ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 3, 2014

# POSITIVE SOLUTIONS TO A THREE POINT FOURTH ORDER FOCAL BOUNDARY VALUE PROBLEM

JOHN R. GRAEF AND BO YANG

ABSTRACT. We consider a three point fourth order boundary value problem of focal type. Some upper and lower estimates for positive solutions of the problem are obtained. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are established. An example is included to illustrate the results.

**1. Introduction.** Boundary value problems are important both from a theoretical perspective as well as for their many applications in the physical and engineering sciences. The study of positive solutions for boundary value problems has been very active for the last two decades. In an interesting paper in this journal **[3]**, Anderson and Avery considered the fourth order four-point right focal boundary value problem

$$x''''(t) + f(x(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$x(0) = x'(q) = x''(r) = x'''(1) = 0, \tag{2}$$

under the assumption that $0 < q < r < 1$. We see that the case $q = r$ was not covered in **[3]** and it is this that motivates our work here.

We consider the three point fourth order boundary value problem

$$u''''(t) + g(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(p) = u''(p) = u'''(1) = 0. \tag{4}$$

Although our primary motivation for this work is to consider the case $q = r$ that is not covered by Anderson and Avery, it is worth pointing out that boundary conditions of the type (4) do have a physical

2010 AMS *Mathematics subject classification.* Primary 34B15, 34B18.
*Keywords and phrases.* Fixed point theorem, cone, nonlinear boundary-value problem, positive solution.
Received by the editors on July 21, 2010, and in revised form on January 20, 2012.
DOI:10.1216/RMJ-2014-44-3-937 

interpretation. For example, if we were modeling an elastic beam under a load, conditions (4) mean that the beam is held at the left end and the load is distributed in such a way that the slope of the beam and the bending moment at $t = p$ are zero, and the shear force at the right end of the beam vanishes. (See, for example, McLachlan **[13]**.) Another place where problems of this type can arise is in Abel-Gontscharoff interpolation (see Agarwal **[1]**).

As is pointed out in **[3]**, if in problem (1)–(2) we let $q \to 0$ and $r \to 1$, then condition (2) becomes

$$x(0) = x'(0) = x''(1) = x'''(1) = 0, \tag{5}$$

which would model a beam with the left end clamped and the right end free with zero bending moment and shear force. Our condition (4) only involves one intermediate point, and letting $p \to 0$ gives the well-known right focal boundary conditions

$$u(0) = u'(0) = u''(0) = u'''(1) = 0. \tag{6}$$

On the other hand, letting $p \to 1$ gives

$$u(0) = u'(1) = u''(1) = u'''(1) = 0, \tag{7}$$

which describes a beam attached at the left end and having a sliding clamp at the right end point.

Throughout the paper, we assume that

(H1) $f : [0,\infty) \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$ are continuous functions, $g(t) \not\equiv 0$ on $[0,1]$ and $p \in (0,1)$ is a constant.

The main purpose of this paper is to prove some upper and lower estimates for positive solutions of the problem (3)–(4). As an application of the estimates, we shall establish some existence and nonexistence results for positive solutions to the problem (3)–(4). Here, by a *positive solution*, we mean a solution $u(t)$ such that $u(t) > 0$ for $0 < t < 1$.

Now we define $G : [0,1] \times [0,1] \to (-\infty,\infty)$ by

$$G(t,s) = \begin{cases} s^3/6, & \text{if } s \le p \text{ and } s \le t, \\ ((t-p)^3 + p^3)/6, & \text{if } s > p \text{ and } s > t, \\ ((t-s)^3 + s^3)/6, & \text{if } s \le p \text{ and } s > t, \\ (p^3 + (s-t)^3 + (t-p)^3)/6, & \text{if } s > p \text{ and } s \le t. \end{cases}$$

Then, $G(t,s)$ is the Green function for the problem (3)–(4), and the problem (3)–(4) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \le t \le 1. \tag{8}$$

We will use the following theorem, known as the Krasnosel'skii fixed point theorem (see **[12]**), to prove some of our results.

