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QUASI-ONE-FIBERED IDEALS OF ORDER ONE IN DIMENSION TWO

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ABSTRACT. Simple complete $\mathfrak{M}$-primary ideals are fundamental in the Zariski-Lipman theory of complete ideals in a two-dimensional regular local ring $(R,\mathfrak{M})$. Complete $\mathfrak{M}$-primary ideals of order one constitute a particular class of such ideals containing, for example, all the first neighborhood ideals of R. A number of interesting properties of complete $\mathfrak{M}$-primary ideals of order one have been proved by several authors. For example, these ideals have only one Rees valuation (and hence are one-fibered) and they have the very simple form $(x_1^n, x_2)R$, $n \in \mathbf{N}_+$, with x_1, x_2 a minimal ideal basis of $\mathfrak{M}$.

In the present paper we investigate how far some results concerning these complete $\mathfrak{M}$-primary ideals of order one can be extended to complete quasi-one-fibered $\mathfrak{M}$-primary ideals of order one in a natural generalization of R: a two-dimensional normal Noetherian local domain with algebraically closed residue field and the associated graded ring an integrally closed domain.

1. Introduction. Let $(R,\mathfrak{M})$ be a two-dimensional normal Noetherian local domain with algebraically closed residue field and with the associated graded ring $\mathrm{gr}_{\mathfrak{M}}(R)$ an integrally closed domain. It follows that the $\mathfrak{M}$-adic order function ord_R is a valuation (mostly denoted by $v_{\mathfrak{M}}$) and the blowup $\mathrm{Bl}_{\mathfrak{M}}(R)$ of R at $\mathfrak{M}$ is a desingularization of R.

These local rings have been studied by Muhly (jointly with Sakuma) in the early 1960s and have therefore been called *two-dimensional Muhly local domains* in [**1, 2**].

This paper is about quasi-one-fibered complete $\mathfrak{M}$-primary ideals of *order one* in R. Let us explain what we mean by "quasi-one-fibered" by describing briefly the motivation for introducing this notion. To do so, we need to recall a few facts from the theory of complete ideals in

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the special case where $(R, \mathfrak{M})$ is *regular*. Let $I \neq \mathfrak{M}$ denote a complete $\mathfrak{M}$-primary ideal of order one in R. It follows that I is simple (i.e., not a product of proper ideals of R) and hence has a unique Rees valuation, say w (see [**6**]). So $T(I) = \{w\}$, where $T(I)$ denotes the set of Rees valuations of I and I is said to be *one-fibered* (cf., [**14**]). This implies that I has a unique immediate base point, namely, the unique two-dimensional local ring $(R', \mathfrak{M}')$ of the blowup $\mathrm{Bl}_{\mathfrak{M}}(R)$ that is dominated by the valuation ring $(W, \mathfrak{M}_W)$ of w. Moreover, the transform $I^{R'}$ of I in R' is again a complete ideal of order one.

By contrast, if the two-dimensional Muhly local domain $(R, \mathfrak{M})$ is *not* regular, then a complete $\mathfrak{M}$-primary ideal I of order one is not necessarily one-fibered. Indeed, it has been proved in [**1**] that, if I is a complete $\mathfrak{M}$-primary ideal adjacent from below to $\mathfrak{M}$ (i.e., length$(\mathfrak{M}/I) = 1$), then $T(I) = \{v_{\mathfrak{M}}, w\}$ with $w \neq v_{\mathfrak{M}}$ a prime divisor of R. This was one of the reasons for introducing the notion of quasi-one-fibered complete $\mathfrak{M}$-primary ideals in R. Here a complete $\mathfrak{M}$-primary ideal I is said to be *quasi-one-fibered* if

$$T(I) \subseteq \{v_{\mathfrak{M}}, w\} \quad \text{and} \quad w \in T(I),$$

for some prime divisor $w \neq v_{\mathfrak{M}}$ of R. Note that $v_{\mathfrak{M}}$ may or may not belong to $T(I)$. More information concerning this sort of ideal can be found in [**4**].

In this paper we will study the simplest such ideals, that is, those of order one. So let I be any quasi-one-fibered complete $\mathfrak{M}$-primary ideal of order one in R (hence, $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$ for some prime divisor $w \neq v_{\mathfrak{M}}$ of R). Then, just as in the regular case, I has precisely one immediate base point, say $(R', \mathfrak{M}')$. But, in contrast to the regular case, the transform $I^{R'}$ does not necessarily have order one ($I^{R'}$ is not even necessarily simple). See [**3**, Example 3.4].

We therefore begin Section 3 by investigating when $\mathrm{ord}_R(I) = 1$ implies $\mathrm{ord}_{R'}(I^{R'}) = 1$, and under what conditions the converse holds. An answer is given in Proposition 3.2 if I satisfies the additional conditions of being normal and minimally generated. A stronger result is obtained if R is regular (see Proposition 3.6).

A special class of quasi-one-fibered complete $\mathfrak{M}$-primary ideals of order one are the first neighborhood complete ideals of R. In [**1**] it has been shown that some properties of R (for example being a rational singularity in case $\operatorname{embdim} R = 3$) are reflected in a certain behavior of its first neighborhood complete ideals. Further, it has been proved in [**2**] that the first neighborhood complete ideals of R are projectively full. The proofs of these results rely on the fact that these complete quasi-one-fibered $\mathfrak{M}$-primary ideals of order one have the very simple form

$$(x_1^2, x_2, \dots, x_d),$$

where $x_1, x_2, \dots, x_d$ denotes a minimal ideal basis of $\mathfrak{M}$. So the natural question arises whether *any* complete quasi-one-fibered $\mathfrak{M}$-primary ideal of order one has the form

$$(x_1^n, x_2, \dots, x_d),$$

for a suitable minimal ideal basis $x_1, x_2, \dots, x_d$ of $\mathfrak{M}$. (Note that this is trivially true if $(R, \mathfrak{M})$ is two-dimensional regular).

In order to answer this question we have in Proposition 3.7 of Section 3 derived necessary and sufficient conditions for a given complete quasi-one-fibered $\mathfrak{M}$-primary I of order one to be of the form $(x_1^n, x_2, \dots, x_d)$, $n \in \mathbf{N}_+$, for some minimal ideal basis $(x_1, x_2, \dots, x_d)$ of $\mathfrak{M}$ such that $x_1 \notin \operatorname{rad}(x_2, \dots, x_d)$. Using this result we have found an example (see Example 3.8) that shows the answer to the above question is negative. Concerning complete $\mathfrak{M}$-primary ideals of order one in a two-dimensional Muhly local domain $(R, \mathfrak{M})$, one can ask the natural question whether there exists an example of such an ideal that is *not* quasi-one-fibered. The author does not know any such example. The reader will find the necessary background material in Section 2.

2. Background. We begin with a brief review of some facts from the theory of degree functions. Degree functions have been defined by Rees in [**12**], and their theory has been developed by Rees and Sharp in [**13**]. With an $\mathfrak{M}$-primary ideal I of Noetherian local domain $(R, \mathfrak{M})$, Rees has associated an integer-valued function on $\mathfrak{M} \setminus \{0\}$:

$$d_I(x) = e\left(\frac{I + xR}{xR}\right),$$

where $e(I + xR/xR)$ denotes the multiplicity of $I + xR/xR$. The function d_I is called the *degree function* defined by I. In [**12**], it is shown that, with every prime divisor v of R, there is associated a non-negative integer $d(I,v)$, with $d(I,v)=0$ for all except finitely many prime divisors, such that

$$d_I(x)=\sum_v d(I,v)v(x)$$

for all $0\neq x\in\mathfrak{M}$ and the sum is over all the prime divisors of R. Here, by a prime divisor v of R, we mean a discrete valuation v of the quotient field K of R with value group $\mathbf{Z}$, whose valuation ring dominates R and the transcendence degree of the residue field of the valuation ring over $R/\mathfrak{M}$ is $\dim R-1$ (see [**13**, pages 454–455]).

In [**13**] Rees and Sharp have proved that the integers $d(I,v)$ occurring in the above sum are uniquely determined, i.e., if $d(I',v)$ are non-negative integers such that $d_I(x)=\sum_v d'(I,v)v(x)$ for all $0\neq x\in\mathfrak{M}$, then $d'(I,v)=d(I,v)$ for all prime divisors v of R. The integers $d(I,v)$ will be called the *degree function coefficients* of I.

If R is analytically unramified, then $d(I,v)\neq 0$ for each prime divisor v of R that is a Rees valuation of I, while $d(I,v')=0$ for all other prime divisors v' of R (see [**12**, Theorem 2.3]). If in addition R is normal and quasi-unmixed, then all the Rees valuations of I are prime divisors of R (cf., [**14**]); hence, $d(I,v)\neq 0$ if and only if v is a Rees valuation of I. For the definition of the Rees valuation rings and the Rees valuations of an ideal I of a Noetherian local domain, the reader is referred to [**14**, page 437] or [**15**, Chapter 10]. Throughout this paper, the set of Rees valuations of I will be denoted by $T(I)$.

In order to recall some other results from the theory of degree functions, let us suppose that $(R,\mathfrak{M})$ is a two-dimensional normal Noetherian local domain that is analytically unramified and with infinite residue field Then we have the following results (see Rees and Sharp [**13**]).

- The multiplicity $e(I)$ of an $\mathfrak{M}$-primary ideal I of R is given by

$$e(I)=\sum_{v\in T(I)} d(I,v)v(I).$$

- If I and J are $\mathfrak{M}$-primary ideals of R, then

$$d(IJ,v)=d(I,v)+d(J,v)$$

for all prime divisors of R. It follows that

$$T(IJ) = T(I) \cup T(J).$$

- If I and J are $\mathfrak{M}$-primary ideals of R, then Rees and Sharp define

$$d_I(J) = \min\{d_I(x) \mid 0 \neq x \in J\},$$

and they prove that

$$d_I(J) = \sum_{v \in T(I)} d(I, v)v(J)$$

and

$$d_I(J) = d_J(I) = e_1(I|J).$$

Here $e_1(I|J)$ denotes the mixed multiplicity of I and J, and it is defined by $e(IJ) = e(I) + 2e_1(I|J) + e(J)$.

