ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 1, 2014

INVARIANT SUBSPACES AND KERNELS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. In this paper we have shown that if $\phi \in (L_h^2)^\perp \cap L^\infty$, $\phi \neq 0$, then $\ker T_\phi = \ker T_\phi^* = \mathrm{sp}\,\{1\}$ and therefore finite-dimensional subspaces of L_a^2. Further, if $\phi \in L^\infty(\mathbf{D})$, $\phi \neq 0$, then it is shown that the Toeplitz operator T_ϕ cannot be of finite rank.

1. Introduction. Let $L^2(\mathbf{D}, dA)$ denote the Hilbert space of complex-valued, absolutely square-integrable, Lebesgue measurable functions f on $\mathbf{D}$ with the inner product

$$\langle f, g\rangle = \int f(z)\overline{g(z)}\, dA(z),$$

where $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is the open unit disc in the complex plane $\mathbf{C}$ and $dA(z)$ is the area measure on $\mathbf{D}$ normalized so that the area of the disc $\mathbf{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} r\, dr\, d\theta.$$

Let $L^\infty(\mathbf{D}, dA)$ denote the Banach space of Lebesgue measurable functions f on $\mathbf{D}$ with

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbf{D}\} < \infty.$$

Let $L_a^2(\mathbf{D})$ (the subscript "a" stands for analytic) be the subspace of $L^2(\mathbf{D}, dA)$ consisting of analytic functions. The space $L_a^2(\mathbf{D})$ is called the Bergman space. Let $H^\infty(\mathbf{D})$ be the space of bounded analytic functions on $\mathbf{D}$. Let $\overline{L_a^2} = \{\overline{f} : f \in L_a^2\}$ and $L_h^2 = L_a^2 \oplus \overline{L_a^2}$. Since point evaluation at $z \in \mathbf{D}$ is a bounded linear functional on the Hilbert space

2010 AMS *Mathematics subject classification.* Primary 47B35.
Keywords and phrases. Toeplitz operators, Hankel operators, Bergman space, finite-rank operators, reproducing kernel.
Received by the editors on July 19, 2010, and in revised form on May 19, 2011.
DOI:10.1216/RMJ-2014-44-1-35

$L_a^2(\mathbf{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(\mathbf{D})$ such that

$$f(z) = \int_{\mathbf{D}} f(w)\overline{K_z(w)}\, dA(w)$$

for all f in $L_a^2(\mathbf{D})$. Let $K(z,w)$ be the function on $\mathbf{D} \times \mathbf{D}$ defined by

$$K(z,w) = \overline{K_z(w)}.$$

The function $K(z,w)$ is thus the reproducing kernel for the Bergman space $L_a^2(\mathbf{D})$ and is called the Bergman kernel. It can be shown that the sequence of functions $\{e_n(z)\} = \{\sqrt{n+1}z^n\}_{n\geq 0}$ forms the standard orthonormal basis for $L_a^2(\mathbf{D})$ and $K(z,w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}$. The Bergman kernel is independent of the choice of orthonormal basis and $K(z,w) = 1/(1-z\overline{w})^2$. Since $L_a^2(\mathbf{D})$ is a closed subspace of $L^2(\mathbf{D}, dA)$ (see [**15**]); there exists an orthogonal projection P from $L^2(\mathbf{D}, dA)$ onto $L_a^2(\mathbf{D})$. For $\phi \in L^\infty(\mathbf{D})$, we define the Toeplitz operator T_ϕ on $L_a^2(\mathbf{D})$ by $T_\phi f = P(\phi f)$, $f \in L_a^2(\mathbf{D})$. The big Hankel operator H_ϕ is a mapping from $L_a^2(\mathbf{D})$ into $(L_a^2(\mathbf{D}))^\perp$ defined by $H_\phi f = (I-P)(\phi f)$, $f \in L_a^2(\mathbf{D})$. The little Hankel operator h_ϕ is a mapping from $L_a^2(\mathbf{D})$ into $\overline{L_a^2(\mathbf{D})}$ defined by $h_\phi f = \overline{P}(\phi f)$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbf{D}, dA)$ onto $\overline{L_a^2(\mathbf{D})}$. There are also many equivalent ways of defining little Hankel operators on the Bergman space. For example, define $S_\phi : L_a^2 \to L_a^2$ as $S_\phi f = P(J(\phi f))$ where $J : L^2 \to L^2$ is such that $Jf(z) = f(\overline{z})$. Observe that, for $f \in L_a^2(\mathbf{D})$, $h_\phi f = \overline{P}(\phi f) = JPJ(\phi f) = JS_\phi f$. Thus, the operators h_ϕ and S_ϕ are unitarily equivalent. Hence, both the operators h_ϕ and S_ϕ are referred to as little Hankel operators in the literature. If P_0 is the rank one projection from $L^2(\mathbf{D}, dA)$ onto the constants, then $\overline{P} - P_0 \leq I - P$, where I is the identity operator. This is the reason why we call h_ϕ the little Hankel operator. The big Hankel operator is defined in terms of the bigger projection $I - P$.

Let $\mathbf{T}$ denote the unit circle in $\mathbf{C}$. Let $L^2(\mathbf{T})$ be the space of complex-valued, absolutely square integrable, Lebesgue measurable functions on $\mathbf{T}$. Let $H^2(\mathbf{T})$ be the corresponding Hardy space of functions on $\mathbf{T}$ with vanishing negative Fourier coefficients. For $\phi \in L^\infty(\mathbf{T})$, the space of essentially bounded measurable functions on $\mathbf{T}$, we define the Toeplitz operator L_ϕ from $H^2(\mathbf{T})$ into $H^2(\mathbf{T})$ as $L_\phi f = \widetilde{P}(\phi f)$, where $\widetilde{P}$ is the orthogonal projection from $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$. It is shown in [**3**] that there exists no compact Toeplitz operator on the Hardy space except the zero Toeplitz operator. In the Bergman space setting, however, there are lots of nontrivial compact Toeplitz operators [**1**], and there are unbounded symbols that induce bounded (even compact) Toeplitz operators. In this paper, we have shown that if $\phi \in (L^2_h)^\perp \cap L^\infty$, $\phi \neq 0$, then $\ker T_\phi = \ker T^*_\phi = \mathrm{sp}\,\{1\}$, and there exists no finite rank nonzero Toeplitz operator with bounded symbol on the Bergman space. Thus, it follows that, if $0 \neq \phi \in (L^2_h)^\perp \cap L^\infty$ and T_ϕ has closed range, then T_ϕ is Fredholm and of index zero but T_ϕ is not invertible.

The proof also extends to $L^2_a(\Omega)$ where Ω is a bounded symmetric domain in $\mathbf{C}$. Luecking [**11, 12**] also showed that there exists no finite rank Toeplitz operators on Bergman space in a setting where he took the Fourier transform of the corresponding measure. Our method is more elementary and explains the situation better. It was also pointed out as a conjecture in [**8**].

2. Intermediate Hankel operators. Hankel operators play an essential role in the theory of Toeplitz operators, and many problems about Toeplitz operators can also be formulated in terms of Hankel operators and vice versa. On the Hardy space of the disk there is essentially only one type of Hankel operator. However, in the Bergman space setting, there are two very different notions of Hankel operators, the so-called big and little Hankel operators. Little Hankel operators on the Bergman space behave more like Hankel operators on the Hardy space. One can also define the intermediate (middle) Hankel operators on the Bergman space. In the present paper, we show that the information about the Fredholmness of T_ϕ can be obtained using intermediate Hankel operators.

For $p \geq 0$, let

$$E_p = \overline{\mathrm{span}}\,\{|z|^{2k}\overline{z}^n,\ k = 0, \dots, p;\ n = 0, 1, 2, \dots\}.$$

For $p \geq 0$, the spaces E_p are closed subspaces of $L^2(\mathbf{D})$. For $\phi \in L^\infty(\mathbf{D})$, we define the intermediate Hankel operator $H^{E_p}_\phi : L^2_a \to E_p$ by $H^{E_p}_\phi(f) = P_p(\phi f)$, $f \in L^2_a$, where P_p is the orthogonal projection from $L^2(\mathbf{D})$ onto E_p. Notice that $\overline{L^2_a} \subseteq E_p \subseteq ((L^2_a)_0)^\perp$ where

$(L^2_a)_0 = \{g \in L^2_a : g(0) = 0\}$. For $n > m$ and $j \in \{0, \dots, p\}$, let

$$A_j^{n,m} = \frac{\prod_{1\le l\le p+1}(n-m+l+j)}{\prod_{1\le l\le p+1}(n+l)} \frac{1}{j!(p-j)!(-1)^{p-j}} \prod_{\substack{0\le l\le p\\ l\ne j}} (m-l).$$

It is not so difficult to check that

$$P_p(\overline{z}^n z^m) = \begin{cases} 0 & \text{if } n < m; \\ \overline{z}^n z^m & \text{if } n \ge m,\ 0 \le m \le p; \\ A_0^{n,m}\overline{z}^{n-m} + A_1^{n,m}\overline{z}^{n-m+1}z + \\ \qquad \cdots + A_p^{n,m}\overline{z}^{n-m+p}z^p & \text{if } n \ge m,\ m > p. \end{cases}$$

Observe that the operator $H_\phi^{E_p} \equiv 0$ if and only if $\phi \in E_p^\perp$ and $(H_\phi^{E_p})^* f = P(\overline{\phi} f)$ for all $f \in E_p$ where P is the Bergman projection. Further, notice that $\ker H_\phi^{E_p} \subset L^2_a$ is invariant under multiplication by z and $\ker H_\phi^{E_p}$ has finite codimension if $H_\phi^{E_p}$ is of finite rank. In this section, we describe the interplay between the kernels of little Hankel operators [**15**] and intermediate Hankel operators to establish the results of the paper.

