$

for some conics G and H. We shall prove that, if $A.B = D + \infty$, then $D \in \mathrm{Div}_0^{3,+}(X)$.

Note first that $\infty \notin \mathrm{Supp}\,(D)$ because the intersection index $I_\infty(A, B) = 1$, since both A and B are non-singular and with different tangent lines at infinity.

Next, we must show that $h^1(X, \mathcal{O}(D)) = 0$. If it is not the case, then there exists a line L such that $X.L \geq D$, but this implies $A.L \geq D$ and $B.L \geq D$. It follows that A and B have a common factor L, contradicting the irreducibility of F. A similar argument implies that $h^1(X, \mathcal{O}(p_i + p_j + \infty)) = 0$. These two constructions are clearly the inverse of each other. □

In order to conclude the construction we need:

Theorem 2.3. *The set Z is a smooth affine variety of dimension 3, isomorphic to an open Zariski set of JX.*

Proof. The proof of this fact is standard. We outline it as follows. The structure of algebraic set $Z \subset \mathbf{A}^{17}$ is given by the expanded relations $F = AG - BH$ with the coefficients of the conics playing the role of affine coordinates. This algebraic set can be interpreted as the image of the map:

$$\phi : \mathrm{Div}_0^{3,+}(X) \longrightarrow Z \subset \mathbf{A}^{17},$$
$$D \longrightarrow (A_D, B_D, G_D, H_D).$$

The condition $h^1(\mathcal{O}(D)) = 0$ for every $D \in \mathrm{Div}_0^{3,+}(X)$ allows us to identify $\mathrm{Div}_0^{3,+}(X)$ with a Zariski open set of the symmetric product of X. It follows at once that Z is irreducible. The fact that Z is smooth can be justified either by making an explicit computation of the Zariski tangent space or by studying the differential $d\phi$. □

The previous construction is valid even if we remove the hypothesis on the existence of a hyper-flex. The easiest way to handle the general case is assuming that the line at infinity is bitangent to X. In fact, choose coordinates such that the line $\{z = 0\}$ is a bitangent line to X and $X.\{z = 0\} = 2q_1 + 2q_2$ with $q_1 = (1 : 0 : 0)$ and $q_2 = (0 : 1 : 0)$.

The definition of $\mathrm{Div}_0^{3,+}(X)$ must be modified as follows:

$$\begin{aligned}\mathrm{Div}_0^{3,+}(X) := \{D = p_1 + p_2 + p_3 \in \mathrm{Div}^{3,+}(X) \mid \\ h^1(X, \mathcal{O}(D)) = 0, \quad \text{and} \quad q_j \neq p_i\} \\ \cap \{D \in \mathrm{Div}^{3,+}(X) \mid p_i \neq q_j \\ \text{and} \quad h^1(X, \mathcal{O}(p_i + p_j + q_1)) = 0 \quad \text{for all } i \neq j\}.\end{aligned}$$

With these modifications, we can proceed analogously, constructing the pencil of conics intersecting X in at least $D + q_1$ and fixing the basis A and B just as before. In fact, since q_i is not in the support of D, the matrix (1) is well defined and invertible. The condition $h^1(X, \mathcal{O}(p_i + p_j + q_1)) = 0$ guarantees that the pencil of conics does not have fixed components and, in consequence, a relation $F = AG - BH$ can be constructed.

3. Law group on JX and equations for Kummer variety. The construction in the previous section can be used for explicitly describing the group structure of JX.

In order to describe multiplication by (-1) consider the surjective map:

$$\pi_\infty : \mathrm{Pic}^3(X) \longrightarrow JX$$
$$D \longrightarrow D - 3\infty.$$

We have:

Theorem 3.1. *The open set Z is invariant under multiplication by (-1) in JX. In the affine coordinates for Z this map is given by:*

$$(A, B, G, H) \longrightarrow (A, H, G, B).$$

Proof. Let $D \in \mathrm{Div}_0^{3,+}(X)$, and denote by D^- the divisor such that $D + D^- - 6\infty \equiv 0$. This divisor can be constructed as follows. Let A be the conic considered before. Then $X.A = D + D' + 2\infty$ with $\deg D' = 3$. Let $L_\infty = \{z = 0\}$. Since ∞ is a hyper-flex, we have:

$$X.A \equiv X.2L_\infty \equiv 8\infty;$$

thus, $D' = D^-$.

Let us prove that $D^- \in \mathrm{Div}_0^{3,+}(X)$. It is easy to see that $I_\infty(X, A) = 2$; thus, $\infty \notin \mathrm{Sup}\, D'$. Moreover, if $h^1(X, \mathcal{O}_X(D')) = 1$, then A must be reducible and the same is valid if $h^1(X, \mathcal{O}_X(q_i + q_j + \infty)) = 1$ with q_i and q_j points in the support of D'. This proves that Z is invariant under multiplication by -1.

Next, as

$$F = AG_D - B_D H_D = AG_{D^-} - B_{D^-} H_{D^-},$$

we have

$$\begin{aligned} A.F = D + D^- + 2\infty &= A.B_{D^-} + A.H_{D^-} \\ &= D^- + \infty + A.H_{D^-}, \end{aligned}$$

and thus

$$A.B_D = A.H_{D^-}.$$

Therefore $H_{D^-} = B + \mu A$ for some $\mu \in \mathbf{C}$. By construction, the coefficients of x^2 are zero for both, H_{D^-} and B are different from zero for A. Thus, we conclude that $\mu = 0$.

In this way, the set of conics associated with D^- is

$$(A_D, H_D, G_D, B_D). \qquad \square$$

We remark that a conic A satisfying $A.X \geq D + 2\infty$ exists independently of being $D \in \mathrm{Div}_0^{3,+}(X)$ or not. Thus, an explicit and simple algorithm for computing the inverse of a divisor $D - 3\infty$ is to take the class of the divisor $A.X - D - 5\infty$.

We obtain as a by-product:

Corollary 3.2. *The open set of the Kummer variety $K(X)$ given by the quotient $Z/\{\pm 1\}$ is determined by the equations:*

$$F = A.G + Q,$$

with Q a quartic expressible as the product of two conics passing through ∞ and $(1:0:0)$.

The condition on reducibility of the quartic Q can be written explicitly in terms of the usual procedure of composing the Segre map with a linear projection.

The addition law on JX, considered to be the image of π_∞, can also be described by a simple algorithm, as follows.

Let $D - 3\infty$, $D' - 3\infty \in JX$. Consider a cubic C such that $C.X = D + D' + E^- + 3\infty$.

Let A_{E^-} be a conic with $A_{E^-}.X = E^- + E + 2\infty$.

Now, $D + D' + E^- \equiv 9\infty$ and

$$E^- + E \equiv 6\infty.$$

It follows that $D + D' - 6\infty \equiv E + 3\infty$, that is,

$$D + D' \equiv E + 3\infty.$$

Of course, Z cannot be invariant under the addition law (it cannot be a subgroup of JX for several reasons), so we cannot expect to find a description of the sum in terms of the affine coordinates for Z.

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