Next we present some background material concerning two-dimensional Muhly local domains. We begin by recalling the definition. By a *two-dimensional Muhly local domain* $(R, \mathfrak{M})$ we mean a two-dimensional integrally closed Noetherian local domain $(R, \mathfrak{M})$ with algebraically closed residue field and with the associated graded ring an integrally closed domain. From this definition if follows that

- the $\mathfrak{M}$-adic order function ord_R is a valuation (mostly denoted by $v_{\mathfrak{M}}$),
- $\mathfrak{M}^n$ is an integrally closed $\mathfrak{M}$-primary ideal for every $n \in \mathbf{N}_+$.

A two-dimensional Muhly local domain $(R, \mathfrak{M})$ can be desingularized by blowing up R at $\mathfrak{M}$. Here the *blowup of R at $\mathfrak{M}$*, denoted $\mathrm{Bl}_{\mathfrak{M}}(R)$, is the following set of local rings lying between R and its quotient field K

$$\left\{ R\left[\frac{\mathfrak{M}}{x}\right]_P \;\middle|\; x \in \mathfrak{M} \setminus \mathfrak{M}^2, P \in \mathrm{Spec}\left(R\left[\frac{\mathfrak{M}}{x}\right]\right)\right\}.$$

For any $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ and any maximal ideal N of $R[\mathfrak{M}/x]$ lying over $\mathfrak{M}$ (i.e., $N \cap R = \mathfrak{M}$), the local ring

$$R' := R\left[\frac{\mathfrak{M}}{x}\right]_N$$

is called a *first* (or an *immediate*) *quadratic transform* of R. Since $\mathrm{Bl}_{\mathfrak{M}}(R)$ is a desingularization, we know that $(R', \mathfrak{M}')$ is a two-dimensional regular local ring and the $\mathfrak{M}'$-adic order function $\mathrm{ord}_{R'}$ is a prime divisor of R, called an *immediate prime divisor* of R.

If I is an $\mathfrak{M}$-primary ideal of R with $\mathrm{ord}_R(I) = r$, then we have in R'

$$IR' = x^r I'$$

with I' an ideal of R'. This ideal I' is called the *transform of I in R'*. If $I' \neq R'$ (equivalently, IR' is not a principal ideal), then $(R', \mathfrak{M}')$ is called an *immediate base point of I*.

Let I be an $\mathfrak{M}$-primary ideal in a two-dimensional Muhly local domain $(R, \mathfrak{M})$, and suppose that $T(I) \neq \{v_{\mathfrak{M}}\}$ (equivalently $\overline{I}$ is not a power of $\mathfrak{M}$). Then in [**4**, Proposition 1.1], the following characterization of the immediate base points of I has been proved.

Every immediate base point of I is a local ring $\in \mathrm{Bl}_{\mathfrak{M}}(R)$ that is dominated by a Rees valuation ring of I. Conversely, a two-dimensional local ring $\in \mathrm{Bl}_{\mathfrak{M}}(R)$ dominated by a Rees valuation ring of I is an immediate base point of I, if $T(I) = \{v_{\mathfrak{M}}, w\}$ (respectively $T(I) = \{w\}$), where w is a prime divisor of R with $w \neq v_{\mathfrak{M}}$.

Since the residue field of a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is infinite, it follows from the preceding characterization of the immediate base points of an $\mathfrak{M}$-primary ideal I, that there exists an element $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ such that all the immediate base points of I are lying on the chart $R[\mathfrak{M}/x]$ (because one can choose an $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ such that $R[\mathfrak{M}/x]$ is contained in every Rees valuation ring of I).

Finally, we recall the notion of the characteristic ideal of an $\mathfrak{M}$-primary ideal I in a two-dimensional Muhly local domain $(R, \mathfrak{M})$ (cf., [**10**, page 214]). Suppose $\mathrm{ord}_R(I) = r$, and let $x_1, x_2, \ldots, x_n$ denote an ideal basis of I. Then, for at least one i, the order of x_i is r and $\mathrm{ord}_R(x_j) \geq r$ for all j. Let x_j^* be the zero element of $\mathrm{gr}_{\mathfrak{M}}(R)$ if $\mathrm{ord}_R(x_j) > r$, and let x_j^* be the leading form of x_j if $\mathrm{ord}_R(x_j) = r$. The elements $x_1^*, x_2^*, \ldots, x_n^*$ generate a homogeneous ideal $\mathfrak{c}(I)$ in $\mathrm{gr}_{\mathfrak{M}}(R)$, which is called the *characteristic ideal* of I. If $x \in I$, then either $\mathrm{ord}_R(x) > r$, and thus x^* is zero by definition or else $\mathrm{ord}_R(x) = r$ and then x^* is a linear combination of $x_1^*, x_2^*, \ldots, x_n^*$ with coefficients in $k = R/\mathfrak{M}$.

3. Quasi-one-fibered ideals of order one. Let us begin by recalling briefly some facts concerning quasi-one-fibered $\mathfrak{M}$-primary ideals in a two-dimensional Muhly local domain $(R, \mathfrak{M})$.

Let I be a quasi-one-fibered $\mathfrak{M}$-primary ideal of R, i.e.,

$$T(I) \subseteq \{v_{\mathfrak{M}}, w\} \quad \text{and} \quad w \in T(I),$$

where w denotes some prime divisor $\neq v_{\mathfrak{M}}$ of R (see the introduction). Then the following assertions hold (see [**4**, Section 1] for details).

• I has only one immediate base point, say $(R', \mathfrak{M}')$, and the integral closure $\overline{I^{R'}}$ of the transform of I in R' is some power of a simple complete $\mathfrak{M}'$-primary ideal of R'. Hence, $T(I^{R'}) = \{w\}$ and w is the unique Rees valuation of that simple complete ideal.

• There corresponds to I a unique finite quadratic sequence starting from $(R, \mathfrak{M})$

$$(R, \mathfrak{M}) < (R_1, \mathfrak{M}_1) < (R_2, \mathfrak{M}_2) < \cdots < (R_s, \mathfrak{M}_s)$$

such that $\overline{I^{R_s}} = \mathfrak{M}_s^n$ for some $n \in \mathbf{N}_+$, and R_1 is the unique immediate base point R' of I. The length s of this sequence is called the *rank* of I.

• The local rings occurring in this sequence are the base points of I. (Recall that an iterated quadratic transform $(S, \mathfrak{M})$ of $(R, \mathfrak{M})$ is said to be a *base point* of I if IS is not a principal ideal).

• Since $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$, it follows that almost all degree function coefficients of I are zero; more precisely, $d(I, v) = 0$ for all prime divisors v of R such that $v \notin \{v_{\mathfrak{M}}, w\}$. Thus, we have to consider only two degree function coefficients of I, namely,

$$d(I, v_{\mathfrak{M}}) \quad \text{and} \quad d(I, w).$$

Note that $d(I, w) \neq 0$ (since $w \in T(I)$), while $d(I, v_{\mathfrak{M}})$ is non-zero or zero according to the fact whether or not $v_{\mathfrak{M}} \in T(I)$.

Our first aim in this section is to answer the question when $\operatorname{ord}_R(I) = 1$ will imply that $\operatorname{ord}_{R'}(I^{R'}) = 1$ and under what conditions the converse will hold. The answer (see Proposition 3.2) is based essentially on the following lemma giving the effect of the quadratic transformation

$(R,\mathfrak{M}) \to (R',\mathfrak{M}')$ on the degree function coefficient $d(I,w)$. Before stating this lemma, we briefly recall the notion of a minimally generated complete $\mathfrak{M}$-primary ideal in R.

A complete $\mathfrak{M}$-primary ideal $\mathfrak{a}$ of order r in R is said to be *minimally generated* if

$$\mu(\mathfrak{a}) = \dim_k\left(\frac{\mathfrak{M}^r}{\mathfrak{M}^{r+1}}\right),$$

where $k = R/\mathfrak{M}$ and $\mu(\mathfrak{a})$ denotes the number of elements in a minimal ideal basis of I.

In [**4**, Section 3] the following facts concerning a minimally generated complete $\mathfrak{M}$-primary ideal $\mathfrak{a}$ have been proved:

• $\mathfrak{a}$ is minimally generated if and only if $\mathfrak{M}^{r+1} = \mathfrak{a}\mathfrak{M} + x\mathfrak{M}^r$ where $x \in \mathfrak{M}$ is such that $xV = \mathfrak{M}V$ for every valuation ring V of $\mathfrak{a}$.

• The natural morphism

$$\varphi\colon \frac{\mathfrak{M}^r}{\mathfrak{a}} \longrightarrow \frac{\mathfrak{M}^r R[\mathfrak{M}/x]}{\mathfrak{a}R[\mathfrak{M}/x]}$$

is an isomorphism.

• $\mathfrak{M}^n\mathfrak{a}$ is complete for all $n \geq 0$.