**Theorem 1.1.** *Let $(X, \|\cdot\|)$ be a Banach space over the reals, and let $P \subset X$ be a cone in $X$. Let $H_1$ and $H_2$ be real numbers such that $H_2 > H_1 > 0$, and let*

$$\Omega_i = \{v \in X \mid \|v\| < H_i\}, \quad i = 1, 2.$$

*If $L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$ is a completely continuous operator such that, either*

(K1) $\|Lv\| \le \|v\|$ *if* $v \in P \cap \partial\Omega_1$, *and* $\|Lv\| \ge \|v\|$ *if* $v \in P \cap \partial\Omega_2$, *or*
(K2) $\|Lv\| \ge \|v\|$ *if* $v \in P \cap \partial\Omega_1$, *and* $\|Lv\| \le \|v\|$ *if* $v \in P \cap \partial\Omega_2$,

*then $L$ has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.*

Throughout the paper, we let $X = C[0,1]$ be equipped with norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad v \in X.$$

We also define

$$F_0 = \limsup_{x\to 0^+}(f(x)/x), \qquad f_0 = \liminf_{x\to 0^+}(f(x)/x),$$
$$F_\infty = \limsup_{x\to +\infty}(f(x)/x), \qquad f_\infty = \liminf_{x\to +\infty}(f(x)/x).$$

These constants will be used later in the statements of our existence and nonexistence theorems.

This paper is organized as follows. In Section 2, we obtain some new upper and lower estimates for positive solutions to the problem (3)-(4). In Sections 3 and 4, we establish some new existence and nonexistence results for positive solutions of the problem. An example is given at the end of the paper to illustrate the main results of the paper.

**2. Estimates for positive solutions.** In this section, we shall prove some upper and lower estimates for positive solutions of the problem (3)–(4). Our first lemma provides some information about the sign property of $G(t,s)$.

**Lemma 2.1.** *The function $G(t,s)$ has the following sign properties.*

(1) *If $(t,s) \in [0,1] \times [0,1]$, then $G(t,s) \geq 0$.*
(2) *If $(t,s) \in (0,1) \times (0,1)$, then $G(t,s) > 0$.*

*Proof.* Let $(t,s) \in [0,1] \times [0,1]$. If $s \leq p$ and $s \leq t$, then

$$G(t,s) = \frac{s^3}{6} \geq 0.$$

If $s > p$ and $s > t$, then

$$G(t,s) = \frac{(t-p)^3 + p^3}{6} = \frac{t}{6}\left(\left(t - \frac{3p}{2}\right)^2 + \frac{3p^2}{4}\right) \geq 0.$$

If $s \leq p$ and $s > t$, then

$$G(t,s) = \frac{(t-s)^3 + s^3}{6} = \frac{t}{6}\left(\left(t - \frac{3s}{2}\right)^2 + \frac{3s^2}{4}\right) \geq 0.$$

If $s > p$ and $s \leq t$, then $s - t > p - t$, and

$$G(t,s) = \frac{p^3 + (s-t)^3 + (t-p)^3}{6} > \frac{p^3 + (p-t)^3 + (t-p)^3}{6} = \frac{p^3}{6} \geq 0.$$

So part (1) of the lemma is proved.