Lemma 1. *Let $\phi \in L^\infty(\mathbf{D})$. Then $\ker h_\phi = \{0\}$ if and only if $\ker h_\phi^* = \{0\}$. That is, $\ker h_\phi = \{0\}$ if and only if $\overline{\operatorname{Range} h_\phi} = \overline{L^2_a}$.*

Proof. Notice that $S_\phi^* = S_{\phi^+}$ where $\phi^+(z) = \overline{\phi(\overline{z})}$. It is not difficult to see that $f \in \ker S_\phi$ if and only if $f^+ \in \ker S_{\phi^+}$. Thus, if $\ker S_\phi = \{0\}$, then $\ker S_\phi^* = \{0\}$. Hence, $\overline{\operatorname{Range} S_\phi} = L^2_a$. Conversely, if $\overline{\operatorname{Range} S_\phi} = L^2_a$, then $\ker S_\phi^* = \{0\}$, and hence $\ker S_\phi = \{0\}$.

If $f \in \ker h_\phi$, then $h_\phi f = 0$. Hence, $JS_\phi f = 0$, and therefore $S_\phi f = 0$. This implies $f^+ \in \ker S_{\phi^+}$. Hence, $Jf^+ \in \ker S_{\phi^+} J = \ker h_\phi^*$ as $S_{\phi^+} J = (JS_\phi)^* = h_\phi^*$. Now suppose $\overline{g} \in \overline{L^2_a}$ and $\overline{g} \in \ker h_\phi^*$. This implies $J\overline{g} \in \ker S_\phi^*$. Hence, $(J\overline{g})^+ \in \ker S_\phi$. Therefore, $(J\overline{g})^+ \in \ker h_\phi$. That is, $g \in \ker h_\phi$. Thus, if $f \in L^2_a$, then $f \in \ker h_\phi$ if and only if $\overline{f} \in \ker h_\phi^*$.

Hence, $\ker h_\phi = \{0\}$ if and only if $\ker h^*_\phi = \{0\}$, and this is true if and only if $\overline{\text{Range}\, h_\phi} = \overline{L^2_a}$. ◻

Lemma 2. *Suppose $f \in L^2_a$ is not a polynomial and $H \in L^2_a$. Then $\ker H^{*E_p}_{\overline{f}H} = \{0\}$ if and only if $\ker H^{E_p}_{\overline{f}H} = \{0\}$. That is, $\ker H^{E_p}_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$ if and only if $\overline{\text{Range}\, H^{E_p}_{\overline{\mathbf{f}}\mathbf{H}}} = E_p$.*

Proof. Notice that

$$\begin{aligned}\ker H^{*E_p}_{\overline{f}H} &= \{g \in E_p : P(f\overline{H}g) = 0\}\\ &\supset \ker h^*_{\overline{f}H}\\ &= \{g \in \overline{L^2_a} : P(f\overline{H}g) = 0\}.\end{aligned}$$

If $\ker H^{*E_p}_{\overline{f}H} = \{0\}$, then $\ker h^*_{\overline{f}H} = \{0\}$. Hence, $\ker h_{\overline{f}H} = \{0\}$. Since, for $h \in L^2_a$, $H^{E_p}_{\overline{f}H}h = h_{\overline{f}H}h + P_{E_p \ominus (\overline{L^2_a})}(\overline{f}Hh)$, we obtain $\ker H^{E_p}_{\overline{f}H} = \{0\}$.

Now, suppose $\ker H^{E_p}_{\overline{f}H} = \{0\}$. This implies $\ker h_{\overline{f}H} = \{0\}$. Because, if $\ker h_{\overline{f}H} \neq \{0\}$, then [**6, 7, 9**] there exists an inner function $G \in L^2_a$ such that $G \in \ker h_{\overline{f}H}$. That is, $h_{\overline{f}HG} \equiv 0$. This is so as $\ker h_{\overline{f}H} = \ker S_{\overline{f}H}$ is an invariant subspace of z. Observe that, for $\psi \in L^\infty(\mathbf{D})$, $h_\psi T_z = h_{\psi z}$ and $(T_{\overline{\mathbf{z}}} h_\psi)^* = h^*_\psi T_z = S_{\psi^+} J T_z$. Further, for $g \in L^2_a$, $S_{\psi^+} J T_z g = S_{\psi^+}(J(zg)) = P(J(\psi^+ \overline{\mathbf{z}} Jg)) = P(\overline{\boldsymbol{\psi}} zg) = h^*_{\psi\overline{\mathbf{z}}} g$.

Similarly, one can show that $(T_{\overline{\mathbf{z}^\mathbf{k}}} h_\psi)^* = h^*_{\psi\overline{\mathbf{z}^\mathbf{k}}}$ for all $k = 0, 1, \ldots, p$. Thus, $h_{\overline{f}HG} \equiv 0$ implies $(T_{\overline{\mathbf{z}}^k} h_{\overline{\mathbf{f}}HG})^* \equiv 0$. Hence, $h^*_{\overline{\mathbf{f}}HG\overline{z}^k} \equiv 0$. This implies $h_{\overline{\mathbf{f}}HG\overline{z}^k} \equiv 0$ for all $k = 0, 1, \ldots, p$. Hence, $\overline{\mathbf{f}}\mathbf{HG}\overline{\mathbf{z}^\mathbf{k}} \in (\overline{L^2_a})^\perp$. That is, $\langle \overline{\mathbf{f}}\mathbf{HG}\overline{\mathbf{z}^\mathbf{k}}, \overline{\mathbf{z}^\mathbf{k}\mathbf{g}}\rangle = 0$ for all $g \in L^2_a$ and $k = 0, 1, \ldots, p$. Thus, $\overline{\mathbf{f}}\mathbf{HG} \in E_p^\perp$. Hence $\ker H^{E_p}_{\overline{\mathbf{f}}\mathbf{H}} \neq \{0\}$. Thus, $\ker H^{E_p}_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$ implies $\ker h_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$. Hence, $\ker h^*_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$. This implies $\ker H^{*E_p}_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$. Because, if $\ker H^{*E_p}_{\overline{\mathbf{f}}\mathbf{H}} \neq \{0\}$, then there exists $0 \neq g \in E_p \cap L^\infty$ such that $\mathbf{f}\overline{\mathbf{H}}\mathbf{g} \in (L^2_a)^\perp$. That is, $\overline{\mathbf{f}}\mathbf{H}\overline{\mathbf{g}} \in (\overline{L^2_a})^\perp$. This implies $h_{\overline{\mathbf{f}}\mathbf{H}\overline{\mathbf{g}}} \equiv 0$, and therefore $\ker h_{\overline{\mathbf{f}}\mathbf{H}} \supseteq \overline{g}L^2_a \cap L^2_a \neq \{0\}$, $\overline{\mathbf{g}} \in \overline{E_p} \cap L^\infty = H^\infty$. Thus, $\ker H^{E_p}_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$ if and only if $\ker H^{*E_p}_{\overline{\mathbf{f}}\mathbf{H}} = \{0\}$. ◻

3. Common zero sets. If N is a subspace of $L_a^2(\mathbf{D})$, let $\mathcal{Z}(N) = \{z \in \mathbf{D} : f(z) = 0 \text{ for all } f \in N\}$, which is called the zero set of functions in N. Here, if z_1 is a zero of multiplicity at most n of all functions in N, then z_1 appears n times in the set $\mathcal{Z}(N)$, and they are treated as distinct elements of $\mathcal{Z}(N)$.

In this section, we relate the concept of common zero set and the rank of a Toeplitz operator. We have shown that, if $\phi \in L^\infty(\mathbf{D})$ is such that T_ϕ is a finite rank Toeplitz operator and $\operatorname{Card}\mathcal{Z}(\ker T_\phi^*) = \text{Rank of}\, T_\phi$, then $\phi \equiv 0$. With the following result, we begin to link the ideas of subspaces and zero-sets.

Proposition 3. *If N is a subspace of $L_a^2(\mathbf{D})$ of finite codimension in $L_a^2(\mathbf{D})$, then*

$$\mathcal{Z}(N) = \{z \in \mathbf{D} : f(z) = 0 \quad \text{for all } f \in N\}$$

is a finite set.

Proof. Suppose $\mathcal{Z}(N)$ is an infinite set. Let $\{z_j\}_{j=1}^\infty$ be distinct points of $\mathcal{Z}(N)$, and let $f_1, f_2, f_3, \dots, f_n$ be functions in $L_a^2(\mathbf{D})$ such that

$$f_i(z_1) = \cdots = f_i(z_{i-1}) = 0, f_i(z_i) = 1 \quad \text{for all } i \geq 2.$$

For example, we could take the functions (f_i) to be polynomials. Then $f_1, f_2, \dots$ are linearly independent modulo N, i.e., if

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n \in N$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{C}$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This contradicts the assumption that N has finite codimension in $L_a^2(\mathbf{D})$. Since each zero of an analytic function has finite multiplicity, the result is proved. □

Let k_z be the normalized reproducing kernel for the Bergman space $L_a^2(\mathbf{D})$. When $|z| \to 1$, $k_z \to 0$ weakly and the normalized reproducing kernels k_z, $z \in \mathbf{D}$ span $L_a^2(\mathbf{D})$ [**15**].