Lemma 3.1. *Let $(R,\mathfrak{M})$ be a two-dimensional Muhly local domain, and let I be a quasi-one-fibered $\mathfrak{M}$-primary ideal of R. Let $(R',\mathfrak{M}')$ denote the unique immediate base point of I, and let w be the unique Rees valuation of $I^{R'}$ (so, $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$). Then we have*

(i) *If I is normal, then $d(I,w) \leq d(I^{R'},w)$.*

(ii) *If I is normal and minimally generated, then $d(I,w) = d(I^R,w)$.*

Proof. (i) As we have observed at the beginning of this section, there corresponds to I a unique finite quadratic sequence

$$(R,\mathfrak{M}) < (R_1,\mathfrak{M}_1) < (R_2,\mathfrak{M}_2) < \cdots < (R_s,\mathfrak{M}_s)$$

with $\overline{I^{R_s}} = \mathfrak{M}_s^n$ for some $n \in \mathbf{N}_+$, and with R_1 the unique immediate base point R' of I. Since the residue field $R/\mathfrak{M}$ is algebraically closed, a minimal ideal basis $x_1, x_2, \ldots, x_d$ of $\mathfrak{M}$ can be chosen such that R' is of the form

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_{M_1} \quad \text{with } M_1 = \left(x_1, \frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right)$$

and such that

$$\mathfrak{M}W = x_1 W \quad \text{and} \quad \mathfrak{M}V_{\mathfrak{M}} = x_1 V_{\mathfrak{M}}.$$

Here $(W, \mathfrak{M}_W)$ denotes the valuation ring of the unique Rees valuation w of $I^{R'}$, while $(V_{\mathfrak{M}}, \mathfrak{M}_{V_{\mathfrak{M}}})$ is the valuation ring of $v_{\mathfrak{M}} = \mathrm{ord}_R$. This shows that $(R', \mathfrak{M}')$, and hence $R[\mathfrak{M}/x_1]$, is contained in every Rees valuation ring of I. Since I^n is complete for every $n \in \mathbf{N}_+$, this implies that I^n is contracted from $R[\mathfrak{M}/x_1]$ for all $n \in \mathbf{N}_+$. If $r := \mathrm{ord}_R(I)$, it follows that the natural morphism

$$\frac{\mathfrak{M}^{rn}}{I^n} \longrightarrow \frac{\mathfrak{M}^{rn} R[\mathfrak{M}/x_1]}{I^n R[\frac{\mathfrak{M}}{x_1}]}$$

is injective for all $n \in \mathbf{N}_+$. Hence,

$$\text{length}\left(\frac{\mathfrak{M}^{rn}}{I^n}\right) \leq \text{length}\left(\frac{\mathfrak{M}^{rn} R[\mathfrak{M}/x_1]}{I^n R[\mathfrak{M}/x_1]}\right) = \text{length}\left(\frac{R'}{(I^{R'})^n}\right).$$

Consequently,

$$\text{length}\left(\frac{R}{I^n}\right) \leq \text{length}\left(\frac{R}{\mathfrak{M}^{rn}}\right) + \text{length}\left(\frac{R'}{(I^{R'})^n}\right)$$

for all $n \in \mathbf{N}_+$. This implies that

$$e(I) \leq e(\mathfrak{M})\mathrm{ord}_R(I)^2 + e(I^{R'}). \tag{1}$$

Next, we recall that we have proved in [**4**, Proposition 2.1] the following formula for $d(I, w)$ using the quadratic sequence corresponding to I:

$$d(I, w) = \frac{e(I) - e(\mathfrak{M})\mathrm{ord}_R(I)^2}{\sum_{j=1}^{s} \mathrm{ord}_{R_j}(I^{R_j}) w(\mathfrak{M}_j)}. \tag{2}$$

Using an adaption of Lipman's lemma [**7**, Lemma 1.11] to our situation, we have that

$$\sum_{j=1}^{s} \operatorname{ord}_{R_j}(I^{R_j}) w(\mathfrak{M}_j) = w(I^{R'}). \tag{3}$$

From (1)–(3), it follows that

$$d(I, w) \leq \frac{e(I^{R'})}{w(I^{R'})}.$$

Since $T(I^{R'}) = \{w\}$, we have by the theory of degree functions (see the background section) that

$$\frac{e(I^{R'})}{w(I^{R'})} = d(I^{R'}, w),$$

which proves (i).

(ii) Since I is normal we have, with the same notations and conventions as in the proof of (i), that the natural morphism

$$\frac{\mathfrak{M}^{rn}}{I^n} \longrightarrow \frac{\mathfrak{M}^{rn} R[\mathfrak{M}/x_1]}{I^n R[\mathfrak{M}/x_1]}$$

is injective for all $n \in \mathbf{N}_+$. Moreover, I is minimally generated (i.e., $\mu(I) = \dim_k(\mathfrak{M}^r/\mathfrak{M}^{r+1})$. Observing that

$$\dim_k\left(\frac{\mathfrak{M}^r}{\mathfrak{M}^{r+1}}\right) = \mu(I) - \operatorname{length}_R\left(\frac{\mathfrak{M}^{r+1}}{I\mathfrak{M} + x_1\mathfrak{M}^r}\right),$$

this is equivalent with

$$\mathfrak{M}^{r+1} = I\mathfrak{M} + x_1\mathfrak{M}^r.$$

Since I is normal, using this characterization of "minimally generated," we have that I^n is minimally generated for all $n \in \mathbf{N}_+$. Because of the observation just before Lemma 3.1, this means that the natural morphism

$$\frac{\mathfrak{M}^{rn}}{I^n} \to \frac{\mathfrak{M}^{rn} R[\mathfrak{M}/x_1]}{I^n R[\mathfrak{M}/x_1]}$$

is an isomorphism. This implies (see proof of (i)) that

$$e(I) = e(\mathfrak{M})\mathrm{ord}_R(I)^2 + e(I^{R'}). \tag{1'}$$

By the same reasoning as in the proof of (i), it follows from (1′), (2) and (3) that

$$d(I, w) = d(I^{R'}, w). \qquad \square$$

For a given quasi-one-fibered complete $\mathfrak{M}$-primary ideal I of order one in R, the above lemma together with results from the theory of degree functions (see the background section) will be used to prove that the transform $I^{R'}$ in the unique immediate base point R' of I also has order one, provided I is normal and minimally generated and two additional conditions are satisfied. We will also investigate when the converse holds.

Proposition 3.2. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain, and let I be a quasi-one-fibered $\mathfrak{M}$-primary ideal of R with unique immediate base point $(R', \mathfrak{M}')$. Let w denote the unique Rees valuation of $I^{R'}$. Suppose that $w(\mathfrak{M}) = w(\mathfrak{M}')$ and $d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$. Then*

(i) *If I is normal, then the implication* $\mathrm{ord}_{R'}(I^{R'}) = 1 \Rightarrow \mathrm{ord}_R(I) = 1$ *holds.*

(ii) *If I is normal and minimally generated, then we have*

$$\mathrm{ord}_R(I) = 1 \Longleftrightarrow \mathrm{ord}_{R'}(I^{R'}) = 1.$$

Proof. (i) We have to prove that $\mathrm{ord}_R(I) = 1$. To do so, we observe that the reciprocity relation

$$d_{\mathfrak{M}}(I) = d_I(\mathfrak{M})$$

implies that

$$d(\mathfrak{M}, v_{\mathfrak{M}})v_{\mathfrak{M}}(I) = d(I, v_{\mathfrak{M}})v_{\mathfrak{M}}(\mathfrak{M}) + d(I, w)w(\mathfrak{M}), \tag{$\star$}$$

since $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$. Further, we know from the theory of degree functions that

$$e(\mathfrak{M}) = d(\mathfrak{M}, v_{\mathfrak{M}})v_{\mathfrak{M}}(\mathfrak{M}),$$

and thus

$$d(\mathfrak{M}, v_{\mathfrak{M}}) = e(\mathfrak{M})$$

since $v_{\mathfrak{M}}(\mathfrak{M}) = 1$.

Next, the assumption $\mathrm{ord}_{R'}(I^{R'}) = 1$ will imply that $w(\mathfrak{M}') = 1$ and $d(I^{R'}, w) = 1$. Indeed, from the reciprocity relation

$$d_{\mathfrak{M}'}(I^{R'}) = d_{I^{R'}}(\mathfrak{M}')$$

we get

$$d(\mathfrak{M}', v_{\mathfrak{M}'})\mathrm{ord}_{R'}(I^{R'}) = d(I^{R'}, w)w(\mathfrak{M}')$$

since $T(\mathfrak{M}') = \{v_{\mathfrak{M}'}\}$, $T(I^{R'}) = \{w\}$, and $v_{\mathfrak{M}'}$ is by definition the $\mathrm{ord}_{R'}$-valuation. Since $\mathrm{ord}_{R'}(I^{R'}) = 1$ and $d(\mathfrak{M}', v_{\mathfrak{M}'}) = e(\mathfrak{M}') = 1$, we have

$$w(\mathfrak{M}') = 1 \quad \text{and} \quad d(I^{R'}, w) = 1.$$

This implies

$$w(\mathfrak{M}) = 1,$$

since we have assumed that $w(\mathfrak{M}) = w(\mathfrak{M}')$. By Lemma 3.1, we have that

$$d(I, w) \leq d(I^{R'}, w).$$

Since $d(I^{R'}, w) = 1$ and $d(I, w) > 0$ (because $w \in T(I)$), it follows that

$$d(I, w) = 1.$$

So the relation $(\star)$ becomes

$$e(\mathfrak{M})\mathrm{ord}_R(I) = d(I, v_{\mathfrak{M}}) + 1.$$

Using the assumption $d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$, this implies that

$$\mathrm{ord}_R(I) = 1,$$

thereby completing the proof of (i).

(ii) It only remains to prove that the implication $\mathrm{ord}_R(I) = 1 \Rightarrow \mathrm{ord}_{R'}(I^{R'}) = 1$ holds. Again we start from the reciprocity relation $(*)$. Since $\mathrm{ord}_R(I) = 1$ and $d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$, we have

$$e(\mathfrak{M}) = e(\mathfrak{M}) - 1 + d(I, w)w(\mathfrak{M}).$$

Hence,

$$d(I, w)w(\mathfrak{M}) = 1,$$

implying that $d(I, w) = 1$ and $w(\mathfrak{M}) = 1$. Since the quasi-one-fibered $\mathfrak{M}$-primary ideal I is supposed to be normal and minimally generated, we have by Lemma 3.1 (ii) that

$$d(I, w) = d(I^{R'}, w).$$

Hence,

$$d(I^{R'}, w) = 1.$$

From the reciprocity relation $d_{\mathfrak{M}'}(I^{R'}) = d_{I^{R'}}(\mathfrak{M}')$, it follows that

$$\mathrm{ord}_{R'}(I^{R'}) = d(I^{R'}, w)w(\mathfrak{M}'),$$

and hence

$$\mathrm{ord}_{R'}(I^{R'}) = 1,$$

since $w(\mathfrak{M}') = w(\mathfrak{M})$ (by assumption) and $w(\mathfrak{M}) = 1$. □

The examples given below illustrate the hypotheses in the previous proposition.

