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REVERSE ORDER LAWS IN RINGS WITH INVOLUTION

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ABSTRACT. Necessary and sufficient conditions for
$$(ab)^{\#} = b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger}$$
and
$$(ab)^{\#} = b^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#}$$
to hold in rings with involution are presented. Also, some equivalent conditions concerning the reverse order laws $(ab)^{\#} = b^{\dagger}a^{\#}$ and $(ab)^{\#} = b^{\#}a^{\dagger}$ are studied.

1. Introduction. Let $\mathcal{R}$ be an associative ring with the unit 1. For invertible elements $a, b \in \mathcal{R}$, the inverse of the product ab satisfied the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. Since this formula cannot trivially be extended to various generalized inverse of the product ab, the reverse order laws for generalized inverses have been investigated in the literature since the 1960s [**1, 2, 3, 4, 6, 7**].

An element $p \in \mathcal{R}$ is idempotent if $p^2 = p$. An element $a \in \mathcal{R}$ is *group invertible* if there is $a^{\#} \in \mathcal{R}$ such that

$$(1)\ aa^{\#}a = a, \quad (2)\ a^{\#}aa^{\#} = a^{\#}, \quad (3)\ aa^{\#} = a^{\#}a;$$

$a^{\#}$ is a group inverse of a and it is uniquely determined by these equations. The group inverse $a^{\#}$ double commutes with a, that is, $ax = xa$ implies $a^{\#}x = xa^{\#}$ [**1**]. Denote by $\mathcal{R}^{\#}$ the set of all group invertible elements of $\mathcal{R}$.

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An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element $a \in \mathcal{R}$ is self-adjoint (or Hermitian) if $a^* = a$.

The *Moore-Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $a^\dagger \in \mathcal{R}$, if the following equations hold [**8**]:

$$(1)\ aa^\dagger a = a, \quad (2)\ a^\dagger a a^\dagger = a^\dagger, \quad (3)\ (aa^\dagger)^* = aa^\dagger, \quad (4)\ (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that above conditions hold. The set of all Moore-Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

Recall that the element $a \in \mathcal{R}$ is Drazin invertible, if there exists some non-negative integer k, and there exists some element $b \in \mathcal{R}$ such that the following hold: $bab = b$, $ab = ba$ and $a^{k+1}b = a$. In this case b is the Drazin inverse of a, and the common notation is $b = a^D$. If the Drazin inverse of a exists, then it is unique. If a is Drazin invertible, then $a^\pi = 1 - aa^D$ is the spectral idempotent of a.

In this paper, we use a similar notation: if a is the Moore-Penrose invertible, then $a_l^\pi = 1 - a^\dagger a$ and $a_r^\pi = 1 - aa^\dagger$. However, there is no connection between a_l^π, a_r^π and a^π.

If $\delta \subset \{1,2,3,4,5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ-inverse of a. The set of all δ-inverse of a is denoted by $a\{\delta\}$. Indeed, if $a \in \mathcal{R}$, then $a\{5\}$ is the commutator of a. Hence, if c commutes with a, then c is an $\{5\}$-inverse of a. Notice that $a\{1,2,5\} = \{a^\#\}$ and $a\{1,2,3,4\} = \{a^\dagger\}$. If a is invertible, then $a^\#$ and $a^\dagger$ coincide with the ordinary inverse a^{-1} of a.

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [**5**]. A Hilbert space version of their result was given by Isumino [**7**]. The results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's paper [**9**]. A natural consideration is to see what will be obtained if we replace the Moore-Penrose inverse by the group inverses.

In this paper, we investigate equivalent conditions for the reverse order laws $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$ and $(ab)^\# = b^\dagger(a^\# abb^\dagger)^\# a^\#$ to be satisfied in rings with involution. Some necessary and sufficient conditions including $a^\dagger abb^\# \in \mathcal{R}^\#$ or $a^\# abb^\dagger \in \mathcal{R}^\#$ for $(ab)^\# = b^\dagger a^\#$ or $(ab)^\# = b^\# a^\dagger$ are obtained. We also give characterizations of the rules

$(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$ and $(abb^\dagger)^\# = (a^\# abb^\dagger)^\# a^\#$. The conditions related to the reverse order laws $(ab)^\# = b^\#(a^*abb^\#)^\# a^*$, $(ab)^\# = b^*(a^\# abb^*)^\# a^\#$, $(ab)^\# = b^\# a^*$ and $(ab)^\# = b^* a^\#$ are presented too.

2. Reverse order laws. In the beginning of this section, we give some characterizations of the reverse order law $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$ in a ring with involution.

Theorem 2.1. *If $a \in \mathcal{R}^\dagger$ and $b, ab, a^\dagger abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$,
(ii) $b^\#(a^\dagger abb^\#)^\# a^\dagger \in (ab)\{5\}$,
(iii) $abaa^\dagger = ab$ *and* $b^\#(a^\dagger abb^\#)^\# a^\dagger abaa^\dagger = abb^\#(a^\dagger abb^\#)^\# a^\dagger$,
(iv) $bb^\# ab = ab$ *and* $b^\# babb^\#(a^\dagger abb^\#)^\# a^\dagger = b^\#(a^\dagger abb^\#)^\# a^\dagger ab$,
(v) $b^\# \cdot (a^\dagger abb^\#)\{1,5\} \cdot a^\dagger \subseteq (ab)\{5\}$.

Proof. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). Observe that $b^\#(a^\dagger abb^\#)^\# a^\dagger \in (ab)\{1\}$, by

$$abb^\#(a^\dagger abb^\#)^\# a^\dagger ab = a(a^\dagger abb^\#(a^\dagger abb^\#)^\# a^\dagger abb^\#)b = aa^\dagger abb^\# b = ab. \tag{1}$$

The condition $b^\#(a^\dagger abb^\#)^\# a^\dagger \in (ab)\{5\}$ gives

$$abaa^\dagger = ababb^\#(a^\dagger abb^\#)^\# a^\dagger aa^\dagger = ababb^\#(a^\dagger abb^\#)^\# a^\dagger = ab$$

and

$$b^\#(a^\dagger abb^\#)^\# a^\dagger(abaa^\dagger) = b^\#(a^\dagger abb^\#)^\# a^\dagger ab = abb^\#(a^\dagger abb^\#)^\# a^\dagger.$$

Hence, statement (iii) holds.