If we take a closer look at the above four cases, we will see that part (2) of the lemma is also true. The proof of the lemma is now complete. $\square$

**Lemma 2.2.** *If $u \in C^4[0,1]$ satisfies the boundary conditions* (4) *and is such that*

$$u''''(t) \leq 0 \quad \text{for } 0 \leq t \leq 1, \tag{9}$$

*then $u'(t) \geq 0$ for $0 \leq t \leq 1$, and*

$$0 \leq u(t) \leq u(1), \quad 0 \leq t \leq 1. \tag{10}$$

*Proof.* Since (9) holds and $u'''(1) = 0$, we have $u'''(t) \geq 0$ for $0 \leq t \leq 1$. This means that $u'$ is concave upward on $[0,1]$. Because $u'(p) = u''(p) = 0$, we have $u'(t) \geq 0$ for $0 \leq t \leq 1$. Therefore, $u(t)$ is nondecreasing on $[0,1]$. Combining this with the fact that $u(0) = 0$, we get (10). The proof is now complete. □

Our next lemma proves to be very useful in our efforts to obtain upper and lower estimates on solutions.

**Lemma 2.3.** *Let $q_1$, $q_2$, and $q_3$ be real numbers such that $q_1 < q_2 < q_3$.*

(1) *If $u \in C^3[q_1, q_3]$ is such that $u(q_1) = 0$, $u(q_2) < 0$, $u(q_3) \geq 0$, $u'(q_3) = u''(q_3) = 0$, then there exist $\beta_1, \beta_2 \in (q_1, q_3)$ such that $\beta_1 < \beta_2$, $u'''(\beta_1) < 0$, and $u'''(\beta_2) > 0$.*
(2) *If $u \in C^3[q_1, q_3]$ is such that $u(q_3) = 0$, $u(q_2) < 0$, $u(q_1) \geq 0$, $u'(q_1) = u''(q_1) = 0$, then there exist $\beta_1, \beta_2 \in (q_1, q_3)$ such that $\beta_1 < \beta_2$, $u'''(\beta_1) < 0$, and $u'''(\beta_2) > 0$.*

*Proof.* We shall prove part (1) of the lemma only. By the mean value theorem, since $u(q_1) = 0 > u(q_2)$ and $u(q_2) < 0 \leq u(q_3)$, there exist $q_4 \in (q_1, q_2)$ and $q_5 \in (q_2, q_3)$ such that $u'(q_4) < 0$ and $u'(q_5) > 0$. Since $u'(q_4) < 0 < u'(q_5)$ and $u'(q_5) > 0 = u'(q_3)$, there exist $q_6 \in (q_4, q_5)$ and $q_7 \in (q_5, q_3)$ such that $u''(q_6) > 0$ and $u''(q_7) < 0$. Since $u''(q_6) > 0 > u''(q_7)$ and $u''(q_7) < 0 = u''(q_3)$, there exist $\beta_1 \in (q_6, q_7)$ and $\beta_2 \in (q_7, q_3)$ such that $u'''(\beta_1) < 0$ and $u'''(\beta_2) > 0$. Thus, we have proved part (1) of the lemma.

Part (2) of the lemma can be proved in a very similar way. □

The next two lemmas will yield lower and upper bounds on solutions to our problem. We define the function $a : [0,1] \to [0, +\infty)$ by

$$a(t) = \frac{(t-p)^3 + p^3}{3p^2 - 3p + 1}, \quad 0 \leq t \leq 1.$$

**Lemma 2.4.** *If $u \in C^4[0,1]$ satisfies* (4) *and* (9), *then*

$$u(t) \geq a(t)u(1) \quad \text{for} \quad 0 \leq t \leq 1.$$

*Proof.* If we let $h(t) = u(t) - a(t)u(1)$, $0 \leq t \leq 1$, then

$$h'(t) = u'(t) - u(1) \cdot 3(t-p)^2/(3p^2 - 3p + 1), \tag{11}$$

$$h''(t) = u''(t) - u(1) \cdot 6(t-p)/(3p^2 - 3p + 1), \tag{12}$$

$$h'''(t) = u'''(t) - u(1) \cdot 6/(3p^2 - 3p + 1), \tag{13}$$

$$h''''(t) = u''''(t) \le 0, \quad 0 \le t \le 1. \tag{14}$$

It follows from the definition of $h(t)$ that $h(0) = h(1) = 0$. We also note that (14) implies that $h''(t)$ is concave downward and $h'''(t)$ is nonincreasing. It is easy to see from (11) and (12) that $h'(p) = h''(p) = 0$. To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$.