Example 3.3. Let $(R, \mathfrak{M})$ be a two-dimensional regular local ring with algebraically closed residue field (thus, R is certainly a two-dimensional Muhly local domain).

We consider the following immediate quadratic transform $(R', \mathfrak{M}')$ of $(R, \mathfrak{M})$:

$$R' = R\left[\frac{\mathfrak{M}}{x}\right]_M \quad \text{with } M = \left(x, \frac{y}{x}\right),$$

where x, y is a minimal ideal basis of $\mathfrak{M}$. Thus,

$$\mathfrak{M}' = \left(x, \frac{y}{x}\right)R',$$

and the ideal

$$I' := \left(x, \left(\frac{y}{x}\right)^2\right)R'$$

is a complete $\mathfrak{M}'$-primary ideal of order 1 in R'. This ideal I' has a unique immediate base point $(R'', \mathfrak{M}'')$ given by

$$R'' = R'\left[\frac{\mathfrak{M}'}{y/x}\right]_{M'} \qquad \text{with } M' = \left(\frac{y}{x}, \frac{x}{y/x}\right),$$

and

$$\mathfrak{M}'' = \left(\frac{y}{x}, \frac{x^2}{y}\right)R''.$$

In R'', we have that

$$I'R'' = \frac{y}{x}\left(\frac{y}{x}, \frac{x^2}{y}\right)R'' = \frac{y}{x}\mathfrak{M}''.$$

Thus, the transform $I'^{R''}$ of I' in R'' is the maximal ideal $\mathfrak{M}''$. This means that the unique quadratic sequence associated with the simple complete $\mathfrak{M}'$-primary ideal I' is as follows

$$(R', \mathfrak{M}') < (R'', \mathfrak{M}'').$$

Hence, the unique Rees valuation w of I' is

$$w = \mathrm{ord}_{R''}\text{-valuation}.$$

Thus, the corresponding valuation ring W is given by

$$W = R''\left[\frac{\mathfrak{M}''}{y/x}\right]_{(y/x)R''[\mathfrak{M}''/y/x]},$$

and the maximal ideal of W is

$$\mathfrak{M}_W = \left(\frac{y}{x}\right)W.$$

Now, let us consider the inverse transform I of I' in R, i.e.,

$$I = x^2 I' \cap R.$$

We claim that

$$I = (x^3, y^2, x^2y).$$

Indeed, let $J := (x^3, y^2, x^2y)R$. Since $\mu(J) = \mathrm{ord}_R(J) + 1$, we have that J is contracted from $R[\mathfrak{M}/x]$ by [**6**, Proposition 2.3]. Further, $JR[\mathfrak{M}/x] = x^2(x, (y/x)^2)$, which shows that $JR[\mathfrak{M}/x]$ is complete. Thus,

$$J = JR\left[\frac{\mathfrak{M}}{x}\right] \cap R$$

is also complete. (Note also that J has $(R', \mathfrak{M}')$ as a unique immediate base point and that $JR' = x^2I'$).

Now we have

$$x^2I' \cap R \subseteq (x^2I' \cap R)R' = x^2I' = JR'.$$

Thus,

$$x^2I' \cap R \subseteq JR' \cap R = J.$$

It is clear that $J = (x^3, y^2, x^2y) \subseteq x^2I' \cap R$, hence

$$J = x^2I' \cap R.$$

So the inverse transform I of I' in R is given by

$$I = (x^3, y^2, x^2y).$$

Thus, $I = (x^3, y^2, x^2y)R$ is a simple (and thus one-fibered) complete $\mathfrak{M}$-primary ideal of *order* 2 with unique immediate base point $(R', \mathfrak{M}')$ and whose transform $I^{R'}$ is the complete $\mathfrak{M}'$-primary ideal $I' = (x, (y/x)^2)R'$ of *order* 1.

Since the implication $\mathrm{ord}_{R'}({I'}^R) = 1 \Rightarrow \mathrm{ord}_R(I) = 1$ does not hold, at least one of the conditions in Proposition 3.2 is not satisfied in this example. To make this clear, we now summarize what we know about the ideal I.

- I is normal, since I is complete and R is two-dimensional regular.
- I is minimally generated since $\mu(I) = 3 = \dim_k(\mathfrak{M}^2/\mathfrak{M}^3)$.
- I is quasi-one-fibered since I has only one immediate base point, namely $(R', \mathfrak{M}')$, and $I^{R'} = I'$ is a simple complete $\mathfrak{M}'$-primary ideal

with $T(I') = \{w\}$ where $w = \mathrm{ord}_{R''}$-valuation. Thus, $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$.

• $d(I, w) = d(I^{R'}, w) = 1$, since I is the inverse transform of I' in R and $I^{R'}$ is a simple complete $\mathfrak{M}'$-primary ideal in the two-dimensional regular local ring $(R', \mathfrak{M}')$ (see [**5**, Proposition 3.4 and Corollary 3.6]).

• From the reciprocity relation $d_I(\mathfrak{M}) = d_{\mathfrak{M}}(I)$ (see the background section) it then follows that

$$d(I, v_{\mathfrak{M}}) + w(\mathfrak{M}) = 2.$$

• Since $w = \mathrm{ord}_{R''}$-valuation and $\mathfrak{M}'' = ((y/x), (x^2/y))R''$, it follows that $w(y/x) = w(x^2/y) = 1$. Using $x = (y/x)(x^2/y)$ and $y = x(y/x)$, we have that

$$w(\mathfrak{M}) = 2.$$

Thus,

$$d(I, v_{\mathfrak{M}}) = 0,$$

and this shows that the condition $d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$ is satisfied, since $e(\mathfrak{M}) = 1$.

• Since $\mathfrak{M}' = (x, (y/x))R'$, we have that

$$w(\mathfrak{M}') = 1.$$

Hence, the condition $w(\mathfrak{M}) = w(\mathfrak{M}')$ is *not* satisfied in this example, while the other conditions hold.

So, this example shows that the assumption "$w(\mathfrak{M}) = w(\mathfrak{M}')$" is indispensable for Proposition 3.2.

Example 3.4. Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain with embedding dimension 3. Suppose R has minimal multiplicity; thus, $(R, \mathfrak{M})$ is a rational singularity (see, for example, [**4**]). Let us consider an immediate quadratic transform $(R', \mathfrak{M}')$ of $(R, \mathfrak{M})$. Then

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_{M_1} \quad \text{with } M_1 = \left(x_1, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$$

and

$$\mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right)R',$$

for a suitable minimal ideal basis x_1, x_2, x_3 of $\mathfrak{M}$. In R we consider the ideal I being defined as the integral closure of the ideal

$$J = (x_1^4, x_1 x_2, x_2^2, x_3^2).$$

Then I is a complete (and hence normal) $\mathfrak{M}$-primary ideal of order 2.

It can readily be seen that J has $(R', \mathfrak{M}')$ as its unique immediate base point and the transform $J^{R'}$ of J in R' is given by

$$J^{R'} = \left(x_1^2, \frac{x_2}{x_1}\right) R'.$$

It follows that $J^{R'}$ is a complete $\mathfrak{M}'$-primary ideal of order 1 in R'; thus, $J^{R'}$ has a unique Rees valuation, which we denote by w.

Consequently,

$$T(J) \subseteq \{v_{\mathfrak{M}}, w\} \quad \text{and} \quad w \in T(J).$$

Since $J^{R'}$ has base points only on the chart $R'[\mathfrak{M}'/x_1]$ and $J^{R'} R'[\mathfrak{M}'/x_1] = x_1(x_1, (x_2/x_1^2))$, the unique quadratic sequence corresponding to $J^{R'}$ is as follows:

$$(R', \mathfrak{M}') < (R'', \mathfrak{M}''),$$

where $R'' = R'[\mathfrak{M}'/x_1]_{M_1'}$ with $M_1' = (x_1, (x_1/x_1^2))$ and the maximal ideal $\mathfrak{M}'' = (x_1, (x_2/x_1^2))R''$. It follows that

$$w = \mathrm{ord}_{R''}\text{-valuation}$$

(see, for example, [**11**, page 608]). Since $I = \overline{J}$, we have that

$$T(I) = T(J);$$

hence,

$$T(I) \subseteq \{v_{\mathfrak{M}}, w\} \quad \text{and} \quad w \in T(I).$$

This shows that I is quasi-one-fibered with $(R', \mathfrak{M}')$ as its unique immediate base point. Moreover,

$$I^{R'} = J^{R'},$$

implying that $\mathrm{ord}_{R'}(I^{R'}) = 1$. As $\mathrm{ord}_R(I) = 2$, it follows that the implication "$\mathrm{ord}_{R'}(I^{R'}) = 1 \Rightarrow \mathrm{ord}_R(I) = 1$" does not hold in this example and thus at least one of the hypotheses in Proposition 3.2 is not satisfied. We already know that I is a quasi-one-fibered normal $\mathfrak{M}$-primary ideal of order 2, and the unique quadratic sequence corresponding to I is

$$(R, \mathfrak{M}) < (R', \mathfrak{M}') < (R'', \mathfrak{M}''),$$

and $I^{R''} = \mathfrak{M}''$. Hence,

$$T(I) \subseteq \{v_{\mathfrak{M}}, w\} \quad \text{and} \quad w \in T(I),$$

where $w = \mathrm{ord}_{R''}$-valuation.