Theorem 4. *If $\phi \in L^\infty(\mathbf{D})$ is such that T_ϕ is a finite rank Toeplitz operator and* $\operatorname{Card}\mathcal{Z}(\ker T_\phi^*) = \text{Rank of}\, T_\phi$, *then $\phi \equiv 0$.*

Proof. Suppose T_ϕ is of finite rank. Then $\operatorname{Range} T_\phi$ is finite dimensional and is a closed subspace of L_a^2. Let $n = \dim \operatorname{Range} T_\phi$. Thus, $\ker T_\phi^* = (\overline{\operatorname{Range} T_\phi})^\perp$ has finite codimension. Therefore, by Proposition 3, $\mathcal{Z}(\ker T_\phi^*)$, the common zero set of $\ker T_\phi^*$ is a finite set. Without loss of generality, we shall assume the elements of $\mathcal{Z}(\ker T_\phi^*)$ are distinct. Suppose $\mathcal{Z}(\ker T_\phi^*) = \{a_1, a_2, \dots, a_n\}$. Here $a_1, a_2, \dots, a_n$ are all distinct. Then $\ker T_\phi^* \subset \{f \in L_a^2 : f(a_i) = 0, i = 1, 2, \dots, n\}$. But $\{f \in L_a^2 : f(a_i) = 0, i = 1, 2, \dots, n\} = \{f \in L_a^2 : \langle f, k_{a_i}\rangle = 0, i = 1, 2, \dots, n\} = \{k_{a_1}, k_{a_2}, \dots, k_{a_n}\}^\perp$. Thus, $\operatorname{sp}\{k_{a_1}, k_{a_2}, \dots, k_{a_n}\} \subset (\ker T_\phi^*)^\perp = \operatorname{Range} T_\phi$. In the case of repeated zeros (if a is a zero of order m, say) the derivatives of the corresponding reproducing kernel up to order $m-1$, i.e., $\mathbf{k_a}, (\partial/\partial\overline{\mathbf{a}})\mathbf{k_a}, \dots, (\partial^{\mathbf{m-1}})/(\partial\overline{\mathbf{a}}^{\mathbf{m-1}})\mathbf{k_a}$ are included in the spanned set [**9**]. Now, since $\operatorname{Range} T_\phi$ has dimension n and $k_{a_1}, k_{a_2}, \dots, k_{a_n}$ are linearly independent, therefore $\operatorname{Range} T_\phi = \operatorname{sp}\{k_{a_1}, k_{a_2}, \dots, k_{a_n}\}$. Thus, $\ker T_\phi^* = \{k_{a_1}, k_{a_2}, \dots, k_{a_n}\}^\perp = \{f \in L_a^2 : f(a_i) = 0, i = 1, 2, \dots, n\}$ is an invariant subspace of the Bergman shift operator T_z defined on $L_a^2(\mathbf{D})$. Since T_ϕ is of finite rank implies T_ϕ^* is of finite rank, therefore one can show that $\ker T_\phi$ is also an invariant subspace of $L_a^2(\mathbf{D})$. Let P be the set of all polynomials in $L_a^2(\mathbf{D})$ and $M = \ker T_\phi$. Since $\operatorname{Range} T_\phi$ has dimension n, therefore $T_\phi 1, T_\phi z, \dots, T_\phi z^n$ are linearly dependent. This implies that there exists a non-zero polynomial p of degree at most n such that $T_\phi p = 0$. That is, $\phi p \in (L_a^2)^\perp$. Using the facts that codimension of M is finite, and $T_z M \subset M$, it follows that $P \cap M$ is a nontrivial ideal of P. Since P is a principal ideal ring, there exists a $q \in P$ such that $P \cap M = qP$, see [**10**]. Thus, $T_{\phi q} g = 0$ for all polynomials $g \in P$. This implies $T_{\phi q} z^k = 0$ for all $k \geq 0$. That is, $\phi q \in (\overline{z}^k L_a^2)^\perp$ for all $k \geq 0$. Hence, $\phi q \in \cap_{k\geq 0}(\overline{z}^k L_a^2)^\perp = (\cup_{k\geq 0}\overline{z}^k L_a^2)^\perp$. Therefore, it follows that $\phi q \perp \overline{z}^k z^n$ for all $k, n \geq 0$. Now, $\phi q \in L^\infty \subset L^2$ implies that $\phi q = 0$. Thus, $\phi = 0$, except at the zeros of q which is a polynomial of degree at most n. Hence, $\phi \equiv 0$ as $\phi \in L^\infty(\mathbf{D})$. □

For z and w in $\mathbf{D}$, let $\phi_z(w) = (z - w)/(1 - \overline{z}w)$. These are involutive Mobius transformations on $\mathbf{D}$. In fact,

(1) $\phi_z \circ \phi_z(w) \equiv w$;

(2) $\phi_z(0) = z$, $\phi_z(z) = 0$;

(3) ϕ_z has a unique fixed point in $\mathbf{D}$.

Given $z \in \mathbf{D}$ and f any measurable function on $\mathbf{D}$, we define a function $U_z f(w) = k_z(w) f(\phi_z(w))$. Since $|k_z|^2$ is the real Jacobian determinant of the mapping ϕ_z (see [**15**]), U_z is easily seen to be a unitary operator on $L^2(\mathbf{D}, dA)$ and $L^2_a(\mathbf{D})$. It is also easy to check that $U_z^* = U_z$; thus, U_z is a self-adjoint unitary operator. If $\phi \in L^\infty(\mathbf{D}, dA)$ and $z \in \mathbf{D}$, then $U_z T_\phi = T_{\phi\circ\phi_z} U_z$. This is because $PU_z = U_z P$ and, for $f \in L^2_a$, $T_{\phi\circ\phi_z} U_z f = T_{\phi\circ\phi_z}((f\circ\phi_z)k_z) = P((\phi\circ\phi_z)(f\circ\phi_z)k_z) = P(U_z(\phi f)) = U_z P(\phi f) = U_z T_\phi f$. Let $\mathrm{Aut}\,(\mathbf{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbf{D}$ and G_0 the isotropy subgroup at 0; i.e., $G_0 = \{\Psi \in \mathrm{Aut}\,(\mathbf{D}) : \Psi(0) = 0\}$. Notice also that $\phi_a(z)$, as a function in a, is one-one and onto for any fixed z in $\mathbf{D}$.

Proposition 5. *If $\phi \in L^\infty(\mathbf{D})$, then T_ϕ is of finite rank if and only if $T_{\phi\circ\phi_z}$ is of finite rank. In this case* Rank of T_ϕ = Rank of $T_{\phi\circ\phi_z}$.

Proof. Note that $f \in \ker T_\phi$ if and only if $U_z f \in \ker T_{\phi\circ\phi_z}$. Further since U_z is unitary, $\dim \ker T^*_\phi = \dim \ker T^*_{\phi\circ\phi_z}$. Thus $\dim \mathrm{Range}\, T_\phi = \mathrm{codim}\, \ker T^*_\phi = \mathrm{codim}\, \ker T^*_{\phi\circ\phi_z} = \dim \mathrm{Range}\, T_{\phi\circ\phi_z}$. □

For $\phi \in L^\infty(\mathbf{D})$, let $\widetilde{\phi}(z) = \langle T_\phi k_z, k_z\rangle$ be the Berezin transform of ϕ. It is easy to check that $H^*_\phi H_\phi = T_{|\phi|^2} - T_{\overline{\phi}} T_\phi$ (see [**15**]). The following also holds. □

Proposition 6. *For $\phi \in L^\infty(\mathbf{D})$,*

$$\mathrm{MO}\,(\phi)^2(z) = \widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2 \le \|H_\phi k_z\|^2 + \|H_{\bar{\phi}} k_z\|^2.$$

Proof. It is easy to observe that

$$\begin{aligned} \|H_\phi k_z\| &= \|(I-P)(\phi k_z)\| = \|(I-P)U_z(\phi\circ\phi_z)\| \\ &= \|U_z(I-P)(\phi\circ\phi_z)\| = \|(I-P)(\phi\circ\phi_z)\| \\ &= \|\phi\circ\phi_z - P(\phi\circ\phi_z)\|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|H_{\overline{\phi}} k_z\| &= \|\overline{\phi}\circ\phi_z - P(\overline{\phi}\circ\phi_z)\| \\ &= \|\phi\circ\phi_z - \overline{P(\overline{\phi}\circ\phi_z)}\|. \end{aligned}$$

Since $\widetilde{\phi}(z) = P(\phi \circ \phi_z)(0)$ and $P\overline{g}(z) = \overline{g}(0)$ for any $g \in L^2_a$ and all $z \in \mathbf{D}$, we have

$$\begin{aligned}
\mathrm{MO}\,(\phi)^2(z) &= \widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2 \\
&= \|\phi \circ \phi_z - P(\phi \circ \phi_z)(0)\|^2 \\
&= \|\phi \circ \phi_z - P(\phi \circ \phi_z)\|^2 + \|P(\phi \circ \phi_z) - P(\phi \circ \phi_z)(0)\|^2 \\
&= \|H_\phi k_z\|^2 + \|P(\phi \circ \phi_z) - \overline{P(\overline{\phi} \circ \phi_z)(0)}\|^2 \\
&= \|H_\phi k_z\|^2 + \|P(\phi \circ \phi_z - \overline{P(\overline{\phi} \circ \phi_z)})\|^2 \\
&\le \|H_\phi k_z\|^2 + \|\phi \circ \phi_z - \overline{P(\overline{\phi} \circ \phi_z)}\|^2 \\
&= \|H_\phi k_z\|^2 + \|H_{\bar{\phi}} k_z\|^2. \qquad \square
\end{aligned}$$