(iii) $\Rightarrow$ (v). For $(a^\dagger abb^\#)^{(1,5)} \in (a^\dagger abb^\#)\{1,5\}$, we obtain

$$\begin{aligned} a^\dagger abb^\#(a^\dagger abb^\#)^{(1,5)} &= (a^\dagger abb^\#)^\# a^\dagger abb^\#(a^\dagger abb^\#(a^\dagger abb^\#)^{(1,5)}) \\ &= (a^\dagger abb^\#)^\#(a^\dagger abb^\#(a^\dagger abb^\#)^{(1,5)} a^\dagger abb^\#) \\ &= (a^\dagger abb^\#)^\# a^\dagger abb^\#. \end{aligned} \tag{2}$$

Then, from $abaa^\dagger = ab$ and $b^\#(a^\dagger abb^\#)^\# a^\dagger abaa^\dagger = abb^\#(a^\dagger abb^\#)^\# a^\dagger$,

$$\begin{aligned} abb^\#(a^\dagger abb^\#)^{(1,5)} a^\dagger &= a(a^\dagger abb^\#(a^\dagger abb^\#)^{(1,5)})a^\dagger \\ &= aa^\dagger abb^\#(a^\dagger abb^\#)^\# a^\dagger = abb^\#(a^\dagger abb^\#)^\# a^\dagger \end{aligned}$$

$$\begin{aligned}
&= b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger}(abaa^{\dagger}) = b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger}ab \\
&= b^{\#}((a^{\dagger}abb^{\#})^{\#}a^{\dagger}abb^{\#})b \\
&= b^{\#}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger}abb^{\#}b \\
&= b^{\#}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger}ab.
\end{aligned}$$

Hence, for any $(a^{\dagger}abb^{\#})^{(1,5)} \in (a^{\dagger}abb^{\#})\{1,5\}$, $b^{\#}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger} \in (ab)\{5\}$ and condition (v) is satisfied.

(v) $\Rightarrow$ (i). By $b^{\#} \cdot (a^{\dagger}abb^{\#})\{1,5\} \cdot a^{\dagger} \subseteq (ab)\{5\}$ and $(a^{\dagger}abb^{\#})^{\#} \in (a^{\dagger}abb^{\#})\{1,5\}$, we deduce that $b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger} \in (ab)\{5\}$. Since the equalities (1) hold and

$$b^{\#}((a^{\dagger}abb^{\#})^{\#}a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#})a^{\dagger} = b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger},$$

we conclude that $b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger} \in (ab)\{1,2\}$. Thus, (i) holds.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Similarly as (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v). □

The next theorems considering the rules $(ab)^{\#} = b^{\#}(a^*abb^{\#})^{\#}a^*$, $(ab)^{\#} = b^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#}$ and $(ab)^{\#} = b^*(a^{\#}abb^*)^{\#}a^{\#}$ can be proved in the same way as in Theorem 2.1.

Theorem 2.2. *If $a \in \mathcal{R}^{\dagger}$ and $b, ab, a^*abb^{\#} \in \mathcal{R}^{\#}$, then the following statements are equivalent*:

(i) $(ab)^{\#} = b^{\#}(a^*abb^{\#})^{\#}a^*$,
(ii) $b^{\#}(a^*abb^{\#})^{\#}a^* \in (ab)\{5\}$,
(iii) $abaa^{\dagger} = ab$ *and* $b^{\#}(a^*abb^{\#})^{\#}a^*abaa^{\dagger} = abb^{\#}(a^*abb^{\#})^{\#}a^*$,
(iv) $bb^{\#}ab = ab$ *and* $b^{\#}babb^{\#}(a^*abb^{\#})^{\#}a^* = b^{\#}(a^*abb^{\#})^{\#}a^*ab$,
(v) $b^{\#} \cdot (a^*abb^{\#})\{1,5\} \cdot a^* \subseteq (ab)\{5\}$.

Proof. Using the equalities $a = (a^{\dagger})^*a^*a$ and $a^* = a^*aa^{\dagger}$, we repeat the argument of the proof of Theorem 2.1. □

Theorem 2.3. *If $b \in \mathcal{R}^{\dagger}$ and $a, ab, a^{\#}abb^{\dagger} \in \mathcal{R}^{\#}$, then the following statements are equivalent*:

(i) $(ab)^{\#} = b^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#}$,
(ii) $b^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#} \in (ab)\{5\}$,
(iii) $b^{\dagger}bab = ab$ *and* $b^{\dagger}babb^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#} = b^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#}ab$,

(iv) $abaa^{\#} = ab$ *and* $b^{\dagger}(a^{\#}abb^{\dagger})^{\#}aa^{\#}baa^{\#} = abb^{\dagger}(a^{\#}abb^{\dagger})^{\#}a^{\#}$,
(v) $b^{\dagger} \cdot (a^{\#}abb^{\dagger})\{1,5\} \cdot a^{\#} \subseteq (ab)\{5\}$.

Theorem 2.4. *If* $b \in \mathcal{R}^{\dagger}$ *and* $a, ab, a^{\#}abb^{*} \in \mathcal{R}^{\#}$, *then the following statements are equivalent*:

(i) $(ab)^{\#} = b^{*}(a^{\#}abb^{*})^{\#}a^{\#}$,
(ii) $b^{*}(a^{\#}abb^{*})^{\#}a^{\#} \in (ab)\{5\}$,
(iii) $b^{\dagger}bab = ab$ *and* $b^{\dagger}babb^{*}(a^{\#}abb^{*})^{\#}a^{\#} = b^{*}(a^{\#}abb^{*})^{\#}a^{\#}ab$,
(iv) $abaa^{\#} = ab$ *and* $b^{*}(a^{\#}abb^{*})^{\#}aa^{\#}baa^{\#} = abb^{*}(a^{\#}abb^{*})^{\#}a^{\#}$,
(v) $b^{*} \cdot (a^{\#}abb^{*})\{1,5\} \cdot a^{\#} \subseteq (ab)\{5\}$.

We prove that the assumption of inclusion $(ab)\{5\} \subseteq b^{\#} \cdot (a^{\dagger}abb^{\#})\{1,5\} \cdot a^{\dagger}$ automatically implies equality.

Theorem 2.5. *If* $a \in \mathcal{R}^{\dagger}$ *and* $b, ab, a^{\dagger}abb^{\#} \in \mathcal{R}^{\#}$, *then the inclusion* $(ab)\{5\} \subseteq b^{\#} \cdot (a^{\dagger}abb^{\#})\{1,5\} \cdot a^{\dagger}$ *is always an equality.*

Proof. If $(ab)\{5\} \subseteq b^{\#} \cdot (a^{\dagger}abb^{\#})\{1,5\} \cdot a^{\dagger}$, by $(ab)^{\#} \in (ab)\{5\}$, there exists $(a^{\dagger}abb^{\#})^{(1,5)} \in (a^{\dagger}abb^{\#})\{1,5\}$ such that $(ab)^{\#} = b^{\#}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger}$. Because the equalities (2) hold again, we have

$$\begin{aligned} b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger} &= b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger} \\ &= (b^{\#}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger})ab(b^{\#}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger}) \\ &= (ab)^{\#}ab(ab)^{\#} = (ab)^{\#}, \end{aligned}$$

implying, by Theorem 2.1, $b^{\#} \cdot (a^{\dagger}abb^{\#})\{1,5\} \cdot a^{\dagger} \subseteq (ab)\{5\}$. Hence, $(ab)\{5\} \subseteq b^{\#} \cdot (a^{\dagger}abb^{\#})\{1,5\} \cdot a^{\dagger}$. □

Similarly as in the proof of Theorem 2.5, we can verify that the inclusions $(ab)\{5\} \subseteq b^{\#} \cdot (a^{*}abb^{\#})\{1,5\} \cdot a^{*}$, $(ab)\{5\} \subseteq b^{\dagger} \cdot (a^{\#}abb^{\dagger})\{1,5\} \cdot a^{\#}$ and $(ab)\{5\} \subseteq b^{*} \cdot (a^{\#}abb^{*})\{1,5\} \cdot a^{\#}$ are always the corresponding equalities.