**Claim I.** $h(p) \ge 0$.

*Proof of Claim* I. Assume to the contrary that $h(p) < 0$. Since $h(0) = 0$, there exists $s \in (0, p)$ such that $h'(s) < 0$. Since $h'(p) = 0$, there exists $r \in (s, p)$ such that $h''(r) > 0$. Since $h''(p) = 0$ and $h''$ is concave downward, we have $h''(t) < 0$ for $p < t < 1$. Because $h'(p) = 0$, we have $h'(t) < 0$ for $p < t < 1$. Because $h(p) < 0$, we have $h(t) < 0$ for $p < t \le 1$. In particular, we have $h(1) < 0$. This contradicts the fact that $h(1) = 0$. The proof of the claim is complete. □

**Claim II.** $h(t) \ge 0$ for $0 \le t \le p$.

*Proof of Claim* II. Assume to the contrary that there exists $t_0 \in (0, p)$ such that $h(t_0) < 0$. Then we have

$$h(0) = 0, \qquad h(t_0) < 0, \qquad h(p) \ge 0, \qquad h'(p) = h''(p) = 0.$$

By part (1) of Lemma 2.3, there exist $\beta_1, \beta_2 \in (0, p)$ such that $\beta_1 < \beta_2$, $h'''(\beta_1) < 0$ and $h'''(\beta_2) > 0$. This contradicts the fact that $h'''$ is nonincreasing. The proof of the claim is complete. □

**Claim III.** $h(t) \ge 0$ for $p \le t \le 1$.

*Proof of Claim* III. Assume there exists $t_1 \in (p, 1)$ such that $h(t_1) < 0$. Then we have

$$h(1) = 0, \qquad h(t_1) < 0, \qquad h(p) \ge 0, \qquad h'(p) = h''(p) = 0.$$

By part (2) of Lemma 2.3, there exist $\tau_1, \tau_2 \in (0, p)$ such that $\tau_1 < \tau_2$, $h'''(\tau_1) < 0$ and $h'''(\tau_2) > 0$. This contradicts the fact that $h'''$ is nonincreasing. The proof of the claim is complete. □

In summary, we have $h(t) \ge 0$ for $0 \le t \le 1$, and this proves the lemma. □

Now we define the function $c:[0,1]\to[0,+\infty)$ by

$$c(t)=\frac{(-12p^2+8p^3)t+(12p-6p^2)t^2-4t^3+t^4}{-18p^2+8p^3+12p-3}.$$

**Lemma 2.5.** *If $u\in C^4[0,1]$ satisfies* (4) *and* (9), *and $u''''(t)$ is nonincreasing on $[0,1]$, then*

$$u(t)\le c(t)u(1)\quad \textit{for } 0\le t\le 1.$$

*Proof.* If we define $h(t)=c(t)u(1)-u(t)$, $0\le t\le 1$, then

$$h'(t)=u(1)\cdot\frac{(-12p^2+8p^3)+(24p-12p^2)t-12t^2+4t^3}{-18p^2+8p^3+12p-3}-u'(t), \tag{15}$$

$$h''(t)=u(1)\cdot\frac{(24p-12p^2)-24t+12t^2}{-18p^2+8p^3+12p-3}-u''(t), \tag{16}$$

$$h'''(t)=u(1)\cdot\frac{-24+24t}{-18p^2+8p^3+12p-3}-u'''(t), \tag{17}$$

$$h''''(t)=u(1)\cdot\frac{24}{-18p^2+8p^3+12p-3}-u''''(t),\quad 0\le t\le 1. \tag{18}$$

It follows from the definition of $h(t)$ that $h(0)=h(1)=0$. We also note that (18) implies that $h''''(t)$ is nondecreasing and $h'''$ is concave upward on $[0,1]$. It is easy to see from (15) and (16) that $h'(p)=h''(p)=0$. To prove the lemma, it suffices to show that $h(t)\ge 0$ for $0\le t\le 1$.