Proposition 7. *If $T \in \mathcal{L}(L^2_a(\mathbf{D}))$ and $\langle Tk_z, k_z\rangle = 0$ for all $z \in \mathbf{D}$, then $T \equiv 0$.*

Proof. Define $\sigma(T)(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbf{D}$. If $\sigma(T) = 0$ identically, then also $\langle TK_z, K_z\rangle = K(z,z)\sigma(T)(z) = 0$ identically where $K_z = K(\cdot, z)$ is the nonnormalized reproducing kernel. Thus, the function $F(x,y) = \langle TK_{\overline{x}}, K_y\rangle$, which is holomorphic in x and y, vanishes on the "anti-diagonal" $x = \overline{y}$. Passing to the variables u, v defined by $x = u + iv$ and $y = u - iv$, we get a holomorphic function $G(u,v)$ of u, v, which vanishes when u, v are real. Thus, $F(x,y) = G(u,v) \equiv 0$. Thus, even $\langle TK_x, K_y\rangle = 0$ for any x, y. Since linear combinations of K_x, $x \in \mathbf{D}$ are dense in $\mathcal{L}(L^2_a)$, it follows that $T \equiv 0$. $\square$

Notice that if, for all $z \in \mathbf{D}$, $P(\phi \circ \phi_z) = P(\phi \circ \phi_z)(0) = 0$, then $U_z T_\phi k_z = U_z T_\phi U_z 1 = T_{\phi \circ \phi_z} 1 = 0$. Hence, $T_\phi k_z = 0$ for all $z \in \mathbf{D}$, and therefore, $\langle T_\phi k_z, k_z\rangle = 0$ for all $z \in \mathbf{D}$. By Proposition 7, $T_\phi \equiv 0$, and thus $\phi \equiv 0$.

Lemma 8. *For any z and w in $\mathbf{D}$, there exists a unitary $U \in G_0$ such that $\phi_w \circ \phi_z = U\phi_{\phi_z(w)}$.*

Proof. Let $U = \phi_w \circ \phi_z \circ \phi_{\phi_z(w)}$. Then $U(0) = \phi_w \circ \phi_z(\phi_z(w)) = \phi_w(w) = 0$; thus, $U \in G_0$. $\square$

4. Kernel of Toeplitz operators. In this section we describe the kernel of a Toeplitz operator on the Bergman space. We have shown that if T_ϕ is a nonzero Toeplitz operator and if constants belong to both $\ker T_\phi$ and $\ker T_\phi^*$, then these kernels contain only constants and are not invariant subspaces of the Bergman shift operator T_z. To establish this, we need to study the properties of Hankel operators defined on the harmonic Bergman space L_h^2 as follows.

For $\psi \in L^\infty(\mathbf{D})$, define $B_{\overline{\boldsymbol{\psi}}} : L_h^2 \to (L_h^2)^\perp$ as $B_{\overline{\boldsymbol{\psi}}} f = (I-Q)(\overline{\boldsymbol{\psi}} f)$ where Q is the orthogonal projection from L^2 onto L_h^2. The operator $B_{\overline{\boldsymbol{\psi}}}$ is well defined, and it is easy to see that

$$Q(\overline{z}^n z^k) = \begin{cases} (n-k+1)/(n+1)\overline{z}^{n-k} & \text{if } k \le n; \\ (k-n+1)/(k+1) z^{k-n} & \text{if } k \ge n. \end{cases}$$

Further, one can verify that $B_{\overline{z}^n}(z^k) \perp B_{\overline{z}^n}(z^j)$ if $j \neq k$ and if $\phi = \sum_{k=s_0}^\infty a_k z^k, a_{s_0} \neq 0$ (that is, $a_0 = \cdots = a_{s_0-1} = 0$) then $B_\phi(\overline{z}^n) = \sum_{k=s_0}^\infty a_k B_{\overline{z}^n}(z^k)$. It is shown in [**14**] that if Ψ and Ω are two functions in L_a^2 such that $\Psi(0) = 0 = \Omega(0)$, then the operators B_Ψ and $B_{\overline{\Omega}}$ are not of finite rank in L_h^2. In fact, the set $\{B_\Psi(\overline{z}^n)\}_{n=1}^p$ is linearly independent for all $p > 0$, and the set $\{B_{\overline{\boldsymbol{\Omega}}}(z^k)\}_{k=1}^p$ is linearly independent for all $p > 0$. If $g \in L_h^2$ and $g = \Psi + \overline{\boldsymbol{\Omega}}$, where Ψ and Ω are from L_a^2, then $B_{\overline{\boldsymbol{\Omega}}} = B_g \mid_{L_a^2}$ is of finite rank if and only if $\Omega \equiv 0$ and similarly $B_g|_{\overline{L_a^2}} = B_\Psi$ is of finite rank if and only if $\Psi \equiv 0$. Notice that for $f \in L_a^2$, $B_{\overline{f}} L_h^2 = B_{\overline{f}} L_a^2$, and we shall also write $B_{\bar{f}}|_{L_a^2} = B_{\overline{f}}$. Thus, $B_{\overline{f}} : L_a^2 \to (L_h^2)^\perp$ is defined as $B_{\overline{f}} h = (I-Q)(\overline{f} h)$ for all $h \in L_a^2$ and $\ker B_{\overline{f}} = \{h \in L_a^2 : \overline{f} h \in L_h^2\}$.

From [**2**], it follows that if f is not constant, $\ker B_{\overline{f}} = \operatorname{sp}\{1\}$. If $f \equiv$ constant, then $\ker B_{\overline{\mathbf{f}}} = L_a^2$, hence $B_{\overline{\mathbf{f}}} \equiv 0$. Now $B_{\overline{\mathbf{f}}}^*$ maps $(L_h^2)^\perp$ into L_a^2 and $B_{\overline{\mathbf{f}}}^* k = P(fk)$ for all $k \in (L_h^2)^\perp$. It therefore follows that, for $f \in L_a^2$,

$$\overline{\operatorname{Range} B_{\overline{f}}^*} = \begin{cases} \{0\} & \text{if } f \equiv \text{constant}; \\ (\operatorname{sp}\{1\})^\perp & \text{if } f \neq \text{constant}. \end{cases}$$

Theorem 9. *If $\phi \in L^\infty(\mathbf{D})$ is such that $T_\phi 1 = T_\phi^* 1 = 0$, then either $\phi \equiv 0$ or $\ker T_\phi^* = \operatorname{sp}\{1\} = \ker T_\phi$. That is, either $\phi \equiv 0$ or T_ϕ is not of finite rank.*

Proof. Given that $T_\phi 1 = T^*_\phi 1 = 0$, hence $\phi \in (L^2_h)^\perp$. If $\psi \in L^\infty(\mathbf{D})$, then it is not difficult to verify that $S_{\overline{\boldsymbol{\psi}}} \equiv 0$ if and only if $\overline{\boldsymbol{\psi}} \in (\overline{L^2_a})^\perp$. Thus $S_{\overline{\boldsymbol{\phi}}} \equiv 0$ and $S_{\overline{\boldsymbol{\phi}}} f = 0$ for all $f \in L^2_a$. Hence, $(\overline{\boldsymbol{\phi}} f) \in (\overline{L^2_a})^\perp$ for all $f \in L^2_a$ and $\ker T^*_\phi = \{f \in L^2_a : (\overline{\boldsymbol{\phi}} f) \in (L^2_a)^\perp\} = \{f \in L^2_a : (\overline{\boldsymbol{\phi}} f) \in (L^2_h)^\perp\}$. Thus,

$$\begin{aligned}
\ker T^*_\phi &= \{f \in L^2_a : \overline{\phi} f \in (L^2_h)^\perp\} \\
&= \{f \in L^2_a : \langle \overline{\phi} f, g\rangle = 0 \quad \text{for all } g \in L^2_h\} \\
&= \{f \in L^2_a : \langle \overline{\phi}, \overline{f} g\rangle = 0 \quad \text{for all } g \in L^2_h\} \\
&= \{f \in L^2_a : \langle \overline{\phi}, (I-Q)(\overline{f} g)\rangle = 0 \quad \text{for all } g \in L^2_h\} \\
&= \{f \in L^2_a : \langle \overline{\phi}, B_{\overline{f}} g\rangle = 0 \quad \text{for all } g \in L^2_h\}.
\end{aligned}$$

Case 1. Let $f \in L^2_a(\mathbf{D})$ be a polynomial of degree $k \geq 1$ and $H(z) = z^{k+1} \in L^2_a$. Then $\overline{f} H \in E_p^\perp$ for all $p \geq 0$ and $H^{E_p}_{\overline{f}H} \equiv 0$. Since $\overline{f} z^k \notin E_p^\perp$ and $\ker H^{E_p}_{\overline{f}}$ is an invariant subspace of z, hence $\ker H^{E_p}_{\overline{f}} = z^{k+1} L^2_a$. Therefore, $H^{*^{E_p}}_{\overline{f}H} \equiv 0$ and $\ker H^{*^{E_p}}_{\overline{f}} = \overline{z}^{k+1} E_p$ for all $p \geq 0$.

