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THE DETERMINANT MAPS AND K_0

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ABSTRACT. Given a commutative ring R, the determinant map $\det_0 : Rk_0(R) \to \mathrm{Pic}\,(R)$ given by $[M] - [R^m] \mapsto \langle \wedge^m M \rangle$ is a homomorphism from the additive group of $Rk_0(R)$ to the multiplicative group $\mathrm{Pic}\,(R)$. In this paper, some properties of the determinant map $\det_0$ are given and some results in [**5**] are extended.

All rings considered are associative with identity 1 unless otherwise specified. All modules will be unitary. We write P^n for the direct sum of n copies of a module P $(= P \oplus \cdots \oplus P,\ n$ times), $\overset{m}{\otimes} P$ for m tensor power of P $(= P \otimes \cdots \otimes P,\ m$ times) and $\wedge^n M$ for the nth exterior power of M. Let R be a ring, the Grothendieck group $K_0(R)$ is the abelian group by the following generators and relations: we take one generator $[P]$ for each isomorphism class of a finitely generated projective R-module P and one relation $[P] + [Q] = [P \oplus Q]$ for each pair P, Q of finitely generated projective R-modules. Each element of $K_0(R)$ can be written in the form $[P] - m[R]$ for some finitely generated projective R-module P and some integer m. An R-module M is called an invertible module if M is finitely generated projective of constant rank 1. If we write $\langle M \rangle$ for the isomorphism class of the invertible module M, it is clear that the set of all such $\langle M \rangle$ forms an abelian group under the operation $\langle M \rangle \langle N \rangle = \langle M \otimes N \rangle$ with $\langle R \rangle = 1$. This group is the Picard group of R, $\mathrm{Pic}\,(R)$.

Let R be a commutative ring, and write $\mathrm{Spec}\,(R)$ for the set of all prime ideals of R, the prime spectrum of R. We write $H_0(R)$ for the set of all continuous maps $\mathrm{Spec}\,(R) \to \mathbf{Z}$, where $\mathbf{Z}$ is given the discrete topology. There is a natural ring homomorphism $K_0(R) \overset{\mathrm{rank}}{\to} H_0(R)$ given by $[M] \mapsto f_M$, where $f_M : \mathrm{Spec}\,(R) \to \mathbf{Z}$ by $f_M(P) = \mathrm{rank}_P(M)$.

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The kernel of the homomorphism $K_0(R) \overset{\text{rank}}{\to} H_0(R)$ is denoted $Rk_0(R)$. For any commutative ring R, $K_0(R) \cong H_0(R) \oplus Rk_0(R)$.

In fact, any element of $Rk_0(R)$ is of the form $[M] - [R^m]$, where M is a finitely generated projective R-module of constant rank m. The map $\det_0 \colon Rk_0(R) \to \mathrm{Pic}\,(R)$ given by $[M] - [R^m] \mapsto \langle \wedge^m M \rangle$ is a homomorphism from the additive group of $Rk_0(R)$ to the multiplicative group $\mathrm{Pic}\,(R)$.

If R is a commutative ring, we can make $K_0(R)$ into a ring by putting $[P][Q] = [P \otimes_R Q]$ for finitely generated projective R-modules P, Q. From [**2**], we know that, if P is a finitely generated projective module over a commutative ring, then some tensor power of P is free if and only if some of the copies of P is free.

Let $y = [P] - m[R] \in \mathrm{Tor}\,(K_0(R))$, then there exists a positive integer n such that $0 = ny = n([P] - m[R])$, i.e., $P^n \oplus R^t \cong R^{mn} \oplus R^t$ for some non-negative t. So $\mathrm{rank}\,(P) = m$ and $y = [P] - m[R] \in Rk_0(R)$, i.e., $\mathrm{Tor}\,(K_0(R)) \subseteq Rk_0(R)$. Considering the map

$$\det_0|_{\mathrm{Tor}\,(K_0(R))} \colon \mathrm{Tor}\,(K_0(R)) \longrightarrow \mathrm{Pic}\,(R),$$

by $[P] - m[R] \mapsto \langle \wedge^m P \rangle$ we have the following result.

Lemma 1. *For a commutative ring R,* $\det_0(\mathrm{Tor}\,(K_0(R))) \subseteq \mathrm{Tor}\,(\mathrm{Pic}\,(R))$. *Moreover,* $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ *is a surjective homomorphism from the additive group* $\mathrm{Tor}\,(K_0(R))$ *to the multiplicative group* $\mathrm{Tor}\,(\mathrm{Pic}\,(R))$.