Now, we characterize the reverse order law $(a^{\dagger}abb^{\#})^{\#} = bb^{\#}a^{\dagger}a$.

Theorem 2.6. *If* $a \in \mathcal{R}^{\dagger}$ *and* $b, a^{\dagger}abb^{\#} \in \mathcal{R}^{\#}$, *then the following statements are equivalent*:

(i) $(a^{\dagger}abb^{\#})^{\#} = bb^{\#}a^{\dagger}a$, (ii) $a^{\dagger}abb^{\#} = bb^{\#}a^{\dagger}a$.

Proof. (i) $\Rightarrow$ (ii). The hypothesis $(a^\dagger abb^\#)^\# = bb^\# a^\dagger a$ gives

$$a^\dagger abb^\# a^\dagger a = a^\dagger abb^\# bb^\# a^\dagger a = bb^\# a^\dagger aa^\dagger abb^\# = bb^\# a^\dagger abb^\#$$

and

$$a^\dagger abb^\# = a^\dagger abb^\# (bb^\# a^\dagger a)a^\dagger abb^\# = a^\dagger abb^\# a^\dagger a$$

implying

$$\begin{aligned} bb^\# a^\dagger a &= bb^\# a^\dagger a(a^\dagger abb^\#)bb^\# a^\dagger a = (bb^\# a^\dagger abb^\#)a^\dagger a \\ &= a^\dagger abb^\# a^\dagger aa^\dagger a = a^\dagger abb^\# a^\dagger a = a^\dagger abb^\#. \end{aligned}$$

(ii) $\Rightarrow$ (i). By the equality $a^\dagger abb^\# = bb^\# a^\dagger a$, observe that $bb^\# a^\dagger a \in (a^\dagger abb^\#)\{1,2\}$ and $(a^\dagger abb^\#)(bb^\# a^\dagger a) = bb^\# a^\dagger aa^\dagger abb^\#$, that is, $bb^\# a^\dagger a \in (a^\dagger abb^\#)\{5\}$. □

The following result can be checked in the same manner as Theorem 2.6.

Theorem 2.7. *If $b \in \mathcal{R}^\dagger$ and $a, a^\# abb^\dagger \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^\# abb^\dagger)^\# = bb^\dagger a^\# a$,
(ii) $a^\# abb^\dagger = bb^\dagger a^\# a$.

Condition (ii) of Theorem 2.6 can be rewritten as $(1 - a_l^\pi)(1 - b^\pi) = (1 - b^\pi)(1 - a_l^\pi)$ or $a_l^\pi b^\pi = b^\pi a_l^\pi$. Condition (ii) of Theorem 2.7 is equivalent to $(1 - a^\pi)(1 - b_r^\pi) = (1 - b_r^\pi)(1 - a^\pi)$ or $a^\pi b_r^\pi = a^\pi b_r^\pi$.

Necessary and sufficient conditions for $(a^* abb^\#)^\# = bb^\# a^\dagger a$ and $(a^\# abb^*)^\# = bb^\dagger a^\# a$ are given in the next theorems.

Theorem 2.8. *If $a \in \mathcal{R}^\dagger$ and $b, a^* abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^* abb^\#)^\# = bb^\# a^\dagger a$,
(ii) $a^* abb^\# = bb^\# a^\dagger a$.

Proof. (i) $\Rightarrow$ (ii). From the assumption $(a^*abb^\#)^\# = bb^\# a^\dagger a$, we obtain

$$\begin{aligned} bb^\# a^\dagger a &= bb^\# a^\dagger a(a^*abb^\#)bb^\# a^\dagger a = bb^\#(a^*abb^\# bb^\# a^\dagger a) \\ &= bb^\# bb^\# a^\dagger aa^*abb^\# = bb^\# a^*abb^\#. \end{aligned}$$

Further, by this equality, we have

$$\begin{aligned} a^*abb^\# &= (a^*abb^\#)^\# a^*abb^\# a^*abb^\# = bb^\# a^\dagger aa^*abb^\# a^*abb^\# \\ &= (bb^\# a^*abb^\#)a^*abb^\# = bb^\# a^\dagger aa^*abb^\# = bb^\# a^*abb^\# = bb^\# a^\dagger a. \end{aligned}$$

(ii) $\Rightarrow$ (i). The equality $a^*abb^\# = bb^\# a^\dagger a$ implies

$$a^*abb^\# = bb^\# a^\dagger a = bb^\#(bb^\# a^\dagger a) = bb^\# a^*abb^\#,$$

which yields

$$\begin{aligned} (a^*abb^\#)^\# &= (a^*abb^\#)[(a^*abb^\#)^\#]^2 = bb^\# a^*abb^\#[(a^*abb^\#)^\#]^2 \\ &= bb^\# a^\dagger a(a^*abb^\#[(a^*abb^\#)^\#]^2) = bb^\# a^\dagger a(a^*abb^\#)^\#. \end{aligned}$$

Therefore, we get

$$\begin{aligned} bb^\# a^\dagger a &= bb^\#(bb^\# a^\dagger a) = bb^\#(a^*abb^\#) \\ &= bb^\#(a^*abb^\#)a^*abb^\#(a^*abb^\#)^\# \\ &= bb^\# bb^\# a^\dagger aa^*abb^\#(a^*abb^\#)^\# \\ &= bb^\#(a^*abb^\#)(a^*abb^\#)^\# \\ &= bb^\# bb^\# a^\dagger a(a^*abb^\#)^\# \\ &= bb^\# a^\dagger a(a^*abb^\#)^\# = (a^*abb^\#)^\#. \end{aligned}$$ □

Exactly as in Theorem 2.8, we can show the next result.

Theorem 2.9. *If $b \in \mathcal{R}^\dagger$ and $a, a^\# abb^* \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $(a^\# abb^*)^\# = bb^\dagger a^\# a$,
(ii) $a^\# abb^* = bb^\dagger a^\# a$.

Now, we consider equivalent conditions for $b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger$ to hold.