Now $\ker B^*_{\overline{f}} = \{g \in (L^2_h)^\perp : P(fg) = 0\} = \{g \in (L^2_h)^\perp : fg \in (L^2_a)^\perp\}$ and $\ker H^{*^{E_p}}_{\overline{f}} = \{g \in E_p : fg \in (L^2_a)^\perp\}$. Hence, $\ker B^*_{\overline{f}} \bigcap E_p = \ker H^{*^{E_p}}_{\overline{f}} \bigcap (L^2_h)^\perp$, and therefore,

$$\begin{aligned}
\ker B^*_{\overline{f}} &= \bigcup_{p \geq 0} \left(\ker B^*_{\overline{f}} \bigcap E_p\right) = \bigcup_{p \geq 0} \left(\ker H^{*^{E_p}}_{\overline{f}} \bigcap (L^2_h)^\perp\right) \\
&= \left(\bigcup_{p \geq 0} \ker H^{*^{E_p}}_{\overline{f}}\right) \bigcap (L^2_h)^\perp = \left(\bigcup_{p \geq 0} \overline{z}^{k+1} E_p\right) \bigcap (L^2_h)^\perp \\
&= \{0\}.
\end{aligned}$$

Thus, if $\overline{\phi} \in (L^2_h)^\perp$ and $\overline{\phi} \neq 0$, then $B^*_{\overline{f}} \overline{\phi} \neq 0$.

Case 2. If $f \in L^2_a(\mathbf{D})$ is a constant, then $B_{\overline{f}} \equiv 0$ and hence $B^*_{\overline{f}} \equiv 0$ and therefore $B^*_{\overline{f}} \overline{\phi} = 0$ if $\overline{\phi} \neq 0$.

Case 3. If $f \in L_a^2(\mathbf{D})$ is not a polynomial, then $\overline{f}g \notin E_p^{\perp}$ for any $g \in L_a^2$, $g \neq 0$, $p \geq 0$. Hence, $\ker H_{\overline{f}}^{E_p} = \{0\}$ and therefore, by Lemma 2, we obtain $\ker H_{\overline{f}}^{*E_p} = \{0\}$ for all $p \geq 0$. This implies $\ker B_{\overline{f}}^{*} = (\bigcup_{p\geq 0} \ker H_{\overline{f}}^{*E_p}) \bigcap (L_h^2)^{\perp} = \{0\}$ and $B_{\overline{f}}^{*}\overline{\phi} \neq 0$ if $\overline{\phi} \in (L_h^2)^{\perp}$ and $\overline{\phi} \neq 0$.

Thus, from the above three cases, it follows that

$$\begin{aligned}\ker T_{\phi}^{*} &= \{f \in L_a^2 : \langle \overline{\phi}, B_{\overline{f}} g\rangle = 0 \quad \text{for all } g \in L_h^2\}\\ &= \{f \in L_a^2 : B_{\overline{f}}^{*}\overline{\phi} = 0\}\\ &= \begin{cases} L_a^2 & \text{if } \phi \equiv 0;\\ \operatorname{sp}\{1\} & \text{if } \phi \neq 0.\end{cases}\end{aligned}$$

Hence, either $\phi \equiv 0$ or T_{ϕ} is not of finite rank. □

Notice that if $\phi \in (L_h^2)_0^{\perp}$, then $P\phi \equiv (P\phi)(0) \equiv b$, a constant. We have the following corollary.

Corollary 10. *If $\phi \in (L_h^2)_0^{\perp} \cap L^{\infty}$, then either $\phi \equiv (P\phi)(0)$ or $\ker T_{\phi-(P\phi)(0)} = \ker T_{\phi-(P\phi)(0)}^{*} = \operatorname{sp}\{1\}$. That is, either $\phi \equiv (P\phi)(0)$ or $T_{\phi-(P\phi)(0)}$ is not of finite rank.*

Proof. Notice that $\phi - (P\phi)(0)$ and $\overline{\phi} - \overline{(P\phi)(0)}$ belong to $(L_h^2)^{\perp}$. By Theorem 9, we have

$$\ker T_{\phi-(P\phi)(0)} = \ker T_{\phi-(P\phi)(0)}^{*} = \begin{cases} L_a^2 & \text{if } \phi \equiv (P\phi)(0);\\ \operatorname{sp}\{1\} & \text{if } \phi \neq (P\phi)(0).\end{cases} \qquad \square$$

If H is a Hilbert space, let $\mathcal{L}(H)$ be the C^* algebra of all bounded linear operators from H into itself. The operator $T \in \mathcal{L}(H)$ is said to be Fredholm if and only if T has closed range, dimensions of kernel T and kernel T^* are finite. The Fredholm index of T is denoted by $J(T)$, and is defined by $J(T) = \dim \ker T - \dim \ker T^*$.

In the case where T is invertible, it is obvious that it is Fredholm and its index is zero. But there are Fredholm operators of index zero that are not invertible. Let $L^2_{\mathbf{C}^n}(\mathbf{T})$ denote the Hilbert space of $\mathbf{C}^n$-valued, norm square integrable, measurable functions on the unit circle $\mathbf{T}$ and $\mathcal{H}^2_{\mathbf{C}^n}(\mathbf{T})$ the corresponding Hardy space of functions in $L^2_{\mathbf{C}^n}(\mathbf{T})$ with vanishing negative Fourier coefficients. If $\phi \in L^\infty_{M_n}(\mathbf{T}) = L^\infty(\mathbf{T}) \otimes M_n$ (where M_n is the algebra of $n \times n$ matrices with complex entries), then B_ϕ denotes the Toeplitz operator defined on $\mathcal{H}^2_{\mathbf{C}^n}(\mathbf{T})$ by $B_\phi f = Q(\phi f)$ for f in $\mathcal{H}^2_{\mathbf{C}^n}(\mathbf{T})$ where Q is the orthogonal projection of $L^2_{\mathbf{C}^n}(\mathbf{T})$ onto $\mathcal{H}^2_{\mathbf{C}^n}(\mathbf{T})$. $\square$

In $\mathcal{H}^2_{\mathbf{C}^n}(\mathbf{T})$, for $n = 2$, let ϕ be defined by

$$\phi = \begin{pmatrix} \chi_m & 0 \\ 0 & \chi_{-m} \end{pmatrix}$$

where $\chi_m(e^{i\theta}) = e^{im\theta}$ and $\chi_{-m}(e^{i\theta}) = e^{-im\theta}$.

Then $\det \phi = 1$ and $\dim \ker B_\phi = m$ for $m \in \mathbf{Z}_+$ as $\ker B_\phi$ contains

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \chi_{m-1} \end{pmatrix}.$$

Thus, B_ϕ is Fredholm [**5**] and of index zero but B_ϕ is not invertible. But for $n = 1$, Coburn [**4**] proved the following. He showed that for ϕ in $L^\infty(\mathbf{T})$, the subspace $\ker B_\phi$ and $\ker B^*_\phi$ cannot both be different from zero. As a corollary to this result, he then showed that if ϕ is in $L^\infty(\mathbf{T})$ such that B_ϕ is a Fredholm operator and of index zero then B_ϕ is invertible. Thus, for $n = 1$, on the Hardy space a nontrivial Toeplitz operator cannot have both a nontrivial kernel and a nontrivial cokernel. But the situation in the Bergman space is rather different. The following examples give some insight into it.

(i) Let $\phi(z) = \log 2 - 1/(1 + |z|^2)$. Notice that $\phi \in C(\overline{\mathbf{D}})$, and $\sigma_e(T_\phi) = \log 2 - 1/2$. Therefore, T_ϕ is Fredholm, and since ϕ is real valued, $J(T_\phi) = 0$. It is not so difficult to verify that $1 \in \ker T_\phi$, and thus T_ϕ is not invertible. Now, since $T_\phi = T^*_\phi = T_{\overline{\phi}}$, therefore $\ker T_\phi \neq \{0\}$ and $\ker T_{\overline{\phi}} \neq \{0\}$.

There are also functions ϕ in $h^\infty(\mathbf{D})$ such that the corresponding Toeplitz operator T_ϕ has nontrivial kernel.

(ii) Let $z \in \mathbf{D}, K_z$ is the corresponding Bergman reproducing kernel and $\phi_z(w) = (z-w)/(1-\overline{z}w)$ for w in $\mathbf{D}$. Then $\ker T^*_{\phi_z} = \text{span}\,(K_z)$. Note that

$$\overline{K_z(w)} = K(z,w) = \frac{1}{(1-z\overline{w})^2}.$$

It is easy to see that $\phi_z(z) = 0$ and $\phi_z(w) \neq 0$ for $z \neq w$. Hence, $\langle \phi_z, K_w\rangle = 0$ if and only if $z = w$. Thus, for all $g \in L^2_a(\mathbf{D}), \langle \phi_z g, K_w\rangle = 0$ if and only if $z = w$. Thus, $K_z \perp \phi_z g$ for every $g \in L^2_a(\mathbf{D})$. Hence, $K_z \in (\text{Range}\, T_{\phi_z})^\perp = \ker T^*_{\phi_z}$. For the inclusion to follow the other way, we first note that $\|z^n\|^2 = 1/(n+1)$ and $P(|w|^{2m}w^n) = (n+1)/(n+m+1)w^n$. Now let $f(w) = \sum_{n=0}^\infty a_n w^n$ and $f \in \ker T^*_{\phi_z}$. Then $\langle T^*_{\phi_z} f, g\rangle = 0$ for every $g \in L^2_a(\mathbf{D})$. Therefore, it follows that $\langle f, \phi_z g\rangle = 0$ for every $g \in L^2_a(\mathbf{D})$. Let $h(w) = 1/(1-\overline{z}w)$. It is easy to see that $hL^2_a = L^2_a$. Thus, we have $P((\overline{z}-\overline{w})f(w)) = 0$. Therefore,

$$\overline{z}\sum_{n=0}^\infty a_n w^n = \sum_{n=1}^\infty \frac{n}{n+1} a_n w^{n-1}$$

and $a_n = (n+1)\overline{z}^n a_0$. Thus,

$$f(z) = \sum_{n=0}^\infty a_0(n+1)(\overline{z}w)^n = \frac{a_0}{(1-\overline{z}w)^2} = a_0 K_z(w).$$

But there exist Fredholm Toeplitz operators $T_\phi \in \mathcal{L}(L^2_a(\mathbf{D}))$, $\phi \in L^\infty(\mathbf{D})$ such that $\ker T_\phi = \ker T^*_\phi = \{0\}$.