Now, it is readily seen that $w(\mathfrak{M}) = 1$ and $w(\mathfrak{M}') = 1$, and thus the condition "$w(\mathfrak{M}) = w(\mathfrak{M}')$" is satisfied.

Using this and the fact that $e(\mathfrak{M}) = 2$, it follows from the reciprocity relation

$$d_I(\mathfrak{M}) = d_{\mathfrak{M}}(I),$$

that

$$d(I, v_{\mathfrak{M}}) + d(I, w) = 4.$$

Further, in the two-dimensional regular local ring $(R', \mathfrak{M}')$ we have the reciprocity relation

$$d_{I^{R'}}(\mathfrak{M}') = d_{\mathfrak{M}'}(I^{R'}).$$

Since $\mathrm{ord}_{R'}(I^{R'}) = 1$ and $e(\mathfrak{M}') = 1$, this implies

$$d(I^{R'}, w) = 1.$$

Since I is a normal quasi-one-fibered $\mathfrak{M}$-primary ideal with unique immediate base point $(R', \mathfrak{M}')$, we have by Lemma 3.1 (i) that

$$d(I, w) \leq d(I^{R'}, w).$$

As $w \in T(I)$, and hence $d(I, w) > 0$, this implies

$$d(I, w) = 1.$$

So

$$d(I, v_{\mathfrak{M}}) = 3,$$

and this shows that the condition "$d(I, v_{\mathfrak{M}}) = e(\mathfrak{M})-1$" is not satisfied here, since $e(\mathfrak{M}) = 2$.

Summarizing, this example shows that the condition "$d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$" cannot be omitted from Proposition 3.2.

Example 3.5. Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain with minimal multiplicity. Suppose there exists a minimal ideal basis

$$x_1, x_2, \dots, x_d$$

of $\mathfrak{M}$ such that $x_1 \notin \operatorname{rad}(x_2, \dots, x_d)$. We then consider the immediate quadratic transform

$$(R_1, \mathfrak{M}_1)$$

where $R_1 = R[\mathfrak{M}/x_1]_{M_1}$ with $M_1 = (x_1, x_2/x_1, \dots, x_d/x_1)$, and (possibly after renumbering $x_2, \dots, x_d$)

$$\mathfrak{M}_1 = \left(x_1, \frac{x_2}{x_1}\right) R_1.$$

Let

$$(V, \mathfrak{M}_V)$$

denote the valuation ring of the ord_R-valuation $v_{\mathfrak{M}}$. For any natural number s there exists a unique quadratic sequence

$$(R, \mathfrak{M}) < (R_1, \mathfrak{M}_1) < (R_2, \mathfrak{M}_2) < \cdots < (R_s, \mathfrak{M}_s),$$

such that $(R_i, \mathfrak{M}_i)$ is contained in $(V, \mathfrak{M}_V)$ (i.e., $(R_i, \mathfrak{M}_i)$ is *proximate* to $(R, \mathfrak{M})$) for $i = 1, 2, \dots, s$. Following Lipman in [**8**, page 240], we then have

$$R_i = R_{i-1}\left[\frac{\mathfrak{M}_{i-1}}{x_2/x_1}\right]_{\left(\frac{x_1}{x_2/x_1^{i-1}}, \frac{x_2}{x_1}\right)}$$

for $i = 2, \dots, s$.

In particular,

$$R_s = R_{s-1}\left[\frac{\mathfrak{M}_{s-1}}{x_2/x_1}\right]_{\left(\frac{x_1}{x_2/x_1^{s-1}}, \frac{x_2}{x_1}\right)}$$

and
$$\mathfrak{M}_s = \left(\frac{x_1}{x_2/{x_1}^{s-1}}, \frac{x_2}{x_1}\right).$$
Let I_{s-1} denote the inverse transform of $\mathfrak{M}_s$ in R_{s-1} (i.e., $I_{s-1} = (x_2/x_1)\mathfrak{M}_s \cap R_{s-1}$). Then we have
$$I_{s-1} = \left(\frac{x_1}{x_2/{x_1}^{s-2}}, \left(\frac{x_2}{x_1}\right)^2\right)R_{s-1}.$$
Descending the quadratic sequence step by step, we find that the inverse transform I_1 of $\mathfrak{M}_s$ in R_1 is given by
$$I_1 = \left(x_1, \left(\frac{x_2}{x_1}\right)^s\right)R_1.$$
Note that I_1 is the simple complete $\mathfrak{M}_1$-primary ideal in the two-dimensional regular local ring $(R_1, \mathfrak{M}_1)$ that corresponds to the prime divisor $w :=$ ord_{R_s}-valuation of R_1 under Zariski's one-to-one correspondence. It follows that w is the unique Rees valuation of I_1, hence
$$T(I_1) = \{w\}.$$
Now let us consider the following ideal I in R:
$$I := x_1^s I_1 \cap R.$$
Then we have

- I is a complete $\mathfrak{M}$-primary ideal,
- $\mathrm{ord}_R(I) = s$,
- the transform of I in R_1 is I_1,
- $(R_1, \mathfrak{M}_1)$ is the unique immediate base point of I.

It follows that the complete $\mathfrak{M}$-primary ideal I is quasi-one-fibered and $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ with $w \in T(I)$.

Since $w = \mathrm{ord}_{R_s}$-valuation and $\mathfrak{M} = (x_1, x_2, \ldots, x_d)$, $\mathfrak{M}' = (x_1, x_2/x_1)R_1$, we have that
$$w(\mathfrak{M}) = s$$

and

$$w(\mathfrak{M}') = 1.$$

Next we determine the degree function coefficients $d(I, w)$ and $d(I, v_{\mathfrak{M}})$ of I.

The two-dimensional Muhly local domain $(R, \mathfrak{M})$ has minimal multiplicity, so it is a rational singularity (see [**3**, Theorem 3.1]). This implies that the complete $\mathfrak{M}$-primary ideal I is in fact normal. Thus,

$$d(I, w) \leq d(I^{R_1}, w),$$

by Lemma 3.1 (i).

Since $I^{R_1} = I_1$ is a simple complete $\mathfrak{M}_1$-primary ideal in the two-dimensional regular local ring $(R_1, \mathfrak{M}_1)$, we have

$$d(I^{R_1}, w) = 1,$$

by [**5**, Corollary 3.6].

Since $w \in T(I)$, we have $d(I, w) > 0$, and thus

$$d(I, w) = 1.$$

In order to determine $d(I, v_{\mathfrak{M}})$, we consider the reciprocity relation

$$d_I(\mathfrak{M}) = d_{\mathfrak{M}}(I).$$

Since $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$, this becomes

$$d(I, v_{\mathfrak{M}})v_{\mathfrak{M}}(\mathfrak{M}) + d(I, w)w(\mathfrak{M}) = d(\mathfrak{M}, v_{\mathfrak{M}})v_{\mathfrak{M}}(I).$$

We already know that

- $v_{\mathfrak{M}}(I) = \mathrm{ord}_R(I) = s$,
- $w(\mathfrak{M}) = s$,
- $d(I, w) = 1$.

Using this together with the fact that $d(\mathfrak{M}, v_{\mathfrak{M}}) = e(\mathfrak{M})$ and $v_{\mathfrak{M}}(\mathfrak{M}) = \mathrm{ord}_R(\mathfrak{M}) = 1$, the reciprocity relation becomes

$$d(I, v_{\mathfrak{M}}) + s = se(\mathfrak{M}).$$

Hence,

$$d(I, v_{\mathfrak{M}}) = s(e(\mathfrak{M}) - 1).$$

Note, if $(R, \mathfrak{M})$ is *not* regular (thus $e(\mathfrak{M}) > 1$), then this shows that $d(I, v_{\mathfrak{M}}) > 0$, which means that $v_{\mathfrak{M}} \in T(I)$, and hence $T(I) = \{v_{\mathfrak{M}}, w\}$ in that case.

From the previous discussion, we can conclude:

• If $s = 1$, then both the conditions "$w(\mathfrak{M}) = w(\mathfrak{M}')$" and "$d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$" are satisfied.

• If $s > 1$, then neither of these conditions is satisfied.

In the next result a stronger version of Proposition 3.2 will be given in the case that the two-dimensional Muhly local domain $(R, \mathfrak{M})$ is *regular* and the quasi-one-fibered complete $\mathfrak{M}$-primary ideal I is *simple* (and hence one-fibered).

Proposition 3.6. *Let $I \neq \mathfrak{M}$ be a simple, complete $\mathfrak{M}$-primary ideal of the two-dimensional regular local ring $(R, \mathfrak{M})$ with algebraically closed residue field. Let $(R', \mathfrak{M}')$ denote the unique immediate base point of I, and let w be the unique Rees valuation of I (so that $T(I) = \{w\}$). Then the following assertions are equivalent*:

(i) $\mathrm{ord}_R(I) = 1$.

(ii) $\mathrm{ord}_{R'}(I^{R'}) = 1$ *and* $w(\mathfrak{M}) = w(\mathfrak{M}')$.

(iii) *There exists a height-one prime $\mathfrak{p}'$ in R' such that $\mathfrak{p}' \subset I^{R'}$ and $R/\mathfrak{p}' \cap R$ is a* DVR.

Proof. (i) $\Rightarrow$ (ii). Since $I \neq \mathfrak{M}$ is an $\mathfrak{M}$-primary ideal of order 1, we have that

$$I = (x_1^n, x_2)$$

for some integer $n > 1$ and with x_1, x_2 an ideal basis of $\mathfrak{M}$. It follows that I has immediate base points only on the chart $R[\mathfrak{M}/x_1]$, since $IR[\mathfrak{M}/x_2] = (x_2)R[\mathfrak{M}/x_2]$. Hence, the unique immediate base point $(R', \mathfrak{M}')$ of I is given by

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_{M_1} \quad \text{with} \quad M_1 = \left(x_1, \frac{x_2}{x_1}\right),$$

since $IR[\mathfrak{M}/x_1] = x_1(x_1^{n-1}, (x_2/x_1))$. Thus, we have in R' that

$$\mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right)R' \quad \text{and} \quad I^{R'} = \left(x_1^{n-1}, \frac{x_2}{x_1}\right)R'.$$

Hence,

$$\mathrm{ord}_{R'}(I^{R'}) = 1.$$

So it remains to show that $w(\mathfrak{M}) = w(\mathfrak{M}')$.