Proof. Clearly, $\det_0|_{\mathrm{Tor}\,(K_0(R))}(\mathrm{Tor}\,(K_0(R))) \subseteq \mathrm{Tor}\,(\mathrm{Pic}\,(R))$. Now we shall prove that $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ is a surjective homomorphism. Let $\langle Q_0 \rangle \in \mathrm{Tor}\,(\mathrm{Pic}\,(R))$. Then there is a positive integer n such that $\langle Q_0 \rangle^n = \langle R \rangle$, i.e., $\otimes^n Q_0 \cong R$. By [**2**], Q_0^m is free for some positive integer m and then $Q_0^m \cong R^m$. Then, $[Q_0] - [R] \in \mathrm{Tor}\,(K_0(R))$ and $\det_0|_{\mathrm{Tor}\,(K_0(R))}([Q_0] - [R]) = \langle Q_0 \rangle$. So $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ is a surjective homomorphism from the additive group $\mathrm{Tor}\,(K_0(R))$ to the multiplicative group $\mathrm{Tor}\,(\mathrm{Pic}\,(R))$. □

Let R be a ring, a finitely generated projective R-module P is called stably free if $P \oplus R^m$ is free. It is well known that the stably free modules are highly important projective modules, and this fact is the crux in solving the Serre's problem. P is called *power stably free* if there exist positive integers m, s, t such that $P^m \oplus R^s \cong R^t$, and $(t-s)/m$ is called the *stably free rank* of P, denoted by $\text{s.f.rank}\,(P) = (t-s)/m$. If R has IBN, the stably free rank of a power stably free module over R is well defined, and if P, Q are power stably free R-modules, then $\text{s.f.rank}\,(P \oplus Q) = \text{s.f.rank}\,(P) + \text{s.f.rank}\,(Q)$. Let P be a finitely generated projective R-module. If there exist a least positive integer r and a finitely generated free R-module F such that $P^r \oplus F$ is free, then r is called the stably free order of P, denoted by $\text{s.f.O}\,(P) = r$; otherwise, we say that P has no finite stably free order and denote it by $\text{s.f.O}\,(P) = \infty$. If $P \oplus Q \cong R^n$ and one of P, Q is a power stably free R-module, so is the other; moreover, $\text{s.f.O}\,(P) = \text{s.f.O}\,(Q)$ (see [**8, 9**]). The category of power stably free modules is larger than the category of stably free modules.

Reference [**1**, Chapter IX, Proposition 3.7] showed that $\det_0\colon Rk_0(R) \to \text{Pic}\,(R)$ is an isomorphism if and only if each finitely generated projective R-module P with constant rank $r > 0$ is stably isomorphic to $\wedge^r P \oplus R^{r-1}$. Using Lemma 1 and [**2**], we can obtain the same result for $\det_0|_{\text{Tor}\,(K_0(R))}$.

Proposition 1. *For a commutative ring R, $\det_0|_{\text{Tor}\,(K_0(R))}\colon \text{Tor}\,(K_0(R)) \to \text{Tor}\,(\text{Pic}\,(R))$ is an isomorphism if and only if, for each power stably free R-module P of* $\text{s.f.rank}\,(P) = m > 0$, *$P$ is stably isomorphic to $\wedge^m P \oplus R^{m-1}$.*

Proof. "$\Rightarrow$." Let P be a power stably free R-module of $\text{s.f.rank}\,(P) = m > 0$. Then

$$[P] - [R^m] \in \text{Tor}\,(K_0(R)) \subseteq Rk_0(R).$$

By Lemma 1, $\det_0|_{\text{Tor}\,(K_0(R))}\colon \text{Tor}\,(K_0(R)) \to \text{Tor}\,(\text{Pic}\,(R))$ is an isomorphism. Then

$$\det_0|_{\text{Tor}\,(K_0(R))}([P] - [R^m]) = \langle \wedge^m P \rangle \in \text{Tor}\,(\text{Pic}\,(R)).$$

So $\langle \wedge^m P \rangle^r = \langle R \rangle$ for some positive integer r, i.e., $\otimes^r (\wedge^m P) \cong R$. By [**2**], $(\wedge^m P)^n \cong R^n$ for some positive integer n. Hence, $[\wedge^m P] - [R] \in \text{Tor}\,(K_0(R))$. We get that

$$\det_0|_{\text{Tor}\,(K_0(R))}([P] - [R^m]) = \det_0|_{\text{Tor}\,(K_0(R))}([\wedge^m P] - [R]).$$

Thus, $[P]-[R^m]) = ([\wedge^m P]-[R])$ since $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ is an isomorphism by the assumption. That is, $P\oplus R^t \cong \wedge^m P\oplus R^{m-1}\oplus R^t$ for some positive integer t.

