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APPROXIMATION BY BÉZIER VARIANT OF THE BASKAKOV-KANTOROVICH OPERATORS IN THE CASE $0 < \alpha < 1$

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ABSTRACT. The present paper deals with the approximation of Bézier variants of Baskakov-Kantorovich operators $V_{n,\alpha}^*$ in the case $0 < \alpha < 1$. Pointwise approximation properties of the operators $V_{n,\alpha}^*$ are studied. A convergence theorem of this type approximation for locally bounded functions is established. This convergence theorem subsumes the approximation of functions of bounded variation as a special case.

1. Introduction. In 2003, Abel and others [**1**] introduced a Bézier variant of the Baskakov-Kantorovich operators $V_{n,\alpha}^*$ defined by

$$V_{n,\alpha}^*(f,x) = n\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\int_{I_k} f(t)\,dt, \tag{1}$$
$$(n \in \mathbf{N},\ \alpha \geq 1,\ \text{or } 0 < \alpha < 1),$$

where

$$\begin{aligned} I_k &= [k/n,\ (k+1)/n], \\ Q_{n,k}^{(\alpha)}(x) &= J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x), \\ J_{n,k}(x) &= \sum_{j=k}^{\infty} b_{n,j}(x), \end{aligned}$$

and

$$b_{n,j}(x) = \binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}},$$

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is the Baskakov basis function. For some basis properties of $J_{n,k}$, one can refer to [**11**]. If we replace the term $n\int_{I_k} f(t)\,dt$ with the term $f(k/n)$ in the definition (1), we obtain the Baskakov-Bézier operators $B_{n,\alpha}$ defined by

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f(k/n), \quad (n \in \mathbf{N},\ \alpha \geq 1,\ \text{or } 0 < \alpha < 1).$$

The operators $B_{n,\alpha}$ were first introduced by Zeng and others [**11**] in 2002. Some important properties of the operators of Baskakov type have been studied by several authors (cf., [**1, 3, 6–9, 11**]).

The authors of [**1**] studied the rate of convergence of the operators (1) for functions of bounded variation in the case $\alpha \geq 1$. Since the other case is equally important, in the present paper we will study the rate of convergence of the Baskakov-Kantorovich operators (1) in the case $0 < \alpha < 1$. We consider the following class of the function Φ_B:

$$\Phi_B = \{f \mid f \text{ is integrable and is bounded on every finite subinterval of } [0,\ \infty),\ \text{for some } r \in \mathbf{N},\ f(t) = O(t^r) \text{ as } t \to \infty\}.$$

For $f \in \Phi_B$, $x \in [0,\infty)$ and $\eta \geq 0$, set

$$\omega_x(f,\eta) = \sup_{t \in [x-\eta,\ x+\eta] \cap [0,\infty)} |f(t) - f(x)|.$$

The basic properties of $\omega_x(f,\eta)$ have been presented in [**11**].

Let the kernel function $K_{n,\alpha}(x,t)$ be defined by

$$K_{n,\alpha}(x,t) = n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \varphi_{n,k}(t), \tag{2}$$

where $\varphi_{n,k}(t)$ denotes the characteristic function of the interval $I_k = [k/n,\ (k+1)/n]$ with respect to $I = [0,\infty)$. Then, by the Lebesgue-Stieltjes integral representation, we have

$$V_{n,\alpha}^{*}(f,x) = \int_0^{\infty} f(t) K_{n,\alpha}(x,t)\,dt. \tag{3}$$

Now we state our main result as follows:

Theorem 1. *Let* $0 < \alpha < 1$, *and let* $f \in \Phi_B$ *and* $f(x+)$, $f(x-)$ *exist at a fixed point* $x \in (0, \infty)$. *Then, for*

$$n > \frac{144(x+1)}{x},$$

we have

$$\begin{aligned}(4)\quad \left| V^*_{n,\alpha}(f,x) - \frac{f(x+) + (2^\alpha - 1)f(x-)}{2^\alpha} \right| &\le \frac{4C_\alpha + 4 + x}{nx} \sum_{k=1}^{n} \omega_x(g_x, x/\sqrt{k}) \\ &\quad + \frac{14\alpha\sqrt{1+x}}{4^\alpha \sqrt{nx}} |f(x+) - f(x-)| + O(n^{-l}),\end{aligned}$$

where C_α *is a constant depending only on* α, $l > r$ *and the auxiliary function* $g_x(t)$ *is defined by*

$$(5)\qquad g_x(t) = \begin{cases} f(t) - f(x+) & x < t < \infty; \\ 0 & t = x; \\ f(t) - f(x-) & 0 \le t < x. \end{cases}$$

Let f be defined on $[0, \infty)$, $f(t) = O(t^r)$, and let f be the bounded variation on every finite subinterval of $[0, \infty)$. Then function f satisfies the conditions of Theorem 1. Therefore, Theorem 1 subsumes the approximation of functions of bounded variation as a special case.