• If $n = 2$, then $I^{R'} = \mathfrak{M}'$, and this implies that the unique Rees valuation w of $I^{R'}$ is $\mathrm{ord}_{R'}$. So it is readily seen that $w(\mathfrak{M}) = w(\mathfrak{M}') = 1$.

• If $n > 2$, then $I^{R'} = (x_1^{n-1}, (x_2/x_1))$ has its unique immediate base point lying on the chart $R'[\mathfrak{M}'/x_1]$. Since this immediate base point of $I^{R'}$ is dominated by the valuation ring W of w (see Section 2 and the Background section), it follows that

$$\frac{x_2}{x_1^2} \in W.$$

This implies that $w(\mathfrak{M}) = w(\mathfrak{M}')$.

(ii) $\Rightarrow$ (iii). In order to prove this implication, we begin by recalling the following properties of the ideal I.

• I is one-fibered since $T(I) = \{w\}$,

• I is normal because, in a two-dimensional regular local ring, any product of complete ideals is complete (see [**6**, Theorem 3.7]).

• Since $I \neq \mathfrak{M}$, we have that

$$w \neq v_{\mathfrak{M}}$$

because of Zariski's one-to-one correspondence between the simple complete $\mathfrak{M}$-primary ideals of R and the prime divisors of R. Hence, $v_{\mathfrak{M}} \notin T(I)$, implying that $d(I, v_{\mathfrak{M}}) = 0$. Thus, the condition "$d(I, v_{\mathfrak{M}}) = e(\mathfrak{M}) - 1$" is satisfied since $e(\mathfrak{M}) = 1$.

Because of assumption (ii) we also know that the condition "$w(\mathfrak{M}) = w(\mathfrak{M}')$" is satisfied. Hence, the ideal I satisfies all the conditions of Proposition 3.2, and thus there exists an element $x_2 \in I$ such that

$x_2 \notin \mathfrak{M}^2$. It follows that there exists an ideal basis x_1, x_2 of $\mathfrak{M}$ such that

$$I = (x_1^n, x_2)R$$

for some $n > 1$.

This implies that I has a unique immediate base point $(R', \mathfrak{M}')$ given by

$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_{\mathfrak{M}'} \quad \text{with} \quad M' = \left(x_1, \frac{x_2}{x_1}\right),$$

and thus

$$\mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right)R'.$$

Further, the transform $I^{R'}$ of I in R' is given by

$$I^{R'} = \left(x_1^{n-1}, \frac{x_2}{x_1}\right)R'.$$

Hence,

$$\mathfrak{p}' := \left(\frac{x_2}{x_1}\right)R'$$

is a height-one prime ideal that is contained in $I^{R'}$.

Moreover, we have

$$\mathfrak{p}' \cap R = (x_2)R,$$

implying that $R/\mathfrak{p}' \cap R$ is a one-dimensional regular local ring (i.e., a DVR). This proves the implication (ii) $\Rightarrow$ (iii). □

(iii) $\Rightarrow$ (i). Let $\mathfrak{p}'$ be a height-one prime in R' such that $\mathfrak{p}' \subset I^{R'}$ and $R/\mathfrak{p}' \cap R$ is a DVR. It follows that there exists a minimal ideal basis x_1, x_2 of $\mathfrak{M}$ such that

$$x_2 \in \mathfrak{p}' \cap R,$$

and thus

$$\mathfrak{p}' \cap R = (x_2)R.$$

This implies that the unique immediate base point $(R', \mathfrak{M}')$ of I cannot be lying on the chart $R[\mathfrak{M}/x_2]$ (since otherwise x_1/x_2 would be an element of R', implying that $x_1 \in \mathfrak{p}' \cap R = (x_2)R$, which is impossible).

So R' is a localization of $R[\mathfrak{M}/x_1]$, and thus x_2/x_1 is an element of R'. It follows that
$$x_1 \frac{x_2}{x_1} = x_2 \in \mathfrak{p}',$$
and since $x_1 \notin \mathfrak{p}'$, we have that $x_2/x_1 \in \mathfrak{p}' \subset I^{R'}$.

As we have already observed, R' is of the form
$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_{M_1}$$
with M_1 a maximal ideal of $R[\mathfrak{M}/x_1]$ lying over $\mathfrak{M}$. Now $x_2/x_1 \in M_1$ (since $x_2/x_1 \in I^{R'} \subset \mathfrak{M}'$) and $x_1 \in M_1$ (since M_1 is lying over $\mathfrak{M}$); thus,
$$\left(x_1, \frac{x_2}{x_1}\right) R\left[\frac{\mathfrak{M}}{x_1}\right] \subseteq M_1.$$
Since $(x_1, x_2/x_1)R[\mathfrak{M}/x_1]$ is a maximal ideal, we have that
$$M_1 = \left(x_1, \frac{x_2}{x_1}\right) R\left[\frac{\mathfrak{M}}{x_1}\right].$$
This shows that the immediate base point $(R', \mathfrak{M}')$ of I is given by
$$R' = R\left[\frac{\mathfrak{M}}{x_1}\right]_{M_1} \quad \text{and} \quad \mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right) R'.$$
Since $x_2/x_1 \in I^{R'}$ and $R'/[(x_2/x_1)R']$ is a DVR, we have that
$$I^{R'} = \left(x_1^n, \frac{x_2}{x_1}\right) R'$$
for some $n \in \mathbf{N}_+$.

From the Zariski-Lipman theory of complete ideals in two-dimensional regular local rings, we know that a simple complete $\mathfrak{M}$-primary ideal I is the inverse transform of its transform $I^{R'}$ in the unique immediate base point $(R', \mathfrak{M}')$ of I. Hence,
$$I = x_1 I^{R'} \cap R = (x_1^{n+1}, x_2)R' \cap R.$$

So $x_2 \in I$, and thus $\operatorname{ord}_R(I) = 1$, which completes the proof of (iii) $\Rightarrow$ (i). $\square$

If a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is regular, then it is easily seen that *any* complete $\mathfrak{M}$-primary ideal I of order one has the very simple form (x_1^n, x_2) with $n \in \mathbf{N}_+$ and x_1, x_2 a minimal ideal basis of $\mathfrak{M}$.

By contrast, if $(R, \mathfrak{M})$ is not regular, then this does not hold in general as we will see in Example 3.8 below.

But, first, we will prove a result (partly inspired by the previous proposition) giving necessary and sufficient conditions for a quasi one-fibered complete $\mathfrak{M}$-primary ideal of order one in a two-dimensional Muhly local domain $(R, \mathfrak{M})$, to be of the form $(x_1^n, x_2, \dots, x_d)$ with $x_1, x_2, \dots, x_d$ a minimal ideal basis of $\mathfrak{M}$ such that $x_1 \notin \operatorname{rad}(x_2, \dots, x_n)$. We therefore begin by recalling some facts needed in what follows.

• The above condition "$x_1 \notin \operatorname{rad}(x_2, \dots, x_n)$" is equivalent to "$(x_2, \dots, x_n)$ is a prime ideal of R", and is also equivalent to "$R/(x_2, \dots, x_d)$ is a one-dimensional regular local ring." It implies that

$$\left(\frac{x_2}{x_1}, \dots, \frac{x_d}{x_1}\right) R\left[\frac{\mathfrak{M}}{x_1}\right] \cap R = (x_2, \dots, x_d).$$

It follows that

$$M_1 := \left(x_1, \frac{x_2}{x_1}, \dots, \frac{x_d}{x_1}\right) R\left[\frac{\mathfrak{M}}{x_1}\right]$$

is a maximal ideal of $R[\mathfrak{M}/x_1]$ lying over $\mathfrak{M}$ (i.e., $M_1 \cap R = \mathfrak{M}$), and the ring

$$R_1 := R\left[\frac{\mathfrak{M}}{x_1}\right]_{M_1}$$

is an immediate quadratic transform of R that is a two-dimensional regular local ring.

• Let $I := (x_1^n, x_2, \dots, x_d)$ with $n > 1$ and $x_1, x_2, \dots, x_d$ a minimal ideal basis of $\mathfrak{M}$ such that $x_1 \notin \operatorname{rad}(x_2, \dots, x_d)$. Since the transform $I^{R_1} \neq R_1$, we see that $(R_1, \mathfrak{M}_1)$ is an immediate base point of I. In fact it is the only immediate base point of I.

• I^{R_1} is a simple complete $\mathfrak{M}_1$-primary ideal, so $I = (x_1^n, x_2, \dots, x_d)$ is quasi one-fibered (see [**4**, Proposition 1.5]). Hence,

$$T(I) \subseteq \{v_{\mathfrak{M}}, w\} \quad \text{and} \quad w \in T(I),$$

where $v_{\mathfrak{M}} = \mathrm{ord}_R$-valuation and w denotes the unique Rees valuation of I^{R_1}.

• I is contracted from $R[\mathfrak{M}/x_1]$ (i.e., $IR[\mathfrak{M}/x_1] \cap R = I$), implying that I is complete.

Now we are ready to state and prove our result.

Proposition 3.7. *Let $I \neq \mathfrak{M}$ be a quasi one-fibered complete $\mathfrak{M}$-primary ideal of order one of the two-dimensional Muhly local domain $(R, \mathfrak{M})$. Then the following assertions are equivalent*:

(i) $I = (x_1^n, x_2, \ldots, x_d)$ *with* $n > 1$ *and* $x_1, x_2, \ldots, x_d$ *a minimal ideal basis of* $\mathfrak{M}$ *such that* $x_1 \notin \mathrm{rad}\,(x_2, \ldots, x_d)$.