**Claim I.** $h(p)\ge 0$.

*Proof of Claim* I. Assume that $h(p)<0$. Since $h(0)=0$, there exists $s\in(0,p)$ such that $h'(s)<0$. Since $h'(p)=0$, there exists $r\in(s,p)$ such that $h''(r)>0$. Since $h''(p)=0$, there exists $\beta\in(s,p)$ such that $h'''(\beta)<0$. It is easy to see that $h'''(1)=0$. Because $h'''$ is concave upward, we have $h'''(t)<0$ for $\beta<t<1$. In particular, $h'''(t)<0$ for $p<t<1$. Since $h''(p)=0$, we have $h''(t)<0$ for $p<t<1$. Because $h'(p)=0$, we have $h'(t)<0$ for $p<t<1$. Since $h(p)<0$, we have $h(t)<0$ for $p<t\le 1$. This contradicts the fact that $h(1)=0$. The proof of the claim is complete. $\square$

**Claim II.** $h(t) \geq 0$ for $0 \leq t \leq p$.

*Proof of Claim* II. Assume there exists $t_0 \in (0, p)$ such that $h(t_0) < 0$. Then we have

$$h(0) = 0, \qquad h(t_0) < 0, \qquad h(p) \geq 0, \qquad h'(p) = h''(p) = 0.$$

By part (1) of Lemma 2.3, there exist $\beta_1, \beta_2 \in (0, p)$ such that $\beta_1 < \beta_2$, $h'''(\beta_1) < 0$ and $h'''(\beta_2) > 0$. Note that (17) implies that $h'''(1) = 0$. Because $h'''(\beta_1) < 0 = h'''(1) = 0$ and $h'''$ is concave upward, we have $h'''(t) < 0$ for $\beta_1 < t < 1$. This contradicts the fact that $h'''(\beta_2) > 0$. This completes the proof of the claim. $\square$

**Claim III.** $h(t) \geq 0$ for $p \leq t \leq 1$.

*Proof of Claim* III. Assume there exists $t_1 \in (p, 1)$ such that $h(t_1) < 0$. Then we have

$$h(1) = 0, \qquad h(t_1) < 0, \qquad h(p) \geq 0, \qquad h'(p) = h''(p) = 0.$$

By part (2) of Lemma 2.3, there exist $\tau_1, \tau_2 \in (p, 1)$ such that $\tau_1 < \tau_2$, $h'''(\tau_1) < 0$ and $h'''(\tau_2) > 0$. Note that (17) implies that $h'''(1) = 0$. Because $h'''(\tau_1) < 0 = h'''(1) = 0$ and $h'''$ is concave upward, we have $h'''(t) < 0$ for $\tau_1 < t < 1$. This contradicts the fact that $h'''(\tau_2) > 0$. This completes the proof of the claim. $\square$

In summary, we have $h(t) \geq 0$ for $0 \leq t \leq 1$, and this completes the proof of the lemma. $\square$

Now we can obtain bounds on solutions to the problem (3)–(4).

**Theorem 2.6.** *Suppose that, in addition to* (H1), *the following condition holds*:

(H2) *Both $f$ and $g$ are non-decreasing functions.*

*If $u \in C^4[0, 1]$ is a non-negative solution of the problem* (3)–(4), *then $u(t) \leq c(t)u(1)$ for $0 \leq t \leq 1$.*

*Proof.* Suppose that $u \in C^4[0, 1]$ is a non-negative solution of the problem (3)–(4). Obviously $u(t)$ satisfies (4) and (9). From Lemma 2.2, we see that $u(t)$ is nondecreasing. If (H2) holds, then

$$u^{(4)}(t) = -g(t)f(u(t))$$

is nonincreasing on $[0, 1]$. Now it follows immediately from Lemma 2.5 that $u(t) \leq c(t)u(1)$ for $0 \leq t \leq 1$. The proof is now complete. $\square$

**Theorem 2.7.** *Suppose that* (H1) *holds. If* $u \in C^4[0, 1]$ *is a non-negative solution of the problem* (3)–(4), *then* $u(t) \geq a(t)u(1)$ *for* $0 \leq t \leq 1$.