2. Some lemmas. In order to prove Theorem 1, we need the following lemmas.

Lemma 1 [**1**, Lemma 4]. *For each fixed* $x \in (0, \infty)$, *let* $T_{n,m}(x) = V^*_{n,1}((t-x)^m, x)$. *Then*

$$V^*_{n,1}(1, x) = 1,$$

$$V^*_{n,1}(t - x, x) = \frac{1}{2n}, \qquad V^*_{n,1}\left((t-x)^2, x\right) = \frac{1 + 3nx(1+x)}{3n^2},$$

and

$$T_{n,m}(x) = O\left(n^{-\lfloor (m+1)/2 \rfloor}\right), \quad (n \to \infty). \tag{6}$$

Lemma 2. *Let the kernel function* $K_{n,\alpha}(x,t), 0 < \alpha < 1$ *be defined as in* (2). *Then*

(i) *for* $0 < y < x$, *we have*

$$\int_0^y K_{n,\alpha}(x,t)\,dt \le \frac{(1+x)^2}{n(x-y)^2}; \tag{7}$$

(ii) *for* $z < x < \infty$, *we have*

$$\int_z^\infty K_{n,\alpha}(x,t)\,dt \le \frac{C_\alpha}{n(z-x)^2}, \tag{8}$$

where C_α *is a constant depending only on* α.

Proof. Choose an integer $k' \in [0,\infty)$ such that $y \in [k'/n, (k'+1)/n)$. Then $y = (k'/n) + (\varepsilon/n)$ and, with some $\varepsilon \in [0,1)$, we have

$$\begin{aligned}
\int_0^y K_{n,\alpha}(x,t)\,dt &= \sum_{k=0}^{k'-1} Q_{n,k}^{(\alpha)}(x) + \varepsilon Q_{n,k'}^{(\alpha)}(x) \\
&= 1 - (1-\varepsilon) J_{n,k'}^{\alpha}(x) - \varepsilon J_{n,k'+1}^{\alpha}(x) \\
&\le 1 - (1-\varepsilon) J_{n,k'}(x) - \varepsilon J_{n,k'+1}(x) \\
&= \sum_{k=0}^{k'-1} Q_{n,k}(x) + \varepsilon Q_{n,k'}(x) \\
&= \int_0^y K_{n,1}(x,t)\,dt \\
&\le \frac{1}{(x-y)^2} T_{n,2}(x) \le \frac{(1+x)^2}{n(x-y)^2}.
\end{aligned}$$

This completes the proof of (7).

Next, if $z \in [k''/n, (k''+1)/n)$, then

$$\begin{aligned}\int_z^\infty K_{n,\alpha}(x,t)\,dt &= nQ^{(\alpha)}_{n,k''}(x)\left(\frac{k''+1}{n} - z\right) \\ &\quad + \sum_{k=k''+1}^{\infty} Q^{(\alpha)}_{n,k}(x) \\ &\le \sum_{k=k''}^{\infty} Q^{(\alpha)}_{n,k}(x) \\ &= \left(\sum_{k=k''}^{\infty} b_{n,k}(x)\right)^{\alpha}.\end{aligned}$$

Now, by applying (6) and the method that was presented in [**10**, Lemma 7], we obtain inequality (8). □

Lemma 3. *Let $0 < \alpha \le 1$. If x belongs to interval $I_{k'}$ for some nonnegative integer k', then for*

$$n > \frac{144(x+1)}{x},$$

we have

$$\left|\left(\sum_{k=k'+1}^{\infty} b_{n,k}(x)\right)^{\alpha} - \frac{1}{2^{\alpha}}\right| \le \frac{12\alpha}{4^{\alpha}}\frac{\sqrt{x+1}}{\sqrt{nx}}. \tag{9}$$

Proof. By the mean value theorem, we have

$$\left|\left(\sum_{k=k'+1}^{\infty} b_{n,k}(x)\right)^{\alpha} - \frac{1}{2^{\alpha}}\right| = \alpha(\xi_{n,k'}(x))^{\alpha-1}\left|\sum_{k=k'+1}^{\infty} b_{n,k}(x) - \frac{1}{2}\right|, \tag{10}$$

where $\xi_{n,k'}(x)$ lies between $1/2$ and $\sum_{k=k'+1}^{\infty} b_{n,k}(x)$. Using [**11**, Lemma 5],

$$\left|\sum_{k=k'+1}^{\infty} b_{n,k}(x) - \frac{1}{2}\right| \le \frac{3\sqrt{x+1}}{\sqrt{nx}} \tag{11}$$

holds. From (11), we get $\sum_{k=k'+1}^{\infty} b_{n,k} > 1/4$ for $n > [144(x+1)]/x$. Thus, $\xi_{n,k'}(x) > 1/4$, for $n > [144(x+1)]/x$. We have shown inequality (9) from (10) and (11).