(ii) *There exists a height-one prime* $\mathfrak{p}_1$ *in the unique immediate base point* R_1 *of* I *such that*

$$\mathfrak{p}_1 \subset I^{R_1} \quad \textit{and} \quad \frac{R}{\mathfrak{p}_1 \cap R} \textit{ is a } \mathrm{DVR}.$$

(iii) *There exists a prime ideal* $\mathfrak{p}$ *in* R *such that* $\mathfrak{p} \subset I$ *with* $\mu(\mathfrak{p}) = \mathrm{embdim}\, R - 1$ *and with* $R/\mathfrak{p}$ *a* DVR.

(iv) *There exists a height-one ideal* I_0 *in* R *such that* $I_0 \subset I$ *and* $\mathfrak{c}(I_0) = \mathfrak{c}(I) = P$, *with* P *a homogeneous prime ideal of height one in* $\mathrm{gr}_{\mathfrak{M}}(R)$.

Proof. (i) $\Rightarrow$ (ii). It is clear that $\mathrm{ord}_R(I) = 1$ and, as we have recalled above, I is a quasi one-fibered complete $\mathfrak{M}$-primary ideal having a unique immediate base point $(R_1, \mathfrak{M}_1)$ and $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ with $w \in T(I)$, where $v_{\mathfrak{M}} = \mathrm{ord}_R$ and w denotes the unique Rees valuation of I^{R_1}.

Since $R_1 = R[\mathfrak{M}/x_1]_{M_1}$ with $M_1 = (x_1, x_2/x_1, \ldots, x_d/x_1)$, we have that the transform I^{R_1} of I in R_1 is given by

$$I^{R_1} = \left(x_1^{n-1}, \frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right) R_1.$$

As we have observed earlier, it follows from the condition $x_1 \notin \mathrm{rad}\,(x_2, \ldots, x_d)$ that

$$\left(\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right) R\left[\frac{\mathfrak{M}}{x_1}\right] \cap R = (x_2, \ldots, x_d).$$

This implies that

$$\frac{R[\mathfrak{M}/x_1]}{(x_2/x_1,\ldots,x_d/x_1)} \cong \frac{R}{(x_2,\ldots,x_d)},$$

and it follows that $(x_2/x_1,\ldots,x_d/x_1)R[\mathfrak{M}/x_1]$ is a prime ideal of height one in $R[\mathfrak{M}/x_1]$. Hence,

$$\mathfrak{p}_1 := \left(\frac{x_2}{x_1},\ldots,\frac{x_d}{x_1}\right)R_1$$

is a height-one prime in R_1 such that $\mathfrak{p}_1 \subset I^{R_1}$ and $\mathfrak{p}_1 \cap R = (x_2,\ldots,x_d)$. Thus, $R/\mathfrak{p}_1 \cap R$ is a DVR.

(ii) $\Rightarrow$ (iii). Since $R/\mathfrak{p}_1 \cap R$ is a DVR, there exists a minimal ideal basis $x_1, x_2, \ldots, x_d$ of $\mathfrak{M}$ such that $(x_2,\ldots,x_d) \subseteq \mathfrak{p}_1 \cap R$. It follows that $R/(x_2,\ldots,x_d)$ is a one-dimensional regular local ring. This implies that $\mathfrak{p}_1 \cap R = (x_2,\ldots,x_d)$ and thus the prime ideal $\mathfrak{p} := \mathfrak{p}_1 \cap R$ satisfies the following conditions from (iii):

$$\mu(\mathfrak{p}) = \operatorname{embdim} R - 1 \quad \text{and} \quad \frac{R}{\mathfrak{p}} \text{ is a DVR.}$$

So, in order to show that (iii) holds, it only remains to prove that $\mathfrak{p} \subset I$.

Since R_1 is the unique immediate base point of I and I is complete, we have that I is contracted from R_1, i.e.,

$$IR_1 \cap R = I.$$

It therefore suffices to show that $\mathfrak{p} = (x_2,\ldots,x_d) \subset IR_1$. Since $x_1 \notin \mathfrak{p}_1$, we see that $(R_1, \mathfrak{M}_1)$ cannot be lying on the chart $R[\mathfrak{M}/x_i]$ for $i = 2,\ldots,d$. Hence, $(R_1,\mathfrak{M}_1)$ is lying on $R[\mathfrak{M}/x_1]$, implying that

$$\frac{x_2}{x_1},\ldots,\frac{x_d}{x_1} \in R_1.$$

Using this, together with the fact that $x_2,\ldots,x_d \in \mathfrak{p}_1$ and $x_1 \notin \mathfrak{p}_1$, we have

$$\frac{x_2}{x_1},\ldots,\frac{x_d}{x_1} \in \mathfrak{p}_1.$$

Since $\mathfrak{p}_1 \subset I^{R_1}$ and $IR_1 = x_1 I^{R_1}$, it follows that

$$\mathfrak{p} = (x_2, \ldots, x_d) \subset IR_1.$$

This completes the proof of the implication (ii) $\Rightarrow$ (iii). $\square$

(iii) $\Rightarrow$ (i). Now, there is given a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \subset I$ with $\mu(\mathfrak{p}) = \operatorname{embdim} R - 1$ and $R/\mathfrak{p}$ is a DVR. It follows that there exist $d-1$ elements in R

$$x_2, \ldots, x_d$$

such that

$$\mathfrak{p} = (x_2, \ldots, x_d).$$

Since $R/\mathfrak{p}$ is a DVR, the maximal ideal $\mathfrak{M}/\mathfrak{p}$ is generated by a single element, say $\overline{x}_1 := x_1 + \mathfrak{p}$. Consequently,

$$\mathfrak{M} = (x_1, x_2, \ldots, x_d),$$

and the minimal ideal basis $x_1, x_2, \ldots, x_d$ of $\mathfrak{M}$ certainly satisfies the condition $x_1 \notin \operatorname{rad}(x_2, \ldots, x_d)$ since $(x_2, \ldots, x_d)$ is the prime ideal $\mathfrak{p}$.

Finally, $I/(x_2, \ldots, x_d)$ is some power of $\mathfrak{M}/(x_2, \ldots, x_d)$, and thus

$$I = (x_1^n, x_2, \ldots, x_d),$$

with $n > 1$ (since $I \neq \mathfrak{M}$). So (iii) $\Rightarrow$ (i) holds.

(i) $\Rightarrow$ (iv). Assume (i) holds, i.e., the ideal $I \neq \mathfrak{M}$ is of the form $I = (x_1^n, x_2, \ldots, x_d)$, where $x_1, x_2, \ldots, x_d$ is a minimal ideal basis of $\mathfrak{M}$ such that $x_1 \notin \operatorname{rad}(x_2, \ldots, x_d)$. Let $I_0 := (x_2, \ldots, x_d)R$. Then I_0 is an ideal of height one such that $I_0 \subset I$. It follows from [**10**, page 214] that

$$\mathfrak{c}(I_0) = \mathfrak{c}(I) = (x_2^*, \ldots, x_d^*),$$

where $x_2^*, \ldots, x_d^*$ denote the leading forms of $x_2, \ldots, x_d$ and $\mathfrak{c}(I_0)$ (respectively, $\mathfrak{c}(I)$) is the characteristic ideal of I_0 (respectively, I). In order to prove that (iv) holds, we only have to show that the homogeneous ideal $(x_2^*, \ldots, x_d^*)$ is a prime ideal of the associated graded ring $\operatorname{gr}_{\mathfrak{M}}(R)$.

Since $(x_2^*, \ldots, x_d^*)$ is a *homogeneous* ideal of $\operatorname{gr}_{\mathfrak{M}}(R)$, it suffices to prove that if the product of two *homogeneous* elements belongs to $(x_2^*, \ldots, x_d^*)$. Then at least one of the factors belongs to this ideal.

Therefore, let $\alpha^* := \alpha + \mathfrak{M}^r$ with $\alpha \in \mathfrak{M}^r \setminus \mathfrak{M}^{r+1}$ and $\beta^* := \beta + \mathfrak{M}^s$ with $\beta \in \mathfrak{M}^s \setminus \mathfrak{M}^{s+1}$ be homogeneous elements of orders r and s, and suppose that

$$\alpha^*\beta^* \in (x_2^*, \dots, x_d^*).$$

Then we have to show that α^* or β^* belongs to $(x_2^*, \dots, x_d^*)$. Suppose not; then we have to show that this leads to a contradiction.

First, we make the following observation. Let $y^* := y + \mathfrak{M}^r$ with $y \in \mathfrak{M}^r \setminus \mathfrak{M}^{r+1}$ a homogeneous element of order r in $\mathrm{gr}_{\mathfrak{M}}(R)$. The assertion that $y^* \in (x_2^*, \dots, x_d^*)$ amounts to the same thing as saying that the natural image $\overline{y}$ of y in $R/(x_2, \dots, x_d)$ belongs to $(\overline{x}_1^{r+1})R/(x_2, \dots, x_d)$. Let us briefly explain this claim.

If $y^* \in (x_2^*, \dots, x_d^*)$, then $y - (a_2x_2 + \dots + a_dx_d) \in \mathfrak{M}^{r+1}$ for some elements $a_2, \dots, a_d$ of R. It follows that $y - (a_2'x_2 + \dots + a_d'x_d) \in (x_1^{r+1})R$, where $a_2', \dots, a_d'$ denote certain elements of R. This implies that $\overline{y} \in (\overline{x}_1^{r+1})R/(x_2, \dots, x_d)$.