Theorem 2.7 follows directly from Lemma 2.4. Note that Theorems 2.6 and 2.7 provide some upper and lower estimates for positive solutions for the boundary value problem (3)–(4). These upper and lower estimates are new and have not been obtained before.

Now we define

$$P = \{v \in X \mid v(1) \geq 0,\ a(t)v(1) \leq v(t) \leq v(1) \quad \text{on } [0, 1]\},$$

and

$$Q = \left\{ v \in X \;\middle|\; \begin{array}{l} v(1) \geq 0, v(t) \text{ is non-decreasing, and} \\ a(t)v(1) \leq v(t) \leq c(t)v(1) \text{ on } [0, 1] \end{array} \right\}.$$

Then, it is easily seen that both $P$ and $Q$ are positive cones of the Banach space $X$. We then have the following result.

**Lemma 2.8.** *If* $u \in P$ *or* $u \in Q$, *then* $u(1) = \|u\|$.

*Proof.* If $u \in P$, then

$$u(1) \geq u(t) \geq u(1)a(t) \geq 0, \quad 0 \leq t \leq 1.$$

Hence $u(1) = \|u\|$.

Note that $Q$ is a subset of $P$. If $u \in Q$, then $u \in P$ and therefore $u(1) = \|u\|$. $\square$

With these definitions of $P$ and $Q$, we can restate Theorems 2.6 and 2.7 as follows.

**Theorem 2.9.** *Suppose that* (H1) *holds. If* $u(t)$ *is a non-negative solution to the problem* (3)–(4), *then* $u \in P$.

**Theorem 2.10.** *Suppose that* (H1) *and* (H2) *hold. If* $u(t)$ *is a non-negative solution to the problem* (3)–(4), *then* $u \in Q$.

Define an operator $T : P \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1.$$

Now the integral equation (8) is equivalent to the equality

$$Tu = u, \quad u \in P.$$

It is well known that $T : P \to X$ is a completely continuous operator. In order to solve the problem (3)–(4), we only need to find a fixed point of $T$.

By arguments similar to those used in the proofs of Theorems 2.6 and 2.7, the next two theorems can be proved without any difficulty.

**Theorem 2.11.** *If* (H1) *holds, then* $T(P) \subset P$.

**Theorem 2.12.** *If* (H1) *and* (H2) *hold, then* $T(Q) \subset Q$.

## 3. Existence results.

We define

$$A = \int_0^1 G(1,s)g(s)a(s)\,ds, \qquad B = \int_0^1 G(1,s)g(s)\,ds,$$

and

$$C = \int_0^1 G(1,s)g(s)c(s)\,ds.$$

The next theorem is our first existence result.

**Theorem 3.1.** *Suppose that* (H1) *holds. If* $BF_0 < 1 < Af_\infty$*, then the problem* (3)–(4) *has at least one positive solution.*

*Proof.* Choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B < 1$. There exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \varepsilon)x \quad \text{for } 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\le \int_0^1 G(1,s)g(s)(F_0+\varepsilon)u(s)\,ds \\ &\le (F_0+\varepsilon)\|u\|\int_0^1 G(1,s)g(s)ds \\ &= (F_0+\varepsilon)\|u\|B \\ &\le \|u\|,\end{aligned}$$

which means $\|Tu\| \le \|u\|$. So, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in P \cap \partial\Omega_1.$$

To construct $\Omega_2$, we choose $\beta \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta)\int_\beta^1 G(1,s)g(s)a(s)\,ds > 1.$$