Theorem 2.10. *If $a \in \mathcal{R}^\dagger$ and $b, a^\dagger abb^\#, bb^\# a^\dagger a \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger$,
(ii) $b^\# a^\dagger abb^\# a^\dagger = b^\# a^\dagger$,
(iii) $bb^\# a^\dagger a$ *is an idempotent,*
(iv) $a(bb^\# a^\dagger a)^\# b = ab$,
(v) $abb^\# a^\dagger ab = ab$,
(vi) $a^\dagger abb^\#$ *is an idempotent.*

Proof. (i) $\Rightarrow$ (ii). Since $b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger$, we observe that

$$\begin{aligned}(b^\# a^\dagger)ab(b^\# a^\dagger) &= b^\#((a^\dagger abb^\#)^\# a^\dagger abb^\#(a^\dagger abb^\#)^\#)a^\dagger \\ &= b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger.\end{aligned}$$

(ii) $\Rightarrow$ (iii). Multiplying the equality $b^\# a^\dagger abb^\# a^\dagger = b^\# a^\dagger$ from the left side by b and from the right side by a, we get $bb^\# a^\dagger abb^\# a^\dagger a = bb^\# a^\dagger a$. Hence, the condition (iii) is satisfied.

(iii) $\Rightarrow$ (i). Assume that $bb^\# a^\dagger a$ is an idempotent. Notice that

$$\begin{aligned}a^\dagger a(a^\dagger abb^\#)^\# &= a^\dagger aa^\dagger abb^\#[(a^\dagger abb^\#)^\#]^2 = a^\dagger abb^\#[(a^\dagger abb^\#)^\#]^2 \\ &= (a^\dagger abb^\#)^\#.\end{aligned}$$

Then, we get

$$\begin{aligned}b^\# a^\dagger &= b^\#(bb^\# a^\dagger a)a^\dagger = b^\#(bb^\# a^\dagger a)^2 a^\dagger \\ &= b^\#(a^\dagger abb^\#)a^\dagger \\ &= b^\#(bb^\# a^\dagger abb^\# a^\dagger a)bb^\#(a^\dagger abb^\#)^\# a^\dagger \\ &= b^\#(bb^\# a^\dagger abb^\# a^\dagger a)(a^\dagger abb^\#)^\# a^\dagger \\ &= b^\#(a^\dagger a(a^\dagger abb^\#)^\#)a^\dagger \\ &= b^\#(a^\dagger abb^\#)^\# a^\dagger.\end{aligned}$$

(iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv). This implications follow as (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (vi). Applying the assumption $b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger$, we have

$$a^\dagger ab(b^\# a^\dagger)abb^\# = a^\dagger abb^\#(a^\dagger abb^\#)^\# a^\dagger abb^\# = a^\dagger abb^\#.$$

So, statement (vi) holds.

(iv) $\Rightarrow$ (iii). In the same manner as (i) $\Rightarrow$ (vi). □

The following theorems can be proved similarly as Theorem 2.10.

Theorem 2.11. *If $a \in \mathcal{R}^\dagger$ and $b, a^*abb^\#, bb^\#a^*a \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $b^\#(a^*abb^\#)^\#a^* = b^\#a^*$,
(ii) $b^\#a^*abb^\#a^* = b^\#a^*$,
(iii) $bb^\#a^*a$ *is an idempotent,*
(iv) $a(bb^\#a^*a)^\#b = ab$,
(v) $abb^\#a^*ab = ab$,
(vi) $a^*abb^\#$ *is an idempotent.*

Theorem 2.12. *If $b \in \mathcal{R}^\dagger$ and $a, a^\#abb^\dagger, bb^\dagger a^\#a \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $b^\dagger(a^\#abb^\dagger)^\#a^\# = b^\dagger a^\#$,
(ii) $b^\dagger a^\#abb^\dagger a^\# = b^\dagger a^\#$,
(iii) $bb^\dagger a^\#a$ *is an idempotent,*
(iv) $a(bb^\dagger a^\#a)^\#b = ab$,
(v) $abb^\dagger a^\#ab = ab$,
(vi) $a^\#abb^\dagger$ *is an idempotent.*

Theorem 2.13. *If $b \in \mathcal{R}^\dagger$ and $a, a^\#abb^*, bb^*a^\#a \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $b^*(a^\#abb^*)^\#a^\# = b^*a^\#$,
(ii) $b^*a^\#abb^*a^\# = b^*a^\#$,
(iii) $bb^*a^\#a$ *is an idempotent,*
(iv) $a(bb^*a^\#a)^\#b = ab$,
(v) $abb^*a^\#ab = ab$,
(vi) $a^\#abb^*$ *is an idempotent.*

Observe that the conditions of Theorem 2.6 (Theorem 2.8, Theorem 2.7 and Theorem 2.9, respectively) imply the conditions of Theorem 2.10 (Theorem 2.11, Theorem 2.12 and Theorem 2.13, respectively). The reverse implication fails.

Since the conditions of Theorem 2.6 give the conditions of Theorem 2.10, combining the conditions of Theorems 2.1 and 2.6, we get the sufficient conditions for the reverse order law $(ab)^\# = b^\# a^\dagger$ to hold. Similarly, we can obtain lists of sufficient conditions for the reverse order laws $(ab)^\# = b^\# a^*$, $(ab)^\# = b^\dagger a^\#$, $(ab)^\# = b^* a^\#$.

In the next theorem, the equivalent condition to $(ab)^\# = b^\# a^\dagger$ is presented.

Theorem 2.14. *If $a \in \mathcal{R}^\dagger$ and $b, ab, a^\dagger abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = b^\# a^\dagger$,
(ii) $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$ *and* $b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger$.

Proof.

(i) ⇒ (ii). The condition $(ab)^\# = b^\# a^\dagger$ implies that $b^\# a^\dagger abb^\# a^\dagger = b^\# a^\dagger$ which is equivalent to $b^\#(a^\dagger abb^\#)^\# a^\dagger = b^\# a^\dagger$, by Theorem 2.10. Thus, $(ab)^\# = b^\# a^\dagger = b^\#(a^\dagger abb^\#)^\# a^\dagger$.

(ii) ⇒ (i). It is trivial. □

Remark 2.15. The next characterizations can be verified in the same way as in Theorem 2.14.