(iii) Let $\phi(z) = |z|^n$, where n is a nonnegative integer. Then, by [**13**], T_ϕ is Fredholm, hence it has closed range. The operator T_ϕ is self adjoint. Further, one can check that $\ker T_\phi = \ker T^*_\phi = \{0\}$. For this, we need to show that T_ϕ is one-to-one. Suppose that $T_\phi f = 0, f \in L^2_a(\mathbf{D})$. Then $\langle T_\phi f, f\rangle = 0$, and hence $\langle \phi f, f\rangle = 0$. Using the fact that $\phi > 0$, this will imply that $\phi|f|^2 = 0$, and hence, $f = 0$. Therefore, T_ϕ is invertible.

There are also functions ϕ in $h^\infty(\mathbf{D})$ such that the corresponding Toeplitz operator T_ϕ has trivial kernel and trivial cokernel.

(iv) Let $\phi(z) = z + \overline{z}$. Then $\ker T_\phi = \{0\}$. To verify this suppose $f \in \ker T_\phi$. Then $\phi f \in (L^2_a)^\perp$ and $\langle \phi f, g\rangle = 0$ for all $g \in L^2_a$. Thus, f is orthogonal to $\overline{z}g+zg$ for all $g \in L_a^2$. Now let $f=\sum C_m z^m$ and $g=z^k$. The function f is orthogonal to $\overline{z}z^k+z^{k+1}$. But $\langle f, \overline{z}z^k\rangle = \langle fz, z^k\rangle = (C_{k-1})/(k+1)$ and $\langle f, z^{k+1}\rangle = (C_{k+1})/(k+2)$. Therefore, we see that $(C_{k-1})/(k+1)+(C_{k+1})/(k+2)=0$ for all $k \geq 1$. In particular, since the function f is orthogonal to $\overline{z}+z$, hence f is orthogonal to z. Therefore, $C_1=0$ and $C_{k+1}=-(k+2)/(k+1)C_{k-1}$. Thus, $|C_{k+1}|>|C_{k-1}|$. It is to note that $C_k=0$ for all $k=2n+1$, $n\geq 0$. If $C_0 \neq 0$, then $C_k \neq 0$ for $k=2n$ and for all $n \geq 0$. But f belongs to L_a^2 implies $\sum (C_n^2)/(n+1)<\infty$. Hence, $f=0$.

(v) Let $\phi(z)=|z|^{2n}$. Then $\phi \in (L_h^2)_0^{\perp} \cap L^\infty$ and $P(|z|^{2n})=1/(n+1)$. Hence $|z|^{2n}-(1/n+1) \in (L_h^2)^{\perp}$. One can easily verify that $1 \in \ker T_{|z|^{2n}-(1/(n+1))}$ as

$$P\bigg(|z|^{2n}-\frac{1}{n+1}\bigg)=P(|z|^{2n})-\frac{1}{n+1}=\frac{1}{n+1}-\frac{1}{n+1}=0.$$

From Corollary 10, it follows that $\ker T_{|z|^{2n}-(1/(n+1))}=\operatorname{sp}\{1\}$.

From Theorem 9, it follows that if $0 \neq \phi \in (L_h^2)^{\perp} \cap L^\infty$ and T_ϕ has closed range then T_ϕ is Fredholm and of index zero but T_ϕ is not invertible as $T_\phi 1=0$. Further, we have shown that unlike the intermediate Hankel operators the kernel of a Toeplitz operator may not be an invariant subspace of $L_a^2(\mathbf{D})$.

If $\phi \in L_h^2(\mathbf{D}) \cap L^\infty(\mathbf{D}) = h^\infty(\mathbf{D})$, the space of bounded harmonic functions on $\mathbf{D}$, then it is not difficult to see that T_ϕ is not of finite rank: let $\phi(z)=\sum_{m=0}^\infty a_m z^m+\sum_{m=1}^\infty a_{-m}\overline{z}^m$. Then

$$\langle T_\phi e_n, e_{n+k}\rangle = \begin{cases} \sqrt{\frac{n+1}{n+k+1}}a_k & \text{if } k \geq 0; \\ \sqrt{\frac{n+k+1}{n+1}}a_k & \text{if } k<0. \end{cases}$$

Since $\{e_n\}$ converges to 0 weakly and T_ϕ is compact, hence $\lim_{n\to\infty} \|T_\phi e_n\|=0$. Thus, $\lim_{n\to\infty}\langle T_\phi e_n, e_{n+k}\rangle=0$ for every integer k. Hence, $a_k=0$ for all $k \in \mathbf{Z}$ and therefore $\phi \equiv 0$. We shall show below that if $\phi \in L^\infty(\mathbf{D})$ and $\phi \neq 0$, then T_ϕ cannot be of finite rank.

Theorem 11. *If $\phi \in L^\infty(\mathbf{D})$ and T_ϕ is a finite rank Toeplitz operator then $\phi \equiv 0$.*

Proof. We shall first show that if T_ϕ is a finite rank Toeplitz operator and $\phi \neq 0$ then for all $z \in \mathbf{D}$, either

$$(*) \qquad P(\phi \circ \phi_z) \neq P(\phi \circ \phi_z)(0)$$

or

$$(**) \qquad P(\overline{\phi} \circ \phi_z) \neq P(\overline{\phi} \circ \phi_z)(0).$$

Suppose T_ϕ is of rank one. By Proposition 5, $T_{\phi\circ\phi_z}$ is of rank one for all $z \in \mathbf{D}$. Let for $z \in \mathbf{D}$,

$$(1) \qquad T_{\phi\circ\phi_z} f = \langle f, g^z \rangle h^z.$$

Hence

$$(2) \qquad T^*_{\phi\circ\phi_z} f = \langle f, h^z \rangle g^z.$$

Suppose for $w \in \mathbf{D}$, $P(\phi \circ \phi_w) = P(\phi \circ \phi_w)(0)$ and $P(\overline{\boldsymbol{\phi}} \circ \phi_w) = P(\overline{\boldsymbol{\phi}} \circ \phi_w)(0)$. Then it follows that

$$(3) \qquad \overline{g^w(0)} h^w = \overline{g^w(0)} h^w(0)$$

and

$$(4) \qquad \overline{h^w(0)} g^w = \overline{h^w(0)} g^w(0).$$

If $g^w(0) = 0$, then $h^w(0) = 0$; otherwise, $g^w = 0$, and therefore $T_\phi \equiv 0$ and hence $\phi \equiv 0$.

If $h^w(0) = 0$, then $g^w(0) = 0$; otherwise, $h^w = 0$. Therefore, $T_\phi \equiv 0$, and hence $\phi \equiv 0$. Suppose now that $g^w(0) = h^w(0) = 0$, but $g^w \neq 0$ and $h^w \neq 0$. Now $T_{\phi\circ\phi_w} 1 = \langle 1, g^w \rangle h^w = \overline{g^w(0)} h^w = 0$ and $T^*_{\phi\circ\phi_w} 1 = \langle 1, h^w \rangle g^w = \overline{h^w(0)} g^w = 0$. Hence, $1 \in \ker T_{\phi\circ\phi_w}$ and $1 \in \ker T^*_{\phi\circ\phi_w}$. Thus, by Theorem 9, either $\phi \circ \phi_w \equiv 0$ or $T_{\phi\circ\phi_w}$ is not of finite rank. If $\phi \circ \phi_w \equiv 0$, then $\phi \equiv 0$ and if $T_{\phi\circ\phi_w}$ is not of finite rank, then by Proposition 5, T_ϕ is not of finite rank.

On the other hand, suppose $g^w(0) \neq 0, h^w(0) \neq 0$. Then, from (3) and (4), it follows that $g^w = g^w(0) \neq 0$ and $h^w = h^w(0) \neq 0$. Hence,

$$\begin{aligned}
\ker T_{\phi\circ\phi_w} &= \{f \in L^2_a : \langle f, g^w \rangle = 0\} \\
&= \{f \in L^2_a : \langle f, g^w(0) \rangle = 0\} \\
&= \{f \in L^2_a : \overline{g^w(0)} f(0) = 0\} \\
&= \{f \in L^2_a : f(0) = 0\}.
\end{aligned}$$

Thus, $\operatorname{Card}(\mathscr{Z}(\ker T_{\phi\circ\phi_w})) = 1 = \text{Rank of } T^*_{\phi\circ\phi_w}$. Therefore, by Theorem 4, $T_{\phi\circ\phi_w} \equiv 0$, which implies $\phi\circ\phi_w \equiv 0$; since $(\phi_w\circ\phi_w)(z) = z$ for all $z \in \mathbf{D}$, hence $\phi \equiv 0$. Then, we obtain that if T_ϕ is a nonzero Toeplitz operator of rank 1, then either $(*)$ or $(**)$ holds.