"⇐." It is sufficient to prove that $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ is injective by Lemma 1. Let $[P]-[R^m]\in \mathrm{Tor}\,(K_0(R))$. Then $r([P]-[R^m])=0$ for some positive integer r and s.f.rank $(P)=m>0$. If $\det_0|_{\mathrm{Tor}\,(K_0(R))}([P]-[R^m]) = \langle\wedge^m P\rangle = \langle R\rangle$, then $\wedge^m P\cong R$. By the assumption, $P\oplus R^t\cong \wedge^m P\oplus R^{m-1}\oplus R^t\cong R^m\oplus R^t$. So $[P]-[R^m]=0$. That is, $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ is injective. Thus, it is an isomorphism. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories with a product, and let $f\colon\mathcal{A}\to\mathcal{B}$ be a product preserving functor. The fibre category ϕf has as objects all triples (M,N,α) with M, $N\in \mathrm{Obj}\,(\mathcal{A})$ and $\alpha\colon \overline{M}\to\overline{N}$ an isomorphism in $\mathcal{B}$, where we are writing $\overline{M}=f(M)$, $\overline{N}=f(N)$. A morphism in ϕf is a pair $(\beta,\gamma)\colon(M,N,\alpha)\to(M',N',\alpha')$, where $\beta\colon M\to M'$ and $\gamma\colon N\to N'$ are isomorphisms in $\mathcal{A}$ such that $\overline{\gamma}\alpha=\alpha'\overline{\beta}$, where we are writing $\overline{\gamma}=f(\gamma)$, $\overline{\beta}=f(\beta)$. If $f\colon\mathcal{A}\to\mathcal{B}$ is a cofinal product preserving functor of categories with a product, then the sequence $K_1\mathcal{A}\xrightarrow{K_1 f}K_1\mathcal{B}\xrightarrow{d'}K_0\phi f\xrightarrow{d}K_0\mathcal{A}\xrightarrow{K_0 f}K_0\mathcal{B}$ is exact (see [**1**, Chapter VII], [**6**, Chapter 3], [**7**, Chapter 7]).

Let R be a commutative ring. Then $\det\colon \mathbf{P}(R)\to\mathbf{P}\mathrm{ic}\,(R)$ is a cofinal product preserving functor, where $\mathbf{P}(R)$ and $\mathbf{P}\mathrm{ic}\,(R)$ are categories of finitely generated projective R-modules and invertible R-modules respectively. Next we shall discuss the problem, for a commutative ring R with $Rk_0(R)\cong\mathbf{Z}$, what is $K_0\phi f$? In fact, there is a commutative ring R such that $Rk_0(R)\cong\mathbf{Z}$. By [**3**], any abelian group is the class group of a suitable Dedekind domain R. But the class group coincides with $\mathrm{Pic}\,(R)$ and $Rk_0(R)\cong\mathrm{Pic}\,(R)$ if R is a Dedekind domain (see [**1, 6, 7**]).

Lemma 2. *Given a commutative ring R,* $\mathrm{Tor}\,(K_0(R))=0$ *if and only if* $\mathrm{Tor}\,(\mathrm{Pic}\,(R))=0$.

Proof. "⇒." By Lemma 1.

"⇐." For any $y=[P]-m[R]\in\mathrm{Tor}\,(K_0(R))$, there exist a positive integer n and non-negative integer t such that $P^n\oplus R^t\cong R^{mn}\oplus R^t$.

Then $\otimes^n(\wedge^m P) \cong R$, i.e., $\langle \wedge^m P\rangle^n = \langle R\rangle$. Since $\mathrm{Tor}\,(\mathrm{Pic}\,(R)) = 0$, $n = 1$. So $y = [P] - m[R] = 0$. Thus, $\mathrm{Tor}\,(K_0(R)) = 0$. □

Lemma 3. *Given a commutative ring R, if $\mathrm{Rk}_0(R) \cong \mathbf{Z}$, then $\mathrm{Pic}\,(R) \cong \mathbf{Z}$ or $\mathrm{Pic}\,(R) = \{1\}$.*

Proof. Suppose that $\varphi: Rk_0(R) \to \mathbf{Z}$ is an isomorphism. Then there exists an element $[P] - m[R] \in Rk_0(R)$ such that $\varphi([P] - m[R]) = 1$.