Lemma 4. *Let $0 < \alpha \leq 1$. If x belongs to interval $I_{k'}$ for some nonnegative integer k', then for $n > [144(x+1)]/x$,*

$$Q_{n,k'}^{(\alpha)}(x) \leq \frac{2\alpha}{4^\alpha} \frac{\sqrt{x+1}}{\sqrt{nx}} \tag{12}$$

holds.

Proof. By using the bound given in [**12**], for any k, we have

$$b_{n,k}(x) \leq \frac{\sqrt{x+1}}{\sqrt{2enx}}. \tag{13}$$

On the other hand, by the mean value theorem, we have

$$\begin{aligned} Q_{n,k'}^{(\alpha)}(x) &= \alpha(\zeta_{n,k'}(x))^{\alpha-1}[J_{n.k'}(x) - J_{n,k'+1}(x)] \\ &= \alpha(\zeta_{n,k'}(x))^{\alpha-1} b_{n,k'}(x), \end{aligned} \tag{14}$$

where $J_{n,k'+1}(x) < \zeta_{n,k'}(x) < J_{n,k'}(x)$. From (11), we get

$$\zeta_{n,k'}(x) > J_{n,k'+1}(x) = \sum_{j=k'+1}^{\infty} b_{n,j}(x) > \frac{1}{4}, \tag{15}$$

for $n > [144(x+1)]/x$. Combining (13), (14) and (15), we obtain inequality (12). Lemma 4 is proved. □

3. Proof of Theorem 1. For any $f \in \Phi_B$, if $f(x+)$ and $f(x-)$ exist at x, then by decomposition (cf., [**11**, page 1449]:

$$\begin{aligned} f(t) = &\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \\ &+ g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha} \operatorname{sgn}_{\alpha,x}(t) \\ &+ \eta_x(t)\left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-)\right], \end{aligned} \tag{16}$$

where $g_x(t)$ is defined in (5) and

$$\operatorname{sgn}_{\alpha,x}(t)=\begin{cases}2^\alpha-1 & t>x,\\ 0 & t=x,\\ -1 & t<x,\end{cases}\qquad \eta_x(t)=\begin{cases}1 & t=x\\ 0 & t\neq x.\end{cases}$$

Obviously,

$$V^*_{n,\alpha}(\eta_x,x)=0. \tag{17}$$

Hence, it follows that

$$\left|V^*_{n,\alpha}(f,x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \le |V^*_{n,\alpha}(g_x,x)|+\left|\frac{f(x+)-f(x-)}{2^\alpha}V^*_{n,\alpha}(\operatorname{sgn}_{x,\alpha},x)\right|. \tag{18}$$

We need to estimate $|V^*_{n,\alpha}(\operatorname{sgn}_{x,\alpha},x)|$ and $|V^*_{n,\alpha}(g_x,x)|$.

Let $x\in I_{k'}$ for some k'. Direct computation gives

$$\begin{aligned}V^*_{n,\alpha}(\operatorname{sgn}_{x,\alpha},x)&=(2^\alpha-1)\sum_{k=k'+1}^{\infty}Q^{(\alpha)}_{n,k}(x)-\sum_{k=0}^{k'-1}Q^{(\alpha)}_{n,k}(x)\\&\quad+nQ^{(\alpha)}_{n,k'}(x)\left(2^\alpha\left(\frac{k'+1}{n}-x\right)-\frac{1}{n}\right)\\&=2^\alpha\sum_{k=k'+1}^{\infty}Q^{(\alpha)}_{n,k}(x)-1+2^\alpha(k'+1-nx)Q^{(\alpha)}_{n,k'}(x).\end{aligned}$$

