THE WEIGHTED WEAK LOCAL HARDY SPACES

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ABSTRACT. In this paper, we establish weighted atomic decomposition characterizations of weighted weak local Hardy spaces Wh_ω^p with local weights. As an application, we show that truncated Riesz transforms and pseudo-differential operators are bounded on weighted weak local Hardy spaces.

1. Introduction. The theory of local Hardy space plays an important role in various fields of analysis and partial differential equations, see [**6, 9, 10, 11, 15**]. Bui [**1**] studied the weighted version h_ω^p of the local Hardy space h^p considered by Goldberg [**6**], where the weights ω belong to the Muckenhoupt class. Recently, Rychkov [**9**] studied some properties of the weighted Besov-Lipschitz and Triebel-Lizorkin spaces with weights that are locally in A_p but may grow or decrease exponentially. In particular, Rychkov explicitly identifies weighted local Hardy space h_ω^p with $F_{p,2}^0(\omega)$ (see [**9**, Theorem 2.25]). Very recently, the author [**14**] established weighted atomic decomposition characterizations for weighted local Hardy space h_ω^p with local weights.

On the other hand, the weak H^1 space theory was first introduced by Fefferman and Soria in [**2**]. Then the weak $H^p(0 < p < 1)$ space theory was studied by Liu in [**8**]. About the local version, Li [**7**] gave an incomplete weak h^p space theory.

The main purpose of this paper is twofold. The first goal is to establish weighted atomic decomposition characterizations for weighted weak local Hardy space with local weights. The second goal is to show that truncated Riesz transforms and pseudo-differential operators are bounded on weighted weak local Hardy spaces.

The paper is organized as follows. In Section 2, we introduce some notation and properties concerning local weights and grand maximal

functions. In Section 3, we establish weighted atomic decomposition characterizations of weighted weak local Hardy space with local weights. Finally, in Section 4, we show that truncated Riesz transforms and pseudo-differential operators are bounded on weighted weak local Hardy spaces.

Throughout this paper, C denotes constants that are independent of the main parameters involved but whose value may differ from line to line. Let $\mathbf{N}$ denote the set $\{1, 2, \ldots\}$ and $\mathbf{N}_0$ denote the set $\mathbf{N} \cup \{0\}$. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Preliminaries. We first introduce weight classes A_p^{loc} from [9].

Let Q be any cube in $\mathbf{R}^n$ (here and below only cubes with sides parallel to the coordinate axes are considered), and let $|Q|$ denote the volume of Q. We define the weight class $A_p^{\mathrm{loc}}(1 < p < \infty)$ to consist of all nonnegative locally integrable functions ω on $\mathbf{R}^n$ for which

$$(2.1) \qquad A_p^{\mathrm{loc}}(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q \omega(x)\, dx \left(\int_Q \omega^{-p'/p}(x)\, dx\mathbf{R} \right)^{p/p'} < \infty,$$
$$1/p + 1/p' = 1.$$

The function ω is said to belong to the weight class of A_1^{loc} on $\mathbf{R}^n$ for which

$$(2.2) \qquad A_1^{\mathrm{loc}}(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q \omega(x)\, dx \left(\sup_{y \in Q} [\omega(y)]^{-1} \right) < \infty.$$

Remark. For any $C > 0$ we could have replaced $|Q| \leq 1$ by $|Q| \leq C$ in (2.1) and (2.2).

In what follows, $Q(x, t)$ denotes the cube centered at x and with sidelength t. Similarly, given $Q = Q(x, t)$ and $\lambda > 0$, we will write λQ for the λ-dilate cube, which is the cube with the same center x and with sidelength λt. Given a Lebesgue measurable set E and a weight ω, let $\omega(E) = \int_E \omega\, dx$. For any weight ω, L_ω^p and $L_\omega^{p;\infty}$ with $p \in (0, \infty)$ denote respectively the set of all measurable functions f such that

$$\|f\|_{L_\omega^p} \equiv \left(\int_{\mathbf{R}^n} |f(x)|^p \omega(x)\, dx \right)^{1/p} < \infty$$

and

$$\|f\|_{L_\omega^{p,\infty}} \equiv \left(\sup_{\lambda>0} \lambda^p \omega(\{x \in \mathbf{R}^n : |f(x)| > \lambda\})\right)^{1/p} < \infty.$$

We define the local Hardy-Littlewood maximal operator by

$$M^{\mathrm{loc}}f(x) = \sup_{x \in Q: |Q| \leq 1} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

Similarly to the classical A_p Muckenhoupt weights, we give some properties for weights $\omega \in A_\infty^{\mathrm{loc}} := \bigcup_{1 \leq p < \infty} A_p^{\mathrm{loc}}$.