(a) If $a \in \mathcal{R}^\dagger$ and $b, ab, a^* abb^\# \in \mathcal{R}^\#$, then:

$$(ab)^\# = b^\# a^* \Longleftrightarrow (ab)^\# = b^\#(a^* abb^\#)^\# a^*$$

and

$$b^\#(a^* abb^\#)^\# a^* = b^\# a^*.$$

(b) If $b \in \mathcal{R}^\dagger$ and $a, ab, a^\# abb^\dagger \in \mathcal{R}^\#$, then:

$$(ab)^\# = b^\dagger a^\# \Longleftrightarrow (ab)^\# = b^\dagger(a^\# abb^\dagger)^\# a^\#$$

and

$$b^\dagger(a^\# abb^\dagger)^\# a^\# = b^\dagger a^\#.$$

(c) If $b \in \mathcal{R}^\dagger$ and $a, ab, a^\# abb^\dagger \in \mathcal{R}^\#$, then:

$$(ab)^\# = b^* a^\# \Longleftrightarrow (ab)^\# = b^*(a^\# abb^*)^\# a^\#$$

and

$$b^*(a^\# abb^*)^\# a^\# = b^* a^\#.$$

If we combine the conditions of Theorems 2.1 and 2.10 (Theorems 2.2 and 2.11, Theorems 2.3 and 2.12, Theorems 2.4 and 2.13, respectively), we obtain a set of equivalent conditions for the reverse order law $(ab)^\# = b^\# a^\dagger$ ($(ab)^\# = b^\# a^*$, $(ab)^\# = b^\dagger a^\#$, $(ab)^\# = b^* a^\#$, respectively) to be satisfied.

Some equivalent conditions for the reverse order law

$$(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$$

to hold are considered in the following theorem.

Theorem 2.16. *If $a \in \mathcal{R}^\dagger$ and $b, a^\dagger ab, a^\dagger abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$,
(ii) $b^\#(a^\dagger abb^\#)^\# \in (a^\dagger ab)\{5\}$,
(iii) $bb^\# a^\dagger ab = a^\dagger ab$ *and* $ba^\dagger abb^\#(a^\dagger abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger ab$,
(iv) $b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\} \subseteq (a^\dagger ab)\{5\}$,
(v) $(a^\dagger abb^\#)^\# = b(a^\dagger ab)^\#$,
(vi) $b(a^\dagger ab)^\# \in (a^\dagger abb^\#)\{5\}$,
(vii) $b \cdot (a^\dagger ab)\{1,5\} \subseteq (a^\dagger abb^\#)\{5\}$.

Proof. (i) $\Rightarrow$ (ii). Obviously.

(ii) $\Rightarrow$ (iii). Assume that $b^\#(a^\dagger abb^\#)^\# \in (a^\dagger ab)\{5\}$. The equality

(3)
$$a^\dagger abb^\#(a^\dagger abb^\#)^\# a^\dagger ab = (a^\dagger abb^\#(a^\dagger abb^\#)^\# a^\dagger abb^\#)b = a^\dagger abb^\# b = a^\dagger ab$$

gives $b^\#(a^\dagger abb^\#)^\# \in (a^\dagger ab)\{1\}$. Therefore, notice that

(4)
$$bb^\#(a^\dagger ab) = bb^\# b^\#(a^\dagger abb^\#)^\#(a^\dagger ab)^2 = b^\#(a^\dagger abb^\#)^\#(a^\dagger ab)^2 = a^\dagger ab$$

and

$$\begin{aligned} ba^\dagger abb^\#(a^\dagger abb^\#)^\# &= bb^\#(a^\dagger abb^\#)^\# a^\dagger ab = bb^\#((a^\dagger abb^\#)^\# a^\dagger abb^\#)b \\ &= (bb^\# a^\dagger ab)b^\#(a^\dagger abb^\#)^\# b = a^\dagger abb^\#(a^\dagger abb^\#)^\# b \\ &= (a^\dagger abb^\#)^\# a^\dagger abb^\# b = (a^\dagger abb^\#)^\# a^\dagger ab. \end{aligned}$$

(iii) $\Rightarrow$ (i). Let $bb^{\#}a^{\dagger}ab = a^{\dagger}ab$ and $ba^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#} = (a^{\dagger}abb^{\#})^{\#}a^{\dagger}ab$. Then, by

$$b^{\#}((a^{\dagger}abb^{\#})^{\#}a^{\dagger}ab) = (b^{\#}ba^{\dagger}ab)b^{\#}(a^{\dagger}abb^{\#})^{\#} = a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#},$$

we conclude that $b^{\#}(a^{\dagger}abb^{\#})^{\#} \in (a^{\dagger}ab)\{5\}$. Since the equalities (3) hold and

$$b^{\#}((a^{\dagger}abb^{\#})^{\#}a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#}) = b^{\#}(a^{\dagger}abb^{\#})^{\#},$$

we have that $b^{\#}(a^{\dagger}abb^{\#})^{\#} \in (a^{\dagger}ab)\{1,2\}$. So, the condition (i) holds.

(ii) $\Rightarrow$ (iv). If $b^{(1,5)} \in b\{1,5\}$ and $(a^{\dagger}abb^{\#})^{(1,5)} \in (a^{\dagger}abb^{\#})\{1,5\}$, then the equalities $b^{(1,5)}b = (b^{(1,5)}bb)b^{\#} = bb^{\#}$, (2) and (4) are satisfied. The hypothesis $b^{\#}(a^{\dagger}abb^{\#})^{\#} \in (a^{\dagger}ab)\{5\}$ implies

$$\begin{aligned}
b^{(1,5)}(a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger}ab &= b^{(1,5)}((a^{\dagger}abb^{\#})^{(1,5)}a^{\dagger}abb^{\#})b \\
&= b^{(1,5)}((a^{\dagger}abb^{\#})^{\#}a^{\dagger}abb^{\#})b \\
&= b^{(1,5)}(a^{\dagger}ab)b^{\#}(a^{\dagger}abb^{\#})^{\#}b \\
&= (b^{(1,5)}b)b^{\#}a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#}b \\
&= b^{\#}bb^{\#}(a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#})b \\
&= b^{\#}(a^{\dagger}abb^{\#})^{\#}a^{\dagger}abb^{\#}b = a^{\dagger}abb^{\#}(a^{\dagger}abb^{\#})^{\#} \\
&= a^{\dagger}a(bb^{\#})(a^{\dagger}abb^{\#})^{(1,5)} \\
&= a^{\dagger}abb^{(1,5)}(a^{\dagger}abb^{\#})^{(1,5)}.
\end{aligned}$$

Hence, for any $b^{(1,5)} \in b\{1,5\}$ and $(a^{\dagger}abb^{\#})^{(1,5)} \in (a^{\dagger}abb^{\#})\{1,5\}$, we get $b^{(1,5)}(a^{\dagger}abb^{\#})^{(1,5)} \in (a^{\dagger}ab)\{5\}$, i.e., the statement (iv) is satisfied.

(iv) $\Rightarrow$ (ii). It follows by $b^{\#} \in b\{1,5\}$ and $(a^{\dagger}abb^{\#})^{\#} \in (a^{\dagger}abb^{\#})\{1,5\}$.