Conversely, suppose that $\overline{y} \in (\overline{x}_1^{r+1})R/(x_2, \dots, x_d)$. Then $y - (a_2x_2 + \dots + a_dx_d) \in (x_1^{r+1})R$ for some $a_2, \dots, a_d \in R$. Hence,

$$y^* = (a_2x_2 + \dots + a_dx_d) + \mathfrak{M}^{r+1}.$$

This implies that $a_2x_2 + \dots + a_dx_d$ is an element of the ideal $(x_2, \dots, x_d)$ of *order* r. From [**10**, page 214], we then know that its leading form $(a_2x_2 + \cdots + a_dx_d) + \mathfrak{M}^{r+1}$ belongs to the characteristic ideal $\mathfrak{c}(x_2, \dots, x_d) = (x_2^*, \dots, x_d^*)$. Thus, $y^* \in (x_2^*, \dots, x_d^*)$, and this proves our observation.

Because of this observation, the assumption that α^* and β^* do not belong to $(x_2^*, \dots, x_d^*)$, means that $\overline{\alpha} \notin (\overline{x}_1^{r+1})R/(x_2, \dots, x_d)$ and $\overline{\beta} \notin (\overline{x}_1^{s+1})R/(x_2, \dots, x_d)$. On the other hand, $\alpha^*\beta^* \in (x_2^*, \dots, x_d^*)$ means that $\overline{\alpha}\overline{\beta} \in (\overline{x}_1^{r+s+1})R/(x_2, \dots, x_d)$. Since $R/(x_2, \dots, x_d)$ is a UFD and $\overline{x}_1$ is a prime element, the desired contradiction follows.

(iv) $\Rightarrow$ (i). Now we assume that there exists a height-one ideal I_0 in R such that $I_0 \subset I$ and $\mathfrak{c}(I_0) = \mathfrak{c}(I)$ is a homogeneous prime ideal P in $\mathrm{gr}_{\mathfrak{M}}(R)$ of height one.

Since the residue field $k = R/\mathfrak{M}$ is algebraically closed, there exists a minimal ideal basis $x_1, x_2, \dots, x_d$ of $\mathfrak{M}$ such that

$$x_1^* \notin P \quad \text{and} \quad P = (x_2^*, \dots, x_d^*).$$

(See, for example, [**9**, page 100].)

Using the fact that P is contained in $\mathfrak{c}(I_0)$, this minimal ideal basis $x_1, x_2, \ldots, x_d$ of $\mathfrak{M}$ can be modified so that a new minimal ideal basis

$$x_1 := x_1, x_2', \ldots, x_d'$$

of $\mathfrak{M}$ is obtained such that

$$(x_2', \ldots, x_d') \subseteq I_0 \quad \text{and} \quad P = (x_2'^{*}, \ldots, x_d'^{*}).$$

In other words, we may suppose that the minimal ideal basis of $\mathfrak{M}$ above (thus with $x_1^* \notin P$ and $P = (x_2^*, \ldots, x_d^*)$), satisfies the additional condition

$$x_2, \ldots, x_d \in I_0.$$

Since height I_0 is one and $(x_2, \ldots, x_d) \subseteq I_0$, no power of x_1 belongs to the ideal $(x_2, \ldots, x_d)$, i.e.,

$$x_1 \notin \operatorname{rad}(x_2, \ldots, x_d).$$

As we have already observed, this means that $R/(x_2, \ldots, x_d)$ is a DVR. So, the ideal $I/(x_2, \ldots, x_d)$ of $R/(x_2, \ldots, x_d)$ is some power of the maximal ideal $\mathfrak{M}/(x_2, \ldots, x_d)$, say, $I/(x_2, \ldots, x_d) = (\mathfrak{M}/(x_2, \ldots, x_d))^n$ with $n > 1$ (since $I \neq \mathfrak{M}$). It follows that $I = (x_1^n, x_2, \ldots, x_d)$ with $n > 1$, and this completes the proof of the proposition. $\square$

In the two-dimensional regular case, every complete $\mathfrak{M}$-primary ideal of order one has the very simple form (x_1^n, x_2), $n \in \mathbf{N}_+$, for a suitable ideal basis x_1, x_2 of $\mathfrak{M}$. By contrast, if a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is *not* regular, then it does not hold in general that any quasi-one-fibered complete $\mathfrak{M}$-primary ideal of order one is of the form $(x_1^n, x_2, \ldots, x_d)$, $n \in \mathbf{N}_+$, with $x_1, x_2, \ldots, x_d$ a minimal ideal basis of $\mathfrak{M}$ such that $x_1 \notin \operatorname{rad}(x_2, \ldots, x_d)$.

In fact, using Proposition 3.7 we will see in Example 3.8 below that this can occur in a two-dimensional Muhly local domain $(R, \mathfrak{M})$ that is the local ring at the vertex of the affine cone over a projective curve in projective 3-space over an algebraically closed field k.

Example 3.8. Let

$$R = \frac{k[X_1, X_2, X_3]_{(X_1, X_2, X_3)}}{(X_1^3 + X_1^2 X_2 - X_2^2 - X_1 X_3)_{(X_1, X_2, X_3)}}$$

with k an algebraically closed field.

Then
$$R = k[x_1, x_2, x_3]_{(x_1,x_2,x_3)}$$
with $x_1^3 = (x_2 - x_1^2)x_2 + x_1x_3$, where x_1, x_2, x_3 denote the natural images X_1, X_2, X_3. It follows that
$$M := \left(x_1, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right) R\left[\frac{\mathfrak{M}}{x_1}\right]$$
is a maximal ideal of $R[\mathfrak{M}/x_1]$ lying over $\mathfrak{M}$, and
$$R' := R\left[\frac{\mathfrak{M}}{x_1}\right]_M$$
is an immediate quadratic transform of R with maximal ideal
$$\mathfrak{M}' = \left(x_1, \frac{x_2}{x_1}\right) R'.$$
We now consider the following $\mathfrak{M}$-primary ideal of order one in R
$$I := (x_1^2, x_1x_2, x_3).$$
In R' we have
$$IR' = x_1\left(x_1, \left(\frac{x_2}{x_1}\right)^2\right).$$
Thus, the transform $I^{R'}$ of I in $(R', \mathfrak{M}')$ is given by
$$I^{R'} = \left(x_1, \left(\frac{x_2}{x_1}\right)^2\right),$$
which is a complete $\mathfrak{M}'$-primary ideal or order one in the two-dimensional regular local ring $(R', \mathfrak{M}')$

In particular, this shows that $(R', \mathfrak{M}')$ is an immediate base point of I. In fact, it is the only immediate base point of I. To see this, note that since (x_1, x_3) is a minimal reduction of $\mathfrak{M}$, all immediate base points of I are lying on $R[\mathfrak{M}/x_1]$ or $R[\mathfrak{M}/x_3]$. Since $IR[\mathfrak{M}/x_3] = (x_3)R[\mathfrak{M}/x_3]$, I has no immediate base points on $R[\mathfrak{M}/x_3]$. Further,

$IR[\mathfrak{M}/x_1] = x_1(x_1, x_2, (x_3/x_1))R[\mathfrak{M}/x_1]$. Thus, the transform of I in $R[\mathfrak{M}/x_1]$ is contained in only one maximal ideal of $R[\mathfrak{M}/x_1]$, namely, $M = (x_1, (x_2/x_1), (x_3/x_1))$. Hence, $R' = R[\mathfrak{M}/x_1]_M$ is the *unique* immediate base point of I. This, together with the fact that $I^{R'}$ is an $\mathfrak{M}'$-primary ideal of order one, implies that I is quasi-one-fibered.

Next, we claim that I is complete. To see this, consider the following inclusion

$$I = (x_1^2, x_1x_2, x_3) \subseteq (x_1^2, x_2, x_3).$$

Since $(x_1^2, x_2, x_3) = x_1\mathfrak{M}' \cap R$, we have that (x_1^2, x_2, x_3) is complete. Hence,

$$I = (x_1^2, x_1x_2, x_3) \subseteq \overline{I} \subseteq (x_1^2, x_2, x_3).$$

Since $(x_1^2, x_1x_2, x_3) \subseteq (x_1^2, x_2, x_3)$ are adjacent ideals, we have $\overline{I} = I$ or $\overline{I} = (x_1^2, x_2, x_3)$. If follows that $\overline{I} = I$, for otherwise (x_1^2, x_1x_2, x_3) would be a reduction of (x_1^2, x_2, x_3) and thus the transform of (x_1^2, x_1x_2, x_3) in R' would also be a reduction of the transform of (x_1^2, x_2, x_3) in R' (that is, $\mathfrak{M}' = (x_1, (x_2/x_1))R'$), which is impossible.

So, $I = (x_1^2, x_1x_2, x_3)$ is a quasi-one-fibered complete $\mathfrak{M}$-primary ideal of order one in the two-dimensional Muhly local domain $(R, \mathfrak{M})$.

We now show that I cannot have the simple form $(x_1'^n, x_2', x_3')$, where x_1', x_2', x_3' denotes a minimal ideal basis of $\mathfrak{M}$ such that $x_1' \notin \operatorname{rad}(x_2', x_3')$. It therefore suffices to show that I does not satisfy condition (iv) of Proposition 3.7. To do so, we begin by observing that the characteristic ideal $\mathfrak{c}(I)$ of $I = (x_1^2, x_1x_2, x_3)$ is the homogeneous ideal (x_3^*) in $\operatorname{gr}_{\mathfrak{M}}(R) = k[X_1, X_2, X_3]/(X_2^2 + X_1X_3)$, where x_3^* denotes the leading form of x_3 in $\operatorname{gr}_{\mathfrak{M}}(R)$. Using the fact that $\operatorname{gr}_{\mathfrak{M}}(R) = k[X_1, X_2, X_3]/(X_2^2 + X_1X_3)$, it is easily seen that $P := (x_2^*, x_3^*)$ is the only homogeneous prime ideal of height one in $\operatorname{gr}_{\mathfrak{M}}(R)$ containing $\mathfrak{c}(I) = (x_3^*)$. So $\mathfrak{c}(I)$ cannot be a homogeneous prime ideal of height one in $\operatorname{gr}_{\mathfrak{M}}(R)$. Thus, I does not satisfy Proposition 3.7 (iv), which proves our claim.

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