If $[Q] - n[R] \in Rk_0(R)$, then $\varphi([Q] - n[R]) = r \in \mathbf{Z}$, and so $[Q] - n[R] = r([P] - m[R])$. We have the following four cases:

Case 1. $Q \oplus R^t \cong P^r \oplus R^{n-rm+t}$, if $r > 0$, $n - rm \geq 0$, for some positive integer t.

Case 2. $Q \oplus R^{rm-n+t} \cong P^r \oplus R^t$, if $r > 0$, $n - rm < 0$, for some positive integer t.

Case 3. $Q \oplus P^{-r} \oplus R^t \cong \oplus R^{n-rm+t}$, if $r < 0$, for some positive integer t.

Case 4. $Q \oplus R^t \cong R^{n+t}$, if $r = 0$.

In Cases 1 and 2, $\wedge^n Q \cong \overset{r}{\otimes}(\wedge^m P)$ with $r > 0$. In Case 3, $\wedge^n Q \otimes (\overset{-r}{\otimes}(\wedge^m P)) \cong R$ with $r < 0$ and $\wedge^n Q \cong R$ in Case 4.

If $\langle \wedge^m P\rangle = \langle R\rangle$, then, for each finitely generated projective R-module Q of constant rank n, we have that $\langle \wedge^n Q\rangle = \langle R\rangle$. So in this case, $\mathrm{Pic}\,(R) = \{1\}$.

If $\langle \wedge^m P\rangle \neq \langle R\rangle$, then, for each finitely generated projective R-module Q of constant rank n, one can get that $\langle \wedge^n Q\rangle = \langle \wedge^m P\rangle^r$, i.e., $\mathrm{Pic}\,(R) = \langle\langle \wedge^m P\rangle\rangle$. Since $\mathrm{Tor}\,(K_0(R)) = 0$, by Lemma 2, $\mathrm{Tor}\,(\mathrm{Pic}\,(R)) = 0$. So the order of $\langle \wedge^m P\rangle$ is infinite in $\mathrm{Pic}\,(R)$. Thus, $\mathrm{Pic}\,(R) \cong \mathbf{Z}$. □

Remark 1. In fact, given a commutative ring R, $\mathrm{Tor}\,(K_0(R)) \subseteq Rk_0(R)$. If $Rk_0(R) \cong \mathbf{Z}$, then $\mathrm{Tor}\,(K_0(R)) = 0$ and, by Lemma 2, $\mathrm{Tor}\,(\mathrm{Pic}\,(R)) = 0$. Note that $\det_0: Rk_0(R) \to \mathrm{Pic}\,(R)$ is surjective; hence, in this case, there exist only two cases for $\mathrm{Pic}\,(R)$, i.e., $\mathrm{Pic}\,(R) \cong \mathbf{Z}$ or $\mathrm{Pic}\,(R) = \{1\}$.

Proposition 2. *Let R be a commutative ring. If* $\mathrm{Rk}_0(R) \cong \mathbf{Z}$*, then either* $K_0(\phi(\det)) \cong K_0(R)$ *or* $K_0(\phi(\det)) \cong H_0(R)$.

Proof. By [**7**, page 153], we have an exact sequence

$$0 \longrightarrow K_0\phi(\det) \xrightarrow{d} K_0(R) \xrightarrow{\det} \mathrm{Pic}\,(R) \longrightarrow 0.$$

By Lemma 3, $\mathrm{Pic}\,(R) \cong \mathbf{Z}$ or $\mathrm{Pic}\,(R) = \{1\}$.

If $\mathrm{Pic}\,(R) = \{1\}$, then $K_0(\phi(\det)) \cong K_0(R)$.

If $\mathrm{Pic}\,(R) \cong \mathbf{Z}$, then $K_0(R) \cong H_0(R) \oplus Rk_0(R)$ and $\det\colon K_0(R) \to \mathrm{Pic}\,(R)$ is the composite of the natural map $K_0(R) \to Rk_0(R)$, and the map $\det_0\colon Rk_0(R) \to \mathrm{Pic}\,(R)(\det_0([M]-m[R]) = \langle \wedge^m M\rangle)$, where $[M]-m[R] \in Rk_0(R)$. Note that $\det_0\colon Rk_0(R) \cong \mathrm{Pic}\,(R)$ and, by the proof of Lemma 3, we have that $\mathrm{Ker}\,(\det) \cong H_0(R)$. So $K_0\phi(\det) \cong \mathrm{Im}\,(d) = \mathrm{Ker}\,(\det) \cong H_0(R)$. □

Proposition 3. *Let R be a commutative ring. If* $\det_0\colon \mathrm{Rk}_0(R) \to \mathrm{Pic}\,(R)$ *is an isomorphism, then for any* $y \in \mathrm{Rk}_0(R)$, $y^2 = 0$.