(i) $\Rightarrow$ (v). The assumption $(a^{\dagger}ab)^{\#} = b^{\#}(a^{\dagger}abb^{\#})^{\#}$ yields $bb^{\#}a^{\dagger}ab = a^{\#}ab$, by (i) $\Rightarrow$ (iii). Thus,

$$\begin{aligned}
(a^{\dagger}abb^{\#})^{\#} &= (a^{\dagger}ab)b^{\#}[(a^{\dagger}abb^{\#})^{\#}]^2 = bb^{\#}(a^{\dagger}abb^{\#}[(a^{\dagger}abb^{\#})^{\#}]^2) \\
&= b(b^{\#}(a^{\dagger}abb^{\#})^{\#}) = b(a^{\dagger}ab)^{\#}.
\end{aligned}$$

(v) $\Rightarrow$ (i). Similarly as (i) $\Rightarrow$ (v).

(v) $\Leftrightarrow$ (vi). Notice that $b(a^{\dagger}ab)^{\#} \in (a^{\dagger}abb^{\#})\{1,2\}$, which gives that $b(a^{\dagger}ab)^{\#} \in (a^{\dagger}abb^{\#})\{5\}$ is equivalent to $(a^{\dagger}abb^{\#})^{\#} = b(a^{\dagger}ab)^{\#}$.

(vi) $\Leftrightarrow$ (vii). This can be checked in the same manner as (ii) $\Leftrightarrow$ (iv). $\square$

Remark 2.17. Similarly as Theorem 2.16, we can show the next results.

(a) If $a \in \mathcal{R}$ and $b, a^*ab, a^*abb^\# \in \mathcal{R}^\#$, then:

$$\begin{aligned}(a^*ab)^\# = b^\#(a^*abb^\#)^\# &\Longleftrightarrow b^\#(a^*abb^\#)^\# \in (a^*ab)\{5\}\\ &\Longleftrightarrow bb^\#a^*ab = a^*ab \text{ and } ba^*abb^\#(a^*abb^\#)^\#\\ &= (a^*abb^\#)^\#a^*ab\\ &\Longleftrightarrow b\{1,5\} \cdot (a^*abb^\#)\{1,5\} \subseteq (a^*ab)\{5\}\\ &\Longleftrightarrow (a^*abb^\#)^\# = b(a^*ab)^\#\\ &\Longleftrightarrow b(a^*ab)^\# \in (a^*abb^\#)\{5\}\\ &\Longleftrightarrow b \cdot (a^*ab)\{1,5\} \subseteq (a^*abb^\#)\{5\}.\end{aligned}$$

(b) If $b \in \mathcal{R}^\dagger$ and $a, abb^\dagger, a^\#abb^\dagger \in \mathcal{R}^\#$, then:

$$\begin{aligned}(abb^\dagger)^\# = (a^\#abb^\dagger)^\#a^\# &\Longleftrightarrow (a^\#abb^\dagger)^\#a^\# \in (abb^\dagger)\{5\}\\ &\Longleftrightarrow abb^\dagger aa^\# = abb^\dagger \text{ and } (a^\#abb^\dagger)^\#a^\#abb^\dagger a\\ &= abb^\dagger(a^\#abb^\dagger)^\#\\ &\Longleftrightarrow (a^\#abb^\dagger)\{1,5\} \cdot a\{1,5\} \subseteq (abb^\dagger)\{5\}\\ &\Longleftrightarrow (a^\#abb^\dagger)^\# = (abb^\dagger)^\#a\\ &\Longleftrightarrow (abb^\dagger)^\#a \in (a^\#abb^\dagger)\{5\}\\ &\Longleftrightarrow (abb^\dagger)\{1,5\} \cdot a \subseteq (a^\#abb^\dagger)\{5\}.\end{aligned}$$

(c) If $b \in \mathcal{R}$ and $a, abb^*, a^\#abb^* \in \mathcal{R}^\#$, then:

$$\begin{aligned}(abb^*)^\# = (a^\#abb^*)^\#a^\# &\Longleftrightarrow (a^\#abb^*)^\#a^\# \in (abb^*)\{5\}\\ &\Longleftrightarrow abb^*aa^\# = abb^* \text{ and } (a^\#abb^*)^\#a^\#abb^*a\\ &= abb^*(a^\#abb^*)^\#\\ &\Longleftrightarrow (a^\#abb^*)\{1,5\} \cdot a\{1,5\} \subseteq (abb^*)\{5\}\\ &\Longleftrightarrow (a^\#abb^*)^\# = (abb^*)^\#a\\ &\Longleftrightarrow (abb^*)^\#a \in (a^\#abb^*)\{5\}\end{aligned}$$

$$\iff (abb^*)\{1,5\} \cdot a \subseteq (a^\# abb^*)\{5\}.$$

Notice that we verify the next characterizations for the rule $(abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger$ in the same manner as in the proof of Theorem 2.16.

Theorem 2.18. *If $a \in \mathcal{R}^\dagger$ and $b, abb^\#, a^\dagger abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $(abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger$,
(ii) $(a^\dagger abb^\#)^\# a^\dagger \in (abb^\#)\{5\}$,
(iii) $abb^\# aa^\dagger = abb^\#$ *and* $(a^\dagger abb^\#)^\# a^\dagger abb^\# a = abb^\#(a^\dagger abb^\#)^\# a^\dagger a$.

Remark 2.19. Now, we present the equivalent conditions for the reverse order laws $(abb^\#)^\# = (a^* abb^\#)^\# a^*$, $(a^\# ab)^\# = b^\dagger(a^\# abb^\dagger)^\#$ and $(a^\# ab)^\# = b^*(a^\# abb^*)^\#$, respectively.

(a) If $a \in \mathcal{R}^\dagger$ and $b, abb^\#, a^* abb^\# \in \mathcal{R}^\#$, then:

$$\begin{aligned}(abb^\#)^\# = (a^* abb^\#)^\# a^* &\Leftrightarrow (a^* abb^\#)^\# a^* \in (abb^\#)\{5\} \\ &\Leftrightarrow abb^\# aa^\dagger = abb^\# \quad \text{and} \quad (a^* abb^\#)^\# a^* abb^\# a \\ &= abb^\#(a^* abb^\#)^\# a^* a.\end{aligned}$$

(b) If $b \in \mathcal{R}^\dagger$ and $a, a^\# ab, a^\# abb^\dagger \in \mathcal{R}^\#$, then:

$$\begin{aligned}(a^\# ab)^\# = b^\dagger(a^\# abb^\dagger)^\# &\Leftrightarrow b^\dagger(a^\# abb^\dagger)^\# \in (a^\# ab)\{5\} \\ &\Leftrightarrow b^\dagger ba^\# ab = a^\# ab \text{ and } ba^\# abb^\dagger(a^\# abb^\dagger)^\# \\ &= bb^\dagger(a^\# abb^\dagger)^\# a^\# ab.\end{aligned}$$

(c) If $b \in \mathcal{R}^\dagger$ and $a, a^\# ab, a^\# abb^* \in \mathcal{R}^\#$, then:

$$\begin{aligned}(a^\# ab)^\# = b^*(a^\# abb^*)^\# &\Leftrightarrow b^*(a^\# abb^*)^\# \in (a^\# ab)\{5\} \\ &\Leftrightarrow b^\dagger ba^\# ab = a^\# ab \quad \text{and} \quad ba^\# abb^*(a^\# abb^*)^\# \\ &= bb^*(a^\# abb^*)^\# a^\# ab.\end{aligned}$$

In the next theorem, we investigate the relation between $(a^\dagger ab)\{5\} \subseteq b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\}$ and $(a^\dagger ab)\{5\} = b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\}$.