Lemma 2.1. *Let $1 \leq p < \infty$, $\omega \in A_p^{\mathrm{loc}}$ and Q be a unit cube, i.e., $|Q| = 1$. Then there exists an $\overline{\omega} \in A_p$ so that $\overline{\omega} = \omega$ on Q and*

(i) $A_p(\overline{\omega}) \leq C A_p^{\mathrm{loc}}(\omega).$

(ii) *if $\omega \in A_p^{\mathrm{loc}}$, then there exists an $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}^{\mathrm{loc}}(\omega)$ for $p > 1$.*

(iii) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\mathrm{loc}} \subset A_{p_2}^{\mathrm{loc}}$.*

(iv) $\omega \in A_p^{\mathrm{loc}}$ *if and only if $\omega^{-1/(p-1)} \in A_{p'}^{\mathrm{loc}}$.*

(v) *If $\omega \in A_p^{\mathrm{loc}}$ for $1 \leq p < \infty$, then*

$$\omega(tQ) \leq exp(c_\omega t)\omega(Q) \quad (t \geq 1, |Q| = 1).$$

(vi) *The local Hardy-Littlewood maximal operator M^{loc} is bounded on L_ω^p if $\omega \in A_p^{\mathrm{loc}}$ with $p \in (1, \infty)$.*

(vii) M^{loc} *is bounded from L_ω^1 to $L_\omega^{1,\infty}$ if $\omega \in A_1^{\mathrm{loc}}$.*

Proof. (i)–(vi) have been proved in [**9**]. (vii) can be proved by the standard method. □

We remark that Lemma 2.1 is also true for $|Q| > 1$ with c depending now on the size of Q. In addition, it is easy to see that $A_p \subset A_p^{\mathrm{loc}}$ for $p \geq 1$ and $e^{c|x|}$, $(1 + |x| \ln^\alpha(2 + |x|))^\beta \in A_1^{\mathrm{loc}}$ with $\alpha \geq 0$, $\beta \in \mathbf{R}$ and $c \in \mathbf{R}$.

As a consequence of Lemma 2.1,

Corollary 2.1. *If* $\omega \in A_\infty^{\mathrm{loc}}$, *then there exists a constant* $C > 0$ *such that*

$$\omega(2Q) \le C\omega(Q), \quad \text{if } |Q| < 1,$$

and

$$\omega(Q(x_0, r+1)) \le C\omega(Q(x_0, r)), \text{ if } |Q(x_0, r)| \ge 1.$$

From Lemma 2.1, for any given $\omega \in A_q^{\mathrm{loc}}$, define the critical index of ω by

$$(2.3) \qquad\qquad q_\omega \equiv \inf\{q \in [1, \infty) : \omega \in A_q^{\mathrm{loc}}\}.$$

Obviously, $q_\omega \in [1, \infty)$. If $q_\omega \in (1, \infty)$, then $\omega \notin A_{q_\omega}^{\mathrm{loc}}$, but $\omega \in A_{q_\omega+\varepsilon}^{\mathrm{loc}}$ for any $\varepsilon > 0$.

The symbol $\mathcal{D}(\mathbf{R}^n) = C_0^\infty(\mathbf{R}^n)$. $\mathcal{D}'(\mathbf{R}^n)$ is the dual space of $\mathcal{D}(\mathbf{R}^n)$. The multi-index notation is usual: for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial^\alpha = (\partial/\partial_{x_1})^{\alpha_1} \cdots (\partial/\partial_{x_n})^{\alpha_n}$, and $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$.

Let $N \in \mathbf{N}_0$ and

$$\mathcal{M}_N^0 f(x) = \sup\{|\varphi_t * f(x)| : \ 0 < t < 1, \varphi \in \mathcal{D}(\mathbf{R}^n), \ \int \varphi \ne 0,$$
$$\operatorname{supp}\varphi \subset B(0,1), \|D^\alpha\varphi\|_\infty \le 1 \ |\alpha| \le N\}$$

and

$$\mathcal{M}_N f(x) = \sup\{|\varphi_t * f(z)| : \ |z - x| < t < 1, \varphi \in \mathcal{D}(\mathbf{R}^n), \ \int \varphi \ne 0,$$
$$\operatorname{supp}\varphi \subset B(0, 2^{3(10+n)}), \|D^\alpha\varphi\|_\infty \le 1 \ |\alpha| \le N\}.$$