Suppose T_ϕ is a Toeplitz operator of rank $n > 1$. Then, by Proposition 5, $T_{\phi\circ\phi_z}$ is of rank n for all $z \in \mathbf{D}$. Suppose there exist $w \in \mathbf{D}$ such that $P(\phi\circ\phi_w) = P(\phi\circ\phi_w)(0)$ and $P(\overline{\boldsymbol{\phi}}\circ\boldsymbol{\phi}_{\mathbf{w}}) = P(\overline{\boldsymbol{\phi}}\circ\boldsymbol{\phi}_{\mathbf{w}})(0)$. Suppose $T_{\phi\circ\phi_w}1 = \langle T_{\phi\circ\phi_w}1, 1\rangle = 0$. This implies $T_{\overline{\boldsymbol{\phi}}\circ\boldsymbol{\phi}_{\mathbf{w}}}1 = \langle \mathbf{T}_{\overline{\boldsymbol{\phi}}\circ\boldsymbol{\phi}_{\mathbf{w}}}\mathbf{1}, \mathbf{1}\rangle = 0$. Thus, by Theorem 9, either $\phi\circ\phi_w \equiv 0$ or $T_{\phi\circ\phi_w}$ is not of finite rank. Hence, either $\phi \equiv 0$ or T_ϕ is not of finite rank.

Now suppose there exists $w \in \mathbf{D}$ such that $P(\phi\circ\phi_w) = P(\phi\circ\phi_w)(0) = c \neq 0$ and $P(\overline{\boldsymbol{\phi}}\circ\boldsymbol{\phi}_{\mathbf{w}}) = P(\overline{\boldsymbol{\phi}}\circ\boldsymbol{\phi}_{\mathbf{w}})(0) = \overline{c} \neq 0$. That is, $T_{\phi\circ\phi_w}1 = \langle T_{\phi\circ\phi_w}1, 1\rangle = c \neq 0$ and $T^*_{\phi\circ\phi_w}1 = \langle T^*_{\phi\circ\phi_w}1, 1\rangle = \overline{c} \neq 0$. Let $\psi_w = \phi\circ\phi_w - c$. Then $T_{\psi_w}1 = 0 = T^*_{\psi_w}1$. Thus, by Theorem 9, either $\psi_w \equiv 0$ or T_{ψ_w} is not of finite rank. Hence, either $\phi \equiv c$ or $T_{\phi-c}$ is not of finite rank. Further, it follows from Theorem 9 that

$$\ker T^*_{\psi_w} = \ker T_{\psi_w} = \begin{cases} \operatorname{sp}\{1\} & \text{if } \psi_w \neq 0;\\ L^2_a & \text{if } \psi_w \equiv 0.\end{cases}$$

Thus, either $\phi \equiv c$ or $\ker T^*_{\psi_w} = \operatorname{sp}\{1\}$. If $\phi \equiv c$, then T_ϕ is not of finite rank. If $\ker T^*_{\psi_w} = \operatorname{sp}\{1\}$, then $\overline{\operatorname{Range} T_{\psi_w}} = (\operatorname{sp}\{1\})^\perp = zL^2_a$. That is, $\overline{\operatorname{Range} T_{\phi\circ\phi_w - c}} = zL^2_a$. Let $\widehat{\phi} = \phi\circ\phi_w$.

Suppose $\widehat{\phi} - c \neq 0$. Then $\ker T_{\widehat{\phi}-c} = \ker T^*_{\widehat{\phi}-c} = (zL^2_a)^\perp = \operatorname{sp}\{1\}$ and $\overline{\operatorname{Range} T^*_{\widehat{\phi}-c}} = \overline{\operatorname{Range} T_{\widehat{\phi}-c}} = zL^2_a$. Thus, $\operatorname{Range} T^*_{\widehat{\phi}-c}$ and $\operatorname{Range} T_{\widehat{\phi}-c}$ are infinite-dimensional vector spaces. Suppose $f \in \ker T_{\widehat{\phi}}$. Then $T_{\widehat{\phi}}f = 0$. That is, $T_{\widehat{\phi}-c}f = -cf$. Hence, $f \in \operatorname{Range} T_{\widehat{\phi}-c} \subseteq zL^2_a$. On the other hand, since $T_{\widehat{\phi}}1 = c \neq 0$ and $T^*_{\widehat{\phi}}1 = \overline{\mathbf{c}} \neq 0$, hence $1 \notin \ker T_{\widehat{\phi}}$ and $1 \notin \ker T^*_{\widehat{\phi}}$. Now $f \in \ker T_{\widehat{\phi}}$ if and only if $\widehat{\phi}f \in (L^2_a)^\perp$. This is true if and only if $f \in (\overline{\widehat{\phi}}L^2_a)^\perp$. Further, $g \in \ker T^*_{\widehat{\phi}}$ if and only if $\overline{\widehat{\phi}}g \in (L^2_a)^\perp$. This is valid if and only if $g \in (\widehat{\phi}L^2_a)^\perp$. Since $\widehat{\phi} \in (L^2_h)^\perp_0$, hence $f \in (\overline{\widehat{\phi}}L^2_a)^\perp$ if and only if $f \in (\widehat{\phi}L^2_a)^\perp$. Thus, $f \in \ker T_{\widehat{\phi}}$ if and only if $f \in \ker T^*_{\widehat{\phi}}$. That is, $\ker T_{\widehat{\phi}} = \ker T^*_{\widehat{\phi}}$. This can also be seen as follows.

For $f \in L^2_a$, define $C_{\overline{\mathbf{f}}} : (L^2_a)_0 \to (L^2_h)_0^{\perp}$ as $C_{\overline{\mathbf{f}}} g = (I - Q_0)(\overline{\mathbf{f}} g)$ where Q_0 is the orthogonal projection from L^2 onto $(L^2_h)_0$. The operator $C_{\overline{\mathbf{f}}}$ is well defined. It can be verified that

$$\ker C_{\overline{\mathbf{f}}} = \begin{cases} (L^2_a)_0 & \text{if } f \equiv \text{constant;} \\ \{0\} & \text{if } f \not\equiv \text{constant.} \end{cases}$$

Further, proceeding as before, one can show that

$$\ker C^*_{\overline{\mathbf{f}}} = \left(\bigcup_{p \geq 0} \ker H^{*^{E_p}}_{\overline{\mathbf{f}}} \right) \bigcap (L^2_h)_0^{\perp}$$

and

$$\ker C^*_{\overline{\mathbf{f}}} = \begin{cases} \{0\} & \text{if } f \not\equiv \text{constant;} \\ ((L^2_h)_0)^{\perp} & \text{if } f \equiv \text{constant.} \end{cases}$$

Now, since $0 \neq \widehat{\phi} \in ((L^2_h)_0)^{\perp}$ and $f \in \ker T_{\widehat{\phi}}$ imply $f = zh$ for some $h \in L^2_a$, we obtain

$$\begin{aligned} \ker T_{\widehat{\phi}} &= \{ f \in L^2_a : \widehat{\phi} f \in (L^2_a)^{\perp} \} \\ &= \{ f \in L^2_a : \widehat{\phi} f \in (L^2_h)^{\perp} \} \\ &= \{ f \in L^2_a : \langle \widehat{\phi} f, g \rangle = 0 \quad \text{for all } g \in L^2_h \} \\ &= \{ f \in L^2_a : \langle \widehat{\phi}, \overline{\mathbf{f}} g \rangle = 0 \quad \text{for all } g \in L^2_h \} \\ &= \{ f \in L^2_a : \langle \widehat{\phi}, (I - Q_0)(\overline{\mathbf{f}} g) \rangle = 0 \quad \text{for all } g \in L^2_h \} \\ &= \{ f \in L^2_a : \langle \widehat{\phi}, C_{\overline{\mathbf{f}}} g \rangle = 0 \quad \text{for all } g \in L^2_h \} \\ &= \{ f \in L^2_a : \langle C^*_{\overline{\mathbf{f}}} \widehat{\phi}, g \rangle = 0 \quad \text{for all } g \in L^2_h \} \\ &= \{ f \in L^2_a : C^*_{\overline{\mathbf{f}}} \widehat{\phi} = 0 \} = \{0\}. \end{aligned}$$

Similarly, one can show that $\ker T^*_{\widehat{\phi}} = \{0\}$. Thus, $\ker T_{\widehat{\phi}} = \ker T^*_{\widehat{\phi}} = \{0\}$, and therefore $\overline{\text{Range}\, T_{\widehat{\phi}}} = \overline{\text{Range}\, T^*_{\widehat{\phi}}} = L^2_a$, and hence $T_{\widehat{\phi}}$ is not of finite rank. This implies T_ϕ is not of finite rank.

Thus, we have shown that, if T_ϕ is a finite rank Toeplitz operator and $\phi \neq 0$, then for all $z \in \mathbf{D}$ either $(*)$ or $(**)$ holds. Now

$$\begin{aligned} &\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2 \\ &\qquad = \|\phi \circ \phi_z - P(\phi \circ \phi_z)\|^2 + \|\overline{\phi} \circ \phi_z - P(\overline{\phi} \circ \phi_z)\|^2 \\ &\qquad < \|\phi \circ \phi_z - P(\phi \circ \phi_z)(0)\|^2 + \|\overline{\phi} \circ \phi_z - P(\overline{\phi} \circ \phi_z)(0)\|^2 \\ &\qquad = 2(\widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2) \end{aligned}$$

because of $(*)$ and $(**)$. Let $c_z = (\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2)/(\widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2)$. Note that $1 \leq c_z < 2$ for all $z \in \mathbf{D}$. It follows from Proposition 6 and from the argument given above.