Theorem 2.20. *If $a \in \mathcal{R}^\dagger$ and $b, a^\dagger ab, a^\dagger abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $(a^\dagger ab)\{5\} \subseteq b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\}$ *and* $bb^\# a^\dagger ab = a^\dagger ab$,
(ii) $(a^\dagger ab)\{5\} = b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\}$.

Proof. (i) $\Rightarrow$ (ii). If $(a^\dagger ab)\{5\} \subseteq b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\}$ and $bb^\# a^\dagger ab = a^\dagger ab$, then there exist $b^{(1,5)} \in b\{1,5\}$ and $(a^\dagger abb^\#)^{(1,5)} \in (a^\dagger abb^\#)\{1,5\}$ such that $(a^\dagger ab)^\# = b^{(1,5)}(a^\dagger abb^\#)^{(1,5)}$. Since the equalities (2) and $bb^\# = bb^{(1,5)}$ hold, then $(a^\dagger abb^\#)^\# = (a^\dagger abb^\#)^{(1,5)} a^\dagger abb^\# (a^\dagger abb^\#)^{(1,5)}$ and

$$\begin{aligned} b^\#(a^\dagger abb^\#)^\# &= b^\#(bb^\#)(a^\dagger abb^\#)^{(1,5)} a^\dagger a(bb^\#)(a^\dagger abb^\#)^{(1,5)} \\ &= b^\# b(b^{(1,5)}(a^\dagger abb^\#)^{(1,5)}) a^\dagger ab(b^{(1,5)}(a^\dagger abb^\#)^{(1,5)}) \\ &= b^\# b(a^\dagger ab)^\# a^\dagger ab(a^\dagger ab)^\# = (b^\# ba^\dagger ab)[(a^\dagger ab)^\#]^2 \\ &= a^\dagger ab[(a^\dagger ab)^\#]^2 = (a^\dagger ab)^\#. \end{aligned}$$

By Theorem 2.16, we conclude that $b\{1,5\} \cdot (a^\dagger abb^\#)\{1,5\} \subseteq (a^\dagger ab)\{5\}$. Hence, statement (ii) holds.

(ii) $\Rightarrow$ (i). This is obvious, by Theorem 2.16. □

In the same way as in Theorem 2.20, we can consider the conditions

$$\begin{aligned} &(a^* ab)\{5\} \subseteq b\{1,5\} \cdot (a^* abb^\#)\{1,5\} \quad \text{and} \quad bb^\# a^* ab = a^* ab; \\ &(abb^\dagger)\{5\} \subseteq (a^\# abb^\dagger)\{1,5\} \cdot a\{1,5\} \quad \text{and} \quad abb^\dagger a^\# a = abb^\dagger; \\ &(abb^*)\{5\} \subseteq (a^\# abb^*)\{1,5\} \cdot a\{1,5\} \quad \text{and} \quad abb^* a^\# a = abb^*. \end{aligned}$$

Also, as in Theorem 2.20, we can obtain that the inclusions $(a^\dagger ab)\{5\} \subseteq b^\# \cdot (a^\dagger abb^\#)\{1,5\}$, $(a^* ab)\{5\} \subseteq b^\# \cdot (a^* abb^\#)\{1,5\}$, $(abb^\dagger)\{5\} \subseteq (a^\# abb^\dagger)\{1,5\} \cdot a^\#$, $(abb^*)\{5\} \subseteq (a^\# abb^*)\{1,5\} \cdot a^\#$ are always the equalities.

Theorem 2.21. *If $a \in \mathcal{R}^\dagger$ and $b, a^\dagger ab, a^\dagger abb^\# \in \mathcal{R}^\#$, then the inclusion $(a^\dagger abb^\#)\{5\} \subseteq b \cdot (a^\dagger ab)\{1,5\}$ is always an equality.*

Proof. Let $(a^\dagger abb^\#)\{5\} \subseteq b \cdot (a^\dagger ab)\{1,5\}$. Then there exists $(a^\dagger ab)^{(1,5)} \in (a^\dagger ab)\{1,5\}$ such that $(a^\dagger abb^\#)^\# = (a^\dagger ab)^{(1,5)} b$. The combination of the equality

$$\begin{aligned} b(a^\dagger ab)^\# &= b(a^\dagger ab)^{(1,5)}a^\dagger ab(a^\dagger ab)^{(1,5)} \\ &= (b(a^\dagger ab)^{(1,5)})a^\dagger abb^\#(b(a^\dagger ab)^{(1,5)}) \\ &= (a^\dagger abb^\#)^\# a^\dagger abb^\#(a^\dagger abb^\#)^\# = (a^\dagger abb^\#)^\# \end{aligned}$$

and Theorem 2.16 give that $(a^\dagger abb^\#)\{5\} = b \cdot (a^\dagger ab)\{1,5\}$ holds. □

For the relations $(a^*abb^\#)\{5\} \subseteq b \cdot (a^*ab)\{1,5\}$, $(a^\# abb^\dagger)\{5\} \subseteq (abb^\dagger)\{1,5\} \cdot a$ and $(a^\# abb^*)\{5\} \subseteq (abb^*)\{1,5\} \cdot a$, we can get the same results as in Theorem 2.18.

In the next theorem, some sufficient conditions for the reverse order law $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$ are presented.

Theorem 2.22. *Suppose that $a \in \mathcal{R}^\dagger$ and $b, ab, a^\dagger ab, abb^\#, a^\dagger abb^\# \in \mathcal{R}^\#$. Then each of the following conditions is sufficient for $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$ to hold:*

(i) $(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$ *and* $(abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger$,
(ii) $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ *and* $(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$,
(iii) $(ab)^\# = b^\#(abb^\#)^\#$ *and* $(abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger$.

Proof.