For any $N \in \mathbf{N}_0$ and $x \in \mathbf{R}^n$, obviously,

$$\mathcal{M}_N^0 f(x) \le \mathcal{M}_N f(x).$$

For convenience, we write

$$\mathcal{D}_N^0 = \{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subset B(0,1), \ \int \varphi \ne 0, \ \|D^\alpha\varphi\|_\infty \le 1 \ |\alpha| \le N\},$$

and

$$\mathcal{D}_N = \{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subset B(0, 2^{3(10+n)}),$$

$$\int \varphi \neq 0, \ \|D^\alpha \varphi\|_\infty \leq 1 \ |\alpha| \leq N\}.$$

Proposition 2.1. *Let $N \geq 0$. Then*

(i) *there exists a positive constant C such that for all $f \in (L^1_{\operatorname{loc}}(\mathbf{R}^n)$ $\bigcap \mathcal{D}'(\mathbf{R}^n))$ and almost all $x \in \mathbf{R}^n$, $|f(x)| \leq \mathcal{M}^0_N f(x) \leq M^{\operatorname{loc}} f(x)$.*

(ii) *If $\omega \in A^{\operatorname{loc}}_q$ with $1 \leq q < \infty$, $N \geq [n(q_\omega/p-1)]+1$, and $p \in (0,1]$, then $\|\mathcal{M}^0_N f\|_{L^{p,\infty}_\omega(\mathbf{R}^n)} \sim \|\mathcal{M}_N f\|_{L^{p,\infty}_\omega(\mathbf{R}^n)}$.*

Proof. The proof of (i) is obvious. For (ii), it is easy to see that

$$(2.4) \qquad \|\mathcal{M}^0_N f\|_{L^{p,\infty}_\omega(\mathbf{R}^n)} \leq \|\mathcal{M}_N f\|_{L^{p,\infty}_\omega(\mathbf{R}^n)}.$$

Hence, it suffices to prove that there exists a positive constant C such that

$$(2.5) \qquad \|\mathcal{M}_N f\|_{L^{p,\infty}_\omega(\mathbf{R}^n)} \leq C\|\mathcal{M}^0_N f\|_{L^{p,\infty}_\omega(\mathbf{R}^n)}.$$

As in [**9**, page 176], we define the "tangential" maximal function by

$$\varphi^{**}_{0,A,B} f(x) = \sup_{y \in \mathbf{R}^n} \frac{|(\varphi_0)_j * f(x-y)|}{m_{j,A,B}(y)} \quad (j \in \mathbf{N}),$$

where

$$m_{j,A,B}(y) = (1 + 2^j|y|)^A 2^{|y|B} \ (A, B > 0)$$

and

$$\varphi_0 \in \mathcal{D}, \operatorname{supp}\varphi \subset B(0,1), \ \int \varphi \neq 0, \ (\varphi_0)_j(x) = 2^{jn}\varphi_0(2^j x).$$

From [**9**, (2.57), page 177], we know that for any $0 < r < 1$ and $A > n/r$, there exists a positive constant C such that

$$(2.6) \qquad \varphi^{**}_{0,A,B} f(x)^r \leq Cl(M^{\operatorname{loc}}(\varphi_0^+ f)^r)(x) + K_{Br}(\varphi_0^+ f)^r(x)\mathbf{R}),$$

where

$$\varphi_0^+ f(x) = \sup_{j \in \mathbf{N}} |(\varphi_0)_j * f(x)|, \ K_B f(x) = \int |f(y)|2^{-B|x-y|}\,dy.$$

From [**9**, (2.58), page 177], we know that

$$\mathcal{M}_N f(x) \leq C \varphi^{**}_{0,A,B} f(x). \tag{2.7}$$

Note that $\varphi_0^+ f(x) \leq \mathcal{M}_N^0 f(x)$, and by (2.6) and (2.7), to show (2.5) holds, it suffices to prove that, for $0 < r < p \leq 1$ and $q < p/r$, there exist constants A, B depending only on n, r, p, ω such that

$$\|(M^{\mathrm{loc}}(g^r))^{1/r}\|_{L^{p,\infty}_\omega(\mathbf{R}^n)} \leq C\|g\|_{L^{p,\infty}_\omega(\mathbf{R}^n)} \tag{2.8}$$

and

$$\|(K_{Br}(g^r))^{1/r}\|_{L^{p,\infty}_\omega(\mathbf{R}^n)} \leq C\|g\|_{L^{p,\infty}_\omega(\mathbf{R}^n)}. \tag{2.9}$$