There exists $H_3 > 0$ such that

$$f(x) \ge (f_\infty - \delta)x \quad \text{for } x \ge H_3.$$

Let $H_2 = \max\{H_3(a(\beta))^{-1},\ 2H_1\}$. If $u \in P$ with $\|u\| = H_2$, then

$$u(t) \ge a(t)H_2 \ge a(\beta)(a(\beta))^{-1}H_3 = H_3 \quad \text{for } \beta \le t \le 1.$$

Therefore, if $u \in P$ with $\|u\| = H_2$, then

$$\begin{aligned}(Tu)(1) &\ge \int_\beta^1 G(1,s)g(s)f(u(s))\,ds \\ &\ge \int_\beta^1 G(1,s)g(s)(f_\infty-\delta)u(s)\,ds \\ &\ge (f_\infty-\delta)\|u\|\int_\beta^1 G(1,s)g(s)a(s)\,ds \\ &\ge \|u\|,\end{aligned}$$

which means $\|Tu\| \ge \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and so

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2.$$

Now condition (K1) of Theorem 1.1 is satisfied, so there exists a fixed point of $T$ in $P$.

This completes the proof of the theorem. $\square$

The proof of the following theorem is very similar to the one above and will therefore be omitted.

**Theorem 3.2.** *Suppose that* (H1) *holds. If* $BF_\infty < 1 < Af_0$, *then the problem* (3)–(4) *has at least one positive solution.*

The proofs of Theorems 3.3 and 3.4 below are very similar to those of Theorems 3.1 and 3.2. The only difference is that we use the positive cone $Q$, instead of $P$, in their proofs.

**Theorem 3.3.** *Suppose that* (H1) *and* (H2) *hold. If* $CF_0 < 1 < Af_\infty$, *then the problem* (3)–(4) *has at least one positive solution.*

**Theorem 3.4.** *Suppose that* (H1) *and* (H2) *hold. If* $CF_\infty < 1 < Af_0$, *then the problem* (3)–(4) *has at least one positive solution.*

**4. Nonexistence results.** In this section, we give some nonexistence results for positive solutions to the problem (3)–(4).

**Theorem 4.1.** *Suppose that* (H1) *holds. If* $Bf(x) < x$ *for all* $x > 0$, *then the problem* (3)–(4) *has no positive solutions.*

*Proof.* Assume to the contrary that $u(t)$ is a positive solution of the problem (3)–(4). Then $u \in P$, $u(t) > 0$ on $(0, 1]$, and

$$\begin{aligned} u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\le B^{-1}u(1)\int_0^1 G(1,s)g(s)\,ds \\ &= u(1). \end{aligned}$$

This contradiction proves the theorem. $\square$

The proofs of the next two theorems are quite similar to that of Theorem 4.1, and so we omit the details.

**Theorem 4.2.** *Suppose that* (H1) *holds. If* $Af(x) > x$ *for all* $x > 0$, *then the problem* (3)–(4) *has no positive solutions.*

**Theorem 4.3.** *Suppose that* (H1) *and* (H2) *hold. If* $Cf(x) < x$ *for all* $x > 0$, *then the problem* (3)–(4) *has no positive solutions.*

We conclude this paper with an example.

**Example 4.4.** Consider the boundary value problem

$$u''''(t) + g(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{19}$$

$$u(0) = u'(2/3) = u''(2/3) = u''(1) = 0, \tag{20}$$

where

$$g(t) = 1 + 2t, \quad 0 \le t \le 1,$$

and

$$f(x) = \frac{\lambda x(1+3x)}{1+x}, \quad x \ge 0.$$

Here $\lambda > 0$ is a parameter. The problem (19)–(20) is a special case of the problem (3)–(4) in which $p = 2/3$.