Now T_ϕ of finite rank implies $T_\phi^* T_\phi = T_{|\phi|^2} - H_\phi^* H_\phi$ is of finite rank. Thus, $0 \leq \langle T_\phi^* T_\phi k_z, k_z \rangle = \widetilde{|\phi|^2}(z) - \|H_\phi k_z\|^2 \to 0$ as $|z| \to 1^-$. Similarly, since $T_\phi T_\phi^*$ is of finite rank, $0 \leq \widetilde{|\phi|^2}(z) - \|H_{\overline{\phi}} k_z\|^2 \to 0$ as $|z| \to 1^-$. Thus, $0 \leq 2\widetilde{|\phi|^2}(z) - (\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2) \to 0$ as $|z| \to 1^-$. Hence, it follows that $0 \leq 2\widetilde{|\phi|^2}(z) - c_z(\widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2) = 2\widetilde{|\phi|^2}(z) - (\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2) \to 0$ as $|z| \to 1^-$. So $0 \leq (2 - c_z)\widetilde{|\phi|^2}(z) + c_z|\widetilde{\phi}(z)|^2 \to 0$ as $|z| \to 1^-$ where $1 \leq c_z < 2$. Since T_ϕ is of finite rank, $|\widetilde{\phi}(z)| \to 0$ as $|z| \to 1^-$. Thus, $\widetilde{|\phi|^2}(z) \to 0$ as $|z| \to 1^-$. Similarly, since $T_{\phi \circ \phi_z}$ is of finite rank, we can show that $\widetilde{|\phi \circ \phi_z|^2}(z) \to 0$ as $|z| \to 1^-$. Thus,

$$\int_{\mathbf{D}} |(\phi \circ \phi_z)(w)|^2 |k_z(w)|^2 dA(w) \longrightarrow 0$$

as $|z| \to 1^-$. Hence, $\int_{\mathbf{D}} |\phi(w)|^2 dA(w) = 0$ as $U_z k_z = k_z(\phi_z(w)) k_z(w) = 1$. It follows therefore that $\phi(w) = 0$ almost everywhere, and hence $\phi \equiv 0$. Thus, there exists no nonzero finite rank Toeplitz operator on the Bergman space. □

Remark 12. Notice that, if $k_z \in \operatorname{Range} T_{\phi \circ \phi_w \circ \phi_z}$, then $U_z T_{\phi \circ \phi_w} U_z f = k_z$ for some $f \in L_a^2$. Hence, $U_w T_\phi U_w U_z f = T_{\phi \circ \phi_w} U_z f = 1$. Hence, $T_\phi U_w U_z f = k_w$ and $k_w \in \operatorname{Range} T_\phi$. Similarly if $k_z \in \ker T_{\phi \circ \phi_w \circ \phi_z}$, then $U_z T_{\phi \circ \phi_w} U_z k_z = T_{\phi \circ \phi_w \circ \phi_z} k_z = 0$. Therefore, $U_z T_{\phi \circ \phi_w} 1 = 0$, and hence $U_w T_\phi U_w 1 = T_{\phi \circ \phi_w} 1 = 0$. Thus, $T_\phi k_w = 0$ and $k_w \in \ker T_\phi$.

Suppose $T_{\phi \circ \phi_w} 1 = T_{\phi \circ \phi_w}^* 1 = 0$ and $w = 0$. Then it follows from [**1**] that $T_\phi 1 = T_\phi^* 1 = 0$. This implies either $\phi \equiv 0$ or T_ϕ is not of finite rank. Further, let $w' = \phi_z(w)$. From Lemma 8, it follows that

$$\begin{aligned}
\operatorname{Range} T_{\phi \circ \phi_w \circ \phi_z} &= \{T_{\phi \circ \phi_w \circ \phi_z} f : f \in L_a^2\} \\
&= \{P((\phi \circ \phi_w \circ \phi_z) f) : f \in L_a^2\} \\
&= \{P((\phi \circ U \phi_{\phi_z(w)}) f) : f \in L_a^2\} \\
&= \{P((\phi \circ U \circ \phi_{w'}) f) : f \in L_a^2\}
\end{aligned}$$

$$
\begin{aligned}
&= \{U_{w'} P[U_{w'}((\phi \circ U \circ \phi_{w'}) f)] : f \in L_a^2\} \\
&= \{U_{w'} P[(\phi \circ U \circ \phi_{w'} \circ \phi_{w'})(f \circ \phi_{w'}) k_{w'}] : f \in L_a^2\} \\
&= \{U_{w'} P[(\phi \circ U)(f \circ \phi_{w'}) k_{w'}] : f \in L_a^2\} \\
&= \{U_{w'} P[(\phi \circ U) U_{w'} f] : f \in L_a^2\} \\
&= \{U_{w'} P[(\phi \circ U) g] : g \in L_a^2\}.
\end{aligned}
$$

Thus, $\operatorname{Range} T_{\phi \circ \phi_w \circ \phi_z} = \operatorname{Range} U_{w'} T_{\phi \circ U}$.

Remark 13. There cannot be an uncountable subset E of $\mathbf{D}$ such that, for all $z \in E$, $k_z \in \operatorname{Range} T_\phi^* = (\ker T_\phi)^\perp$. Because, if $k_z \perp \ker T_\phi$ for all $z \in E$, then $\mathcal{Z}(\ker T_\phi)$ is an uncountable set and that implies $\ker T_\phi = \{0\}$. Thus, if $\ker T_\phi$ contains nonzero elements of L_a^2, then $\mathcal{Z}(\ker T_\phi)$ is at most a countable set and $\operatorname{Range} T_\phi^*$ contains only a countable number of the normalized reproducing kernels $k_z, z \in \mathbf{D}$.

Remark 14. If $T_\phi 1 = T_\phi^* 1 = 0$, then $\phi \in (L_h^2)^\perp$ and $\mathcal{Z}(\ker T_\phi^*) = \varnothing$. This implies $k_z \notin \operatorname{Range} T_\phi$ for all $z \in \mathbf{D}$, $z \neq 0$. For, if $k_z \in \operatorname{Range} T_\phi = (\ker T_\phi^*)^\perp$, then $\langle f, k_z \rangle = 0$ for all $f \in \ker T_\phi^*$. That is, then $f(z) = 0$ for all $f \in \ker T_\phi^*$ and $z \in \mathcal{Z}(\ker T_\phi^*)$.

Let Ω be a bounded symmetric domain in $\mathbf{C}$. We assume that Ω is in its standard (Harish-Chandra) realization so that $0 \in \Omega$ and Ω is circular. The domain Ω is also starlike, i.e., $z \in \Omega$ implies that $tz \in \Omega$ for all $t \in [0,1]$. Let $\operatorname{Aut}(\Omega)$ be the Lie group of all automorphisms (biholomorphic mappings) of Ω, and G_0 the isotropy subgroup at 0; i.e., $G_0 = \{\Psi \in \operatorname{Aut}(\Omega) : \Psi(0) = 0\}$. Since Ω is bounded symmetric, we can canonically define [**16**] for each a in Ω an automorphism ϕ_a in $\operatorname{Aut}(\Omega)$ such that

(1) $\phi_a \circ \phi_a(z) \equiv z$;

(2) $\phi_a(0) = a, \phi_a(a) = 0$;

(3) ϕ_a has a unique fixed point in Ω.

Actually, the above three conditions completely characterize the ϕ_a's as the set of all (holomorphic) geodesic symmetries of Ω. When $\Omega = \mathbf{D}$, we have noted that

$$\phi_a(z) = \frac{a - z}{1 - \overline{a} z}$$

for all a and z in $\mathbf{D}$. They are involutive Möbius transformations on $\mathbf{D}$.

Let dA be the normalized Lebesgue measure on Ω. We consider the Bergman space $L_a^2(\Omega)$ of holomorphic functions in $L^2(\Omega, dA)$. The reproducing kernel $K(z,w)$ of $L_a^2(\Omega, dA)$ is holomorphic in z and anti-holomorphic in w, and

$$\int_\Omega |K(z,w)|^2 dA(w) = K(z,z) > 0$$

for all z in Ω. Thus, we can define for each $\lambda \in \Omega$ a unit vector k_λ in $L_a^2(\Omega)$ by $k_\lambda(z) = (K(z,\lambda))/\sqrt{K(\lambda,\lambda)}$. Given $\lambda \in \Omega$ and f any measurable function on Ω, we define the operator U_λ on $L_a^2(\Omega)$ by $U_\lambda f(z) = k_\lambda(z) f(\phi_\lambda(z))$. Since $|k_\lambda|^2$ is the real Jacobian determinant of the mapping ϕ_λ (see [**16**]), the operator U_λ is easily seen to be a unitary operator on $L_a^2(\Omega)$. It is also easy to check that $U_\lambda^* = U_\lambda$; thus, U_λ is a self-adjoint unitary operator. For any $\Psi \in \mathrm{Aut}\,(\Omega)$, we denote by $J_\Psi(z)$ the complex Jacobian determinant of the mapping $\Psi : \Omega \to \Omega$. If $a \in \Omega$, then (for reference, see [**16**]), there exists a unimodular constant $\theta(a)$ such that

$$J_{\phi_a}(z) = \theta(a) k_a(z)$$

for all $z \in \Omega$. In the simplest case $\Omega = \mathbf{D}$, we have $\phi_a(z) = (a-z)/(1-\overline{a}z)$ and $J_{\phi_a}(z) = \phi_a'(z) = -k_a(z)$; thus, $\theta(a) = -1$ is independent of a. All the results proved in this work also carry over to any bounded symmetric domain in $\mathbf{C}$ described above. Thus, there exists no nonzero finite rank Toeplitz operator on the Bergman space $L_a^2(\Omega)$.

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