We first prove (2.8). Indeed, for any $t > 0$, set $g := g_1 + g_2$ and $g_1(x) = g(x)$ if $|g(x)| \leq t/2$, otherwise is zero. Without loss of generality, we can assume that $q > 1$. By the weighted $L^q(\omega)$ boundedness of M^{loc} (see Lemma 2.1 (vi)) and the fact that $rq < p$, we see that

$$
\begin{aligned}
\omega(\{x \in \mathbf{R}^n &: (M^{\mathrm{loc}}(g^r)(x))^{1/r} > t\}) \\
&= \omega(\{x \in \mathbf{R}^n : (M^{\mathrm{loc}}(g_2^r)(x)) > (t/2)^r\}) \\
&\leq Ct^{-rq} \int_{\{x \in \mathbf{R}^n : |g(x)| \geq t/2\}} |g(x)|^{rq} \omega(x)\, dx \\
&\leq Ct^{-rq} \int_{t/2}^\infty \omega(\{x \in \mathbf{R}^n : |g(x)| > \lambda\})\, d\lambda^{rq} \\
&\quad + Ct^{-rq} \int_0^{t/2} \omega(\{x \in \mathbf{R}^n : |g(x)| > t/2\})\, d\lambda^{rq} \\
&\leq Ct^{-rq} \int_{t/2}^\infty \lambda^{rq-1-p} \lambda^p \omega(\{x \in \mathbf{R}^n : |g(x)| > \lambda\})\, d\lambda \\
&\quad + t^{-p}\|g\|^p_{L^{p,\infty}_\omega(\mathbf{R}^n)} \leq Ct^{-p}\|g\|^p_{L^{p,\infty}_\omega(\mathbf{R}^n)}.
\end{aligned}
$$

It remains to prove (2.9). From [**9**, Lemma 2.11], we know that

$$\|K_{Br}g\|_{L^q_\omega(\mathbf{R}^n)} \leq C\|g\|_{L^q_\omega(\mathbf{R}^n)} \tag{2.10}$$

if Br is taken to be sufficiently large.

Similarly to the proof of (2.8) and by (2.10), we can prove (2.9). The proof is complete. ∎

3. The decomposition theorem. Let $0 < p \leq 1$, $\omega \in A_q^{\text{loc}}$ with $1 \leq q < \infty$ and $N \geq [n(q_\omega/p - 1)] + 1$ with q_ω defined as in (2.3). The weighted local Hardy spaces h_ω^p are defined by $h_\omega^p = \{f \in \mathcal{D}' : M_N f \in L_\omega^p\}$ and $\|f\|_{h_\omega^p} = \|M_N f\|_{L_\omega^p}$, see [**9**, **14**]. Similarly, the weighted weak local spaces Wh_ω^p are defined by $Wh_\omega^p = \{f \in \mathcal{D}' : M_N f \in L_\omega^{p,\infty}\}$ and $\|f\|_{Wh_\omega^p} = \|M_N f\|_{L_\omega^{p,\infty}}$. In this section, we shall establish a decomposition theorem of weighted weak local spaces Wh_ω^p.

Theorem 3.1. *Given $f \in Wh_\omega^p$, $0 < p \leq 1$ and $\omega \in A_q^{\text{loc}}$ with $1 \leq q < \infty$, there exists a sequence of bounded functions $\{f_k\}_{k=k'}^{+\infty}$ with the following properties*:

(1) *$f = \sum_{k=k'}^{+\infty} f_k$ is in the sense of distributions.*

(2) *Each h_i^k can be further decomposed as $f_k = \sum_i h_i^k$, where h_i^k satisfies*

(a) *Each h_i^k is supported in a cube Q_i^k with $|Q_i^k| \leq 2$, $\sum_i \omega(Q_i^k) \leq A2^{-kp}$ and $\sum_i \chi_{Q_i^k}(x) \leq C$, where A is a constant depending on f and $\chi_{Q_i^k}$ denote the characteristic functions.*

(b) *$\|h_i^k\|_{L^\infty} \leq C2^k$ and $\int_{\mathbf{R}^n} h_i^k(x) x^\alpha dx = 0$ for $\alpha \in (\mathbf{N}_0)^n$ with $|\alpha| \geq [n(q_w/p - 1)] + 1$, if $|Q_i^k| < 1$.*

Conversely, if a distribution f satisfies (1) and (2), then $f \in Wh_\omega^p$.

Moreover, we have $\|f\|_{Wh_\omega^p}^p \sim A$.

Proof. Suppose $f \in Wh_\omega^p$. Set $\Omega_k = \{x \in \mathbf{R}^n : M_N f(x) > 2^k\}