Proof. Let $y = [P] - m[R] \in Rk_0(R)$. Then P is a finitely generated projective R-module of constant rank m. Since $\det_0$ is an isomorphism, by [**1**, Chapter IX, Proposition 3.7], P is stably isomorphic to $\wedge^m P \oplus R^{m-1}$, i.e., $P \oplus R^t \cong \wedge^m P \oplus R^{m-1} \oplus R^t$ for some non-negative integer t. Then

$$(P \oplus R^t) \otimes (P \oplus R^t) \cong (\wedge^m P \oplus R^{m-1} \oplus R^t) \\ \otimes (\wedge^m P \oplus R^{m-1} \oplus R^t),$$

i.e.,

$$P \otimes P \oplus P^{2t} \oplus R^{t^2} \cong (\wedge^m P) \otimes (\wedge^m P) \\ \oplus (\wedge^m P)^{2(m-1+t)} \oplus R^{(m-1+t)^2}.$$

Applying $\wedge^{(m+t)^2}$ to both sides, we get

$$\wedge^{m^2}(P \otimes P) \otimes \overset{2t}{\otimes}(\wedge^m P) \cong (\wedge^m P) \otimes (\wedge^m P) \otimes \overset{2(m-1+t)}{\otimes}(\wedge^m P).$$

So $\langle \wedge^{m^2}(P\otimes P)\rangle = \langle \wedge^m P\rangle^{2m}$ in $\mathrm{Pic}\,(R)$. Thus,

$$\begin{aligned}\det_0(y^2) &= \det_0([P\otimes P] - 2m[P] + m^2[R])\\ &= \det_0(([P\otimes P] - m^2[R]) - 2m([P] - m[R]))\\ &= \langle \wedge^{m^2}(P\otimes P)\rangle\langle \wedge^m P\rangle^{-2m} = \langle R\rangle.\end{aligned}$$

Therefore, $y^2 = 0$ as $\det_0$ is an isomorphism by the assumption. □

Definition 1. Let R be a commutative ring and M a finitely generated projective R-module of constant rank. If there is a finitely generated projective R-module Q such that M is stably isomorphic to $Q^{\mathrm{rank}\,(M)}$, then M is said to be divided by $\mathrm{rank}\,(M)$, denoted by $\mathrm{rank}\,(M) \mid M$. If each finitely generated projective R-module of constant rank can be divided by its rank, then we say that R is locally divisible.

A ring R is called an MM (Morita matrix) ring if its only Morita equivalent rings are (up to isomorphism) the matrix rings over R (see **5**]). By [**5**, Proposition 1.1], if R is an indecomposable commutative MM ring and M is a finitely generated projective R-module of constant rank, then $M \cong Q^{\mathrm{rank}\,(M)}$, for some invertible R-module Q. So the indecomposable commutative MM rings are local divisible (some examples can be found in [**5**]).

We denote by $\mathrm{rk}_1(R) = \{[P]-[R] \mid P$ is a finitely generated projective R-module of constant rank $1\}$. If R is a commutative ring with $\dim\,(R) \le d$, then, by [**1**, Chapter IV, Proposition 4.4], $\mathrm{rk}_1(R)^{d+1} = 0$.. Applying some methods of [**2**], we can prove the following result.

Proposition 4. *Let R be a commutative ring and locally divisible. If, for any $x \in \mathrm{rk}_1(R)$, there is some positive integer t such that $x^t = 0$, then* $\ker\,(\det_0\colon \mathrm{Rk}_0(R) \to \mathrm{Pic}\,(R)) \subseteq \mathrm{Tor}\,(K_0(R))$. *In this case,* $\det_0$ *is an isomorphism if and only if* $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ *is an isomorphism. In particular, if for any $x \in \mathrm{rk}_1(R)$, $x^2 = 0$, then* $\det_0\colon \mathrm{Rk}_0(R) \to \mathrm{Pic}\,(R)$ *is an isomorphism.*

Proof. Let $[M] - m[R] \in \mathrm{Rk}_0(R)$. Then M is a finitely generated projective R-module of constant rank m. If $\det_0([M] - m[R]) =$