Note that $0<k'+1-nx<1$. By Lemmas 3 and 4, we have

$$\begin{aligned}|V^*_{n,\alpha}(\operatorname{sgn}_{x,\alpha},x)|&\le 2^\alpha\left|\sum_{k=k'+1}^{\infty}Q^{(\alpha)}_{n,k}(x)-\frac{1}{2^\alpha}\right|+2^\alpha Q^{(\alpha)}_{n,k'}(x)\\&=2^\alpha\left|\left(\sum_{k=k'+1}^{\infty}b_{n,k}(x)\right)^\alpha-\frac{1}{2^\alpha}\right|+2^\alpha Q^{(\alpha)}_{n,k'}(x)\\&\le\frac{12\alpha}{2^\alpha}\frac{\sqrt{1+x}}{\sqrt{nx}}+\frac{2\alpha}{2^\alpha}\frac{\sqrt{1+x}}{\sqrt{nx}}=\frac{14\alpha}{2^\alpha}\frac{\sqrt{1+x}}{\sqrt{nx}}.\end{aligned} \tag{19}$$

Next we estimate $|V^*_{n,\alpha}(g_x, x)|$. Using the Bojanic-Cheng decomposition [**2, 4, 5**], we write

$$V^*_{n,\alpha}(g_x, x) = \int_{[0,\infty)} g_x(t) K_{n,\alpha}(x,t)\, dt = \sum_{j=1}^{4} \int_{A_j} g_x(t) K_{n,\alpha}(x,t)\, dt, \tag{20}$$

where

$$A_1 := [0,\ x - x/\sqrt{n}], \qquad A_2 := (x - x/\sqrt{n},\ x + x/\sqrt{n}],$$
$$A_3 := (x + x/\sqrt{n},\ 2x], \qquad A_4 := (2x,\ \infty).$$

Firstly, note that $g_x(x) = 0$; thus,

$$\left| \int_{A_2} g_x(t) K_{n,\alpha}(x,t)\, dt \right| \le \omega_x(g_x, x/\sqrt{n}) \le \frac{1}{n} \sum_{k=1}^{n} \omega_x(g_x, x/\sqrt{k}). \tag{21}$$

To estimate

$$\left| \int_{A_1} g_x(t) K_{n,\alpha}(x,t)\, dt \right|,$$

note that $\omega_x(g_x, \eta)$ is monotone increasing with respect to η; thus, it follows that

$$\left| \int_0^{x - x/\sqrt{n}} g_x(t) K_{n,\alpha}(x,t)\, dt \right| \le \int_0^{x - x/\sqrt{n}} \omega_x(g_x, x-t) K_{n,\alpha}(x,t)\, dt.$$

Integrating by parts with $y = x - x/\sqrt{n}$, we have

$$\begin{aligned} &\int_0^{x - x/\sqrt{n}} \omega_x(g_x, x-t) H_{n,\alpha}(x,t)\, dt \\ &\qquad \le \omega_x(g_x, x-y) \int_0^y K_{n,\alpha}(x,t)\, dt \\ &\qquad\quad + \int_0^y \left(\int_0^t K_{n,\alpha}(x,u)\, du \right) d\left(-\omega_x(g_x, x-t)\right). \end{aligned} \tag{22}$$

From (22) and Lemma 2, it follows that

$$(23)\quad \left|\int_{A_1} g_x(t)\,d_t\lambda_{n,\alpha}(x,t)\right| \le \omega_x\left(g_x, x/\sqrt{n}\right)\frac{3nx+1}{2n^2(x-y)^2} + \frac{3nx+1}{2n^2}\int_0^y \frac{1}{(x-t)^2}d(-\omega_x(g_x,x-t)).$$

$$\int_0^y \frac{1}{(x-t)^2}d(-\omega_x(g_x,x-t)) = -\frac{\omega_x(g_x,x-y)}{(x-y)^2} + \frac{\omega_x(g_x,x)}{x^2} + \int_0^y \omega_x(g_x,x-t)\frac{2}{(x-t)^3}\,dt.$$

We have, from (23),

$$\left|\int_0^{x-x/\sqrt{n}} g_x(t)d_tK_{n,\alpha}(x,t)\right| \le \frac{3nx+1}{2n^2x^2}\omega_x(g_x,x) + \frac{3nx+1}{2n^2}\int_0^{x-x/\sqrt{n}}\omega_x(g_x,x-t)\frac{2}{(x-t)^3}\,dt.$$

Using the substitution $t = x - x/\sqrt{u}$ for the last integral, we get

$$\int_0^{x-x/\sqrt{n}}\omega_x(g_x,x-t)\frac{2}{(x-t)^3}\,dt = \frac{1}{x^2}\int_1^n \omega_x(g_x,x/x\sqrt{u})\,du \le \frac{1}{x^2}\sum_{k=1}^n \omega_x(g_x,x/\sqrt{k}).$$

Consequently,

$$(24)\quad \left|\int_{A_1} g_x(t)K_{n,\alpha}(x,t)\,dt\right| \le \frac{3nx+1}{2n^2x^2}\left(\omega_x(g_x,x)+\sum_{k=1}^n\omega_x(g_x,x/\sqrt{k})\right) \le \frac{3nx+1}{n^2x^2}\sum_{k=1}^n\omega_x(g_x,x/\sqrt{k}).$$

Using a similar method to estimate

$$\left|\int_{A_3} g_x(t)K_{n,\alpha}(x,t)\,dt\right|,$$

we get

$$\left|\int_{A_3} g_x(t)K_{n,\alpha}(x,t)\,dt\right| \le C_\alpha \frac{3\alpha+1}{nx}\sum_{k=1}^{n}\omega_x(g_x, x/\sqrt{k}), \tag{25}$$

where C_α is the constant in Lemma 2.