It is easy to see that, for the problem (19)–(20), we have $f_0 = F_0 = \lambda$ and $f_\infty = F_\infty = 3\lambda$. It is also easy to see that $\lambda x < f(x) < 3\lambda x$ for $x > 0$. Using the software *Maple* or *Mathematica*, we can easily compute the constants

$$A = \frac{803}{13608}, \quad B = \frac{211}{3240}, \quad \text{and} \quad C = \frac{429011}{6940080}.$$

From Theorem 3.1, we see that if

$$5.64882 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 15.35545,$$

then the problem (19)–(20) has at least one positive solution. From Theorems 4.1 and 4.2 we see that if

$$\text{either } \lambda \le \frac{1}{3B} \approx 5.11848 \text{ or } \lambda \ge \frac{1}{A} \approx 16.94646,$$

then the problem (19)–(20) has no positive solutions.

Note that the function $g(t)$ is increasing in $t$, and $f(x)$ is increasing in $x$ for each fixed $\lambda > 0$; therefore, Theorems 3.3 and 4.3 apply. From Theorem 3.3, we see that if

$$5.64882 \approx \frac{1}{3A} < \lambda < \frac{1}{C} \approx 16.17692,$$

then the problem (19)–(20) has at least one positive solution. From Theorem 4.3, we see that if

$$\lambda \leq \frac{1}{3C} \approx 5.39230,$$

then the problem (19)–(20) has no positive solutions.

This example shows that our existence and nonexistence results work very well.

## REFERENCES

**1**. R.P. Agarwal, *Focal boundary value problems for differential and difference equations*, Kluwer, Dordrecht, 1998.

**2**. R.P. Agarwal, D. O'Regan and V. Lakshmikantham, *Singular $(p, n-p)$ focal and $(n, p)$ higher order boundary value problems*, Nonlinear Anal. **42** (2000), 215–228.

**3**. D.R. Anderson and R.I. Avery, *A fourth-order four-point right focal boundary value problem*, Rocky Mountain J. Math. **36** (2006), 367–380.

**4**. D.R. Anderson and J.M. Davis, *Multiple solutions and eigenvalues for third-order right focal boundary value problems*, J. Math. Anal. Appl. **267** (2002), 135–157.

**5**. J.V. Baxley and C.R. Houmand, *Nonlinear higher order boundary value problems with multiple positive solutions*, J. Math. Anal. Appl. **286** (2003), 682–691.

**6**. C.J. Chyan and J. Henderson, *Multiple solutions for $(n, p)$ boundary value problems*, Dynam. Syst. Appl. **10** (2001), 53–61.

**7**. P.W. Eloe and J. Henderson, *Singular nonlinear boundary value problems for higher order ordinary differential equations*, Nonlinear Anal. **17** (1991), 1–10.

**8**. ———, *Positive solutions for higher order ordinary differential equations*, Elect. J. Diff. Equat. **1995** (1995), 1–8.

**9**. L.H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743–748.

**10**. J.R. Graef and B. Yang, *Boundary value problems for second order nonlinear differential equations*, Comm. Appl. Anal. **6** (2002), 273–288.

**11**. J. Henderson and H. Wang, *Positive solutions for nonlinear eigenvalue problems*, J. Math. Anal. Appl. **208** (1997), 252–259.

**12**. M.A. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

**13**. N.W. McLachlan, *Theory of vibrations*, Dover, New York, 1951.

**14**. B. Yang, *Positive solutions for the beam equation under certain boundary conditions*, Elect. J. Diff. Equat. **2005** (2005), 1–8.

**15**. ______, *Estimates of positive solutions to a boundary value problem for the beam equation*, Comm. Math. Anal. **2** (2007), 13–21.

**16**. ______, *Positive solutions to a boundary value problem for the beam equation*, Z. Anal. Anwend. **26** (2007), 221–230.

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403
**Email address**: **John-Graef@utc.edu**

Department of Mathematics and Statistics, Kennesaw State University, Kennesaw, GA 30144
**Email address**: **byang@kennesaw.edu**