$\langle \wedge^m M \rangle = \langle R \rangle$, we shall prove that $[M] - m[R] \in \mathrm{Tor}\,(K_0(R))$. By the assumption, R is locally divisible. Then there exists a finitely generated projective R-module Q such that M is stably isomorphic to Q^m. So $\wedge^m M \cong \overset{m}{\otimes} Q$. Thus, $[Q]^m = [\wedge^m M] = [R]$ in $K_0(R)$ and $[Q] - [R] \in \mathrm{rk}_1(R)$. Let $x = [Q] - [R]$, then

$$0 = [Q]^m - [R]^m = xz, \tag{1}$$

where $u = [Q]^{m-1} + [Q]^{m-2} + \cdots + [Q] + [R]$. We have

$$u - m[R] = \sum_{i=1}^{m-1} ([Q]^i - [R]^i) = ([Q] - [R])(-y) = -xy,$$

for some $y \in K_0(R)$, whence

$$m[R] = u + xy, \tag{2}$$

We have

$$m^t[R] = (u + xy)^t = uz + x^t y^t$$

for some $z \in K_0(R)$. Since $x \in \mathrm{rk}_1(R)$, by assumption, $x^t = 0$ for some positive integer t. Note that (1) holds. Then $m^t x = 0$, i.e., $m^t[Q] = m^t[R]$. That is, $m^{t-1}([M] - m[R]) = 0$. Thus, $[M] - m[R] \in \mathrm{Tor}\,(K_0(R))$.

Clearly, by Lemma 1, $\det_0$ is an isomorphism if and only if $\det_0|_{\mathrm{Tor}\,(K_0(R))}$ is an isomorphism under the assumption of this proposition.

In particular, by (2), $mx = xu + x^2 y$. Then $mx = 0$ as for any $x \in \mathrm{rk}_1(R)$, $x^2 = 0$ and (1). So $[M] - m[R] = m[Q] - m[R] = mx = 0$. Thus, in this case, $\det_0 \colon \mathrm{Rk}_0(R) \to \mathrm{Pic}\,(R)$ is an isomorphism. □

Corollary 1. *Let R be a commutative ring which is either a ring whose maximal ideal spectrum is Noetherian of dimension ≤ 1 or a Prüfer domain of Krull dimension ≤ 1. If R is indecomposable and* $\mathrm{Pic}\,(R)$ *is divisible, then* $\det_0 \colon \mathrm{Rk}_0(R) \to \mathrm{Pic}\,(R)$ *is an isomorphism.*

Proof. By [**1**, Chapter IX, Proposition 4.4] and [**5**, Corollary 2.6], Proposition 4 has been proved. □

Remark 2. Algebraic geometry provides some examples of commutative MM rings R with $\dim(R) \leq 1$ and $\mathrm{Pic}(R)$ divisible. For instance, if K is an algebraically closed field of characteristic zero, then the rings $S = K[x,y]/(y^2 - x^3)$ and $T = K[x,y]/(y^2 - x^3 - x^2)$ are the coordinate rings of the cuspidal and nodal cubic, respectively. These are Noetherian domains of Krull dimension 1, but they are not Dedekind domains, the origin being a double point for both curves. The group $\mathrm{Pic}(S)$ is isomorphic to the additive group of K and therefore divisible if the characteristic of K is zero; $\mathrm{Pic}(T)$ is isomorphic to the multiplicative group of K and so is divisible because K is algebraically closed (see [**5**, page 305]). In this case, S and T are commutative MM rings. Then S and T are locally divisible and $\mathrm{rk}_1(S)^2 = 0$, $\mathrm{rk}_1(T)^2 = 0$. So $\mathrm{Rk}_0(S) \cong \mathrm{Pic}(S)$ and $\mathrm{Rk}_0(T) \cong \mathrm{Pic}(T)$.

If R is an indecomposable commutative ring, let $\mathcal{P}^*(R) = \mathbf{P}(R) \setminus 0$. For $P \in \mathcal{P}^*(R)$, $r(P)$ denotes the rank of P. The pair $(r(-), \langle \wedge^{r(-)}(-)\rangle) \in \mathbf{N} \times \mathrm{Pic}(R)$ is called a complete invariant for $\mathcal{P}^*(R)$ if, for any $P, Q \in \mathcal{P}^*(R)$, $P \cong Q \Leftrightarrow (r(P), \langle \wedge^{r(P)}(P)\rangle) = (r(Q), \langle \wedge^{r(Q)}(Q)\rangle)$ (see [**5**]). Merisi and Vámos showed that, if R is an indecomposable ring with divisible Picard group, then a sufficient condition for R to be an MM ring is that the pair $(r(-), \langle \wedge^{r(-)}(-)\rangle)$ be a complete invariant for $\mathcal{P}^*(R)$. This condition is also necessary if the Picard group is torsion free (see [**5**, Proposition 2.5]). Next we shall consider the pair $(r(-), \langle \wedge^{r(-)}(-)\rangle)$. Some properties of this pair are obtained and [**5**, Proposition 2.5] is extended.