(i) Assume that $(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$ and $(abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger$. Then, $b^\#(a^\dagger abb^\#)^\# a^\dagger \in (ab)\{5\}$, by

$$\begin{aligned} abb^\#(a^\dagger abb^\#)^\# a^\dagger &= (a^\dagger abb^\#)^\# a^\dagger abb^\# = a^\dagger abb^\#(a^\dagger abb^\#)^\# \\ &= b^\#(a^\dagger abb^\#)^\# a^\dagger ab. \end{aligned}$$

It is easy to verify that $b^\#(a^\dagger abb^\#)^\# a^\dagger \in (ab)\{1,2\}$ which gives $(ab)^\# = b^\#(a^\dagger abb^\#)^\# a^\dagger$.

(ii) The equalities $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ and $(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$ imply

$$(ab)^\# = (a^\dagger ab)^\# a^\dagger = b^\#(a^\dagger abb^\#)^\# a^\dagger.$$

(iii) In the same manner as part (ii). □

The conditions $(a^\dagger ab)^\# = b^\#(a^\dagger abb^\#)^\#$ and $(abb^\#)^\# = (a^\dagger abb^\#)^\# a^\dagger$ in Theorem 2.22 can be replaced by some equivalent conditions from Theorems 2.16 and 2.18.

Remark 2.23. Analogously to Theorem 2.22, we obtain the next results.

(a) If $a \in \mathcal{R}^\dagger$ and $b, ab, a^*ab, abb^\#, a^*abb^\# \in \mathcal{R}^\#$, then each of the following conditions is sufficient for $(ab)^\# = b^\#(a^*abb^\#)^\# a^*$ to hold:
   (i) $(a^*ab)^\# = b^\#(a^*abb^\#)^\#$ and $(abb^\#)^\# = (a^*abb^\#)^\# a^*$,
   (ii) $(ab)^\# = (a^*ab)^\# a^*$ and $(a^*ab)^\# = b^\#(a^*abb^\#)^\#$,
   (iii) $(ab)^\# = b^\#(abb^\#)^\#$ and $(abb^\#)^\# = (a^*abb^\#)^\# a^*$.

(b) If $b \in \mathcal{R}^\dagger$ and $a, ab, a^\#ab, abb^\dagger, a^\#abb^\dagger \in \mathcal{R}^\#$, then each of the following conditions is sufficient for $(ab)^\# = b^\dagger(a^\#abb^\dagger)^\# a^\#$ to hold:
   (i) $(a^\#ab)^\# = b^\dagger(a^\#abb^\dagger)^\#$ and $(abb^\dagger)^\# = (a^\#abb^\dagger)^\# a^\#$,
   (ii) $(ab)^\# = (a^\#ab)^\# a^\#$ and $(a^\#ab)^\# = b^\dagger(a^\#abb^\dagger)^\#$,
   (iii) $(ab)^\# = b^\dagger(abb^\dagger)^\#$ and $(abb^\dagger)^\# = (a^\#abb^\dagger)^\# a^\#$.

(c) If $b \in \mathcal{R}^\dagger$ and $a, ab, a^\#ab, abb^*, a^\#abb^* \in \mathcal{R}^\#$, then each of the following conditions is sufficient for $(ab)^\# = b^*(a^\#abb^*)^\# a^\#$ to hold:
   (i) $(a^\#ab)^\# = b^*(a^\#abb^*)^\#$ and $(abb^*)^\# = (a^\#abb^*)^\# a^\#$,
   (ii) $(ab)^\# = (a^\#ab)^\# a^\#$ and $(a^\#ab)^\# = b^*(a^\#abb^*)^\#$,
   (iii) $(ab)^\# = b^*(abb^*)^\#$ and $(abb^*)^\# = (a^\#abb^*)^\# a^\#$.

Now, we study the relation between the reverse order laws $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ and $(a^\dagger ab)^\# = (ab)^\# a$.

Theorem 2.24. *If $b \in \mathcal{R}$, $a \in \mathcal{R}^\dagger$ and if $ab, a^\dagger ab \in \mathcal{R}^\#$, then the following statements are equivalent*:

(i) $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ *and* $a^\dagger aba^\dagger a = a^\dagger ab$,
(ii) $(a^\dagger ab)^\# = (ab)^\# a$ *and* $abaa^\dagger = ab$.

Proof. (i) $\Rightarrow$ (ii). Using $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ and $a^\dagger aba^\dagger a = a^\dagger ab$, we get

$$abaa^\dagger = (ab)^2(ab)^\# aa^\dagger = (ab)^2(a^\dagger ab)^\# a^\dagger aa^\dagger = (ab)^2(a^\dagger ab)^\# a^\dagger = ab$$

and

$$\begin{aligned}(ab)^\# a &= (a^\dagger ab)^\# a^\dagger a = [(a^\dagger ab)^\#]^2 (a^\dagger aba^\dagger a)\\ &= [(a^\dagger ab)^\#]^2 a^\dagger ab = (a^\dagger ab)^\#.\end{aligned}$$

Hence, item (ii) is satisfied.

(ii) $\Rightarrow$ (i). The equalities $(a^\dagger ab)^\# = (ab)^\# a$ and $abaa^\dagger = ab$ give

$$\begin{aligned}a^\dagger aba^\dagger a &= (a^\dagger ab)^2 (a^\dagger ab)^\# a^\dagger a = (a^\dagger ab)^2 (ab)^\# aa^\dagger a\\ &= (a^\dagger ab)^2 (ab)^\# a = a^\dagger ab\end{aligned}$$

and

$$(a^\dagger ab)^\# a^\dagger = (ab)^\# aa^\dagger = [(ab)^\#]^2 (abaa^\dagger) = [(ab)^\#]^2 ab = ab.$$

So, statement (i) holds. □

Remark 2.25. As in Theorem 2.24, we can show the following.

(a) If $b \in \mathcal{R}$, $a \in \mathcal{R}^\dagger$ and if $ab, a^*ab \in \mathcal{R}^\#$, then the following statements are equivalent:

$$(ab)^\# = (a^*ab)^\# a^\dagger$$

and

$$a^*aba^\dagger a = a^*ab \Longleftrightarrow (a^*ab)^\# = (ab)^\# a$$

and

$$abaa^\dagger = ab.$$

(b) If $a \in \mathcal{R}$, $b \in \mathcal{R}^\dagger$ and if $ab, abb^\dagger \in \mathcal{R}^\#$, then:

$$(ab)^\# = b^\dagger (abb^\dagger)^\#$$

and

$$bb^\dagger abb^\dagger = abb^\dagger \Longleftrightarrow (abb^\dagger)^\# = b(ab)^\#$$

and

$$b^\dagger bab = ab.$$

(c) If $a \in \mathcal{R}$, $b \in \mathcal{R}^\dagger$ and if $ab, abb^* \in \mathcal{R}^\#$, then:

$$(ab)^\# = b^\dagger (abb^*)^\#$$

and
$$bb^\dagger abb^* = abb^* \Longleftrightarrow (abb^*)^\# = b(ab)^\#$$
and
$$b^\dagger bab = ab.$$

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