Finally, we estimate

$$\left|\int_{A_4} g_x(t)K_{n,\alpha}(x,t)\,dt\right|.$$

Since $f(t) = O(t^r)$, there exists a constant $M > 0$ such that $|f(t)| \le Mt^r$. Thus, we have

$$\begin{aligned}\left|\int_{A_4} g_x(t)K_{n,\alpha}(x,t)\,dt\right| &\le Mn\sum_{k=[2nx]}^{\infty} Q_{n,k}^{(\alpha)}(x)\int_{k/n}^{(k+1)/n} t^r\,dt\\ &= M\sum_{k=[2nx]}^{\infty} Q_{n,k}^{(\alpha)}(x)\frac{(k+1)^{r+1}-k^{r+1}}{(r+1)n^r}.\end{aligned}$$

By binomial expansion,

$$(k+1)^{r+1}-k^{r+1} = \sum_{i=0}^{r}\frac{(r+1)!}{i!(r+1-i)!}k^i.$$

If we take

$$M_r = \frac{M}{r+1}\sum_{i=0}^{r}\frac{(r+1)!}{i!(r+1-i)!},$$

then it follows that

$$M\sum_{k=[2nx]}^{\infty} Q_{n,k}^{(\alpha)}(x)\frac{(k+1)^{r+1}-k^{r+1}}{(r+1)n^r} \le M_r\sum_{k=[2nx]}^{\infty} Q_{n,k}^{(\alpha)}(x)(k/n)^r.$$

Now, by the results of Lemma 3 and [**7**, equation (11)], we obtain

$$\left|\int_{A_4} g_x(t)K_{n,\alpha}(x,t)\,dt\right| \le \frac{M(f,\alpha,r,x)}{n^l}, \tag{26}$$

where $M(f, \alpha, r, x)$ is a constant depending only on f, α, r, x. Theorem 1 now follows from (18)–(21) and (24)–(26), along with some simple computations.

REFERENCES

1. U. Abel and V. Gupta, *An estimate of the rate of convergence of a Bézier variant of the Baskakov-Kantorovich operators for bounded variation functions*, Demo. Math. **36** (2003), 124–136.

2. R. Bojanic and F. Cheng, *Rate of convergence of Bernstein polynomials of functions with derivative of bounded variation*, J. Math. Anal. Appl. **141** (1989), 136–151.

3. F.L. Cao and C.M. Ding, *Lp approximation by multivariate Baskakov-Kantorovich operators*, J. Math. Anal. Appl. **348** (2008), 856–861.

4. F. Cheng, *On the rate of convergence of Bernstein polynomials of functions of bounded variation*, J. Approx. Theor. **39** (1983), 259–274.

5. ———, *On the rate of convergence of Szász-Mirakyan operator for functions of bounded variation*, J. Approx. Theor. **40** (1984), 226–241.

6. S. Guo, Q. Qi and G. Liu, *The central approximation theorems for Baskakov-Bézier operators* J. Approx. Theor. **147** (2007), 112–124.

7. V. Gupta, *An estimate on the convergence of Baskakov-Bézier operators*, J. Math. Anal. Appl. **312** (2005), 280–288.

8. P. Pych-Taberska, *Some properties of the Bézier-Kantorovich type operators*, J. Approx. Theor. **123** (2003), 256–269.

9. Z. Walczak, *A note on the convergence of Baskakov type operators*, Appl. Math. Comp. **202** (2008), 370–375.

10. X.M. Zeng, *On the rate of two Bernstein-Bézier type operators for bounded variation functions* II, J. Approx. Theor. **104** (2000), 330–344.

11. X.M. Zeng and V. Gupta, *Rate of convergence of Baskakov-Bézier type operators for locally bounded functions*, Comp. Math. Appl. **44** (2002), 1445–1453.

12. X.M. Zeng and J.N. Zhao, *Exact bounds for some basis functions of approximation operators*, J. Ineq. Appl. **6** (2001), 563–575.

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