A ring R is said to have left stable range 1 if, whenever $Ra + Rb = R\,(a, b \in R)$, there exists an $e \in R$ such that $a + eb \in U(R)$ (the group of units of R) (see [**4**, subsection 20]). $\mathbf{P}(R)$ is called *cancellative* if, for $P, Q, M \in \mathbf{P}(R)$, $P \oplus M \cong Q \oplus M$ implies $P \cong Q$. By the cancellation theorem (see [**4**, 20.11]), if R has left stable range 1, then $\mathbf{P}(R)$ is cancellative.

Proposition 5. *Let R be an indecomposable commutative ring. Then the pair $(r(-); \langle \wedge^{r(-)}(-)\rangle)$ is a complete invariant for $\mathcal{P}^*(R)$ if and only if* $\det_0\colon \mathrm{Rk}_0(R) \to \mathrm{Pic}(R)$ *is an isomorphism and* $\mathbf{P}(R)$ *is cancellative.*

Proof. "⇒." Firstly, we shall prove that $\det_0\colon \mathrm{Rk}_0(R) \to \mathrm{Pic}(R)$ is an isomorphism. In fact, for $[P] - m[R] \in \mathrm{Rk}_0(R)$, if $\det_0([P] - m[R]) =$

$\langle\wedge^m P\rangle = \langle R\rangle$, note that $r(P) = m$. Then

$$(r(P),\ \langle\wedge^m P\rangle) = (r(\wedge^m P \oplus R^{m-1}),\ \langle\wedge^m(\wedge^m P \oplus R^{m-1})\rangle).$$

So $P \cong \wedge^m P \oplus R^{m-1}$ as the pair $(r(-); \langle\wedge^{r(-)}(-)\rangle)$ is a complete invariant for $\mathcal{P}^*(R)$, i.e., $P \cong R^m$. Thus, $[P] - m[R] = 0$, and then $\det_0$ is an isomorphism as $\det_0$ is always surjective. Secondly, we shall show that $\mathbf{P}(R)$ is cancellative. It is sufficient to prove that, for $P,\ Q \in \mathbf{P}(R)$, $P \oplus R^n \cong Q \oplus R^n$ implies $P \cong Q$ for any positive integer n. In fact, for $P,\ Q \in \mathbf{P}(R)$, if $P \oplus R^n \cong Q \oplus R^n$, then $r(P) = r(Q)$ and $\wedge^{r(P)}P \cong \wedge^{r(Q)}Q$, i.e., $(r(P); \langle\wedge^{r(P)}P\rangle) = (r(Q); \langle\wedge^{r(Q)}Q\rangle)$. So $P \cong Q$ as this pair is completely invariant.

"$\Leftarrow$." Suppose that $\det_0\colon Rk_0(R) \to \operatorname{Pic}(R)$ is an isomorphism and $\mathbf{P}(R)$ is cancellative. It is sufficient to prove that, for any $P,\ Q \in \mathbf{P}(R)$, if

$$(r(P); \langle\wedge^{r(P)}P\rangle) = (r(Q); \langle\wedge^{r(Q)}Q\rangle),$$

then $P \cong Q$. Since $r(P) = r(Q)$ and $\langle\wedge^{r(P)}P\rangle = \langle\wedge^{r(Q)}Q\rangle$, $\det_0([P] - r(P)[R]) = \det_0([Q] - r(Q)[R])$. But $\det_0$ is an isomorphism; hence, $[P] - r(P)[R] = [Q] - r(Q)[R]$. In other words, $[P] = [Q]$, i.e., $P \oplus R^n \cong Q \oplus R^n$ for some non-negative integer n. So $P \cong Q$ as $\mathbf{P}(R)$ is cancellative. Thus, the pair $(r(-),\ \langle\wedge^{r(-)}(-)\rangle)$ is completely invariant for $\mathcal{P}^*(R)$. ☐

Corollary 2. *If R is an indecomposable commutative ring with stable range* 1*, then the pair $(r(-),\ \langle\wedge^{r(-)}(-)\rangle)$ is a complete invariant for $\mathcal{P}^*$ if and only if* $\det_0\colon \mathrm{Rk}_0(R) \to \operatorname{Pic}(R)$ *is an isomorphism.*

Proposition 6. *If R is an indecomposable commutative ring with divisible Picard group, then the pair $(r(-),\ \langle\wedge^{r(-)}(-)\rangle)$ is a complete invariant for $\mathcal{P}^*$ if and only if R is an MM ring,* $\mathbf{P}(R)$ *is cancellative and, for each $y \in \mathrm{rk}_1(R)$, $y^2 = 0$.*

Proof. By Propositions 3, 4, 5 and [**4**, Proposition 2.53], Proposition 6 has been proved. ☐

Corollary 3. *Let R be an indecomposable commutative ring with divisible Picard group. If the dimension of $R \leq 1$, then the pair $(r(-),\ \langle\wedge^{r(-)}(-)\rangle)$ is a complete invariant for $\mathcal{P}^*$ if and only if R is an* MM *ring.*

Proof. By [**1**, Chapter IV, Corollary 3.5 and Chapter IX, Proposition 4.4], $\mathbf{P}(R)$ is cancellative and $\mathrm{rk}_1(R)^2 = 0$. So the result immediately follows from Proposition 6. □

Proposition 7. *Let R be a commutative ring and $\mathrm{Rk}_0(R)^2 = 0$. If M, N are two power stably free R-modules with $\text{s.f.O}\,(M) = r_1$, $\text{s.f.O}\,(N) = r_2$, $\text{s.f.rank}\,(M) = m$ and $\text{s.f.rank}\,(N) = n$, then*

(1) $[\wedge^{mn}(M \otimes N)]^r = [\wedge^n N]^{t_1 m}[\wedge^m M]^{t_2 n}$, *where $r = (r_1, r_2)$ and $t_1 + t_2 = r$.*

(2) $r[\wedge^{mn}(M \otimes N)] = t_1[\wedge^n N]^m + t_2[\wedge^m M]^n$, *where $r = (r_1, r_2)$ and $t_1 + t_2 = r$.*

Proof. If M, N are two finitely generated projective R-modules, then

$$M^{r_1} \oplus R^{m_1} \cong R^{n_1} \quad \text{and} \quad N^{r_2} \oplus R^{m_2} \cong R^{n_2} \tag{3}$$

for some non-negative integers m_1, m_2, n_1, n_2. From (3), we have

$$(M \otimes_R N)^{r_1} \oplus N^{m_1} \cong N^{n_1} \quad \text{and} \quad (M \otimes_R N)^{r_2} \oplus M^{m_2} \cong M^{n_2}. \tag{4}$$

Then

$$[\wedge^{mn}(M \otimes_R N)]^{r_1} = [\wedge^n N]^{r_1 m} \tag{5}$$

and

$$[\wedge^{mn}(M \otimes_R N)]^{r_2} = [\wedge^m M]^{r_2 n}.$$

We take $s_1 = r_1/r$, $s_2 = r_2/r$, and put $a = [\wedge^{m+n}(M \otimes_R N)]$, $b = [\wedge^n N]^m$, $c = [\wedge^m M]^n$ and $[R] = 1$ in $K_0(R)$.

(1) Since $(s_1, s_2) = 1$, there exist some integers h_1, h_2 such that $h_1 s_1 + h_2 s_2 = 1$. Then $h_1 r_1 + h_2 r_2 = r(h_1 s_1 + h_2 s_2) = r$. By (5), we have

$$[\wedge^{mn}(M \otimes_R N)]^{h_1 r_1} = [\wedge^n N]^{h_1 r_1 m}$$

and

$$[\wedge^{mn}(M \otimes_R N)]^{h_2 r_2} = [\wedge^m M]^{h_2 r_2 n}.$$

Then $[\wedge^{mn}(M \otimes N)]^r = [\wedge^n N]^{t_1 m}[\wedge^m M]^{t_2 n}$, where $t_1 = h_1 r_1$, $t_2 = h_2 r_2$.

(2) By (5), $a^{r_1} = b^{r_1}$ and $a^{r_2} = c^{r_2}$, i.e., $(ab^{-1})^{r_1} - 1 = 0$ and $(ac^{-1})^{r_2} - 1 = 0$. But $Rk_0(R)^2 = 0$; hence, $r_1(ab^{-1} - 1) = 0$ and $r_2(ac^{-1} - 1) = 0$, i.e., $r_1 a = r_1 b$ and $r_2 a = r_2 c$. Note that $(s_1, s_2) = 1$, and there exist some integers h_1, h_2 such that $h_1 s_1 + h_2 s_2 = 1$. So $ra = h_1 r_1 b + h_2 r_2 c$ and $h_1 r_1 + h_2 r_2 = r(h_1 s_1 + h_2 s_2) = r$. That is, $r[\wedge^{mn}(M \otimes_R N)] = t_1[\wedge^n N]^m + t_2[\wedge^m M]^n$, where $t_1 = h_1 r_1$, $t_2 = h_2 r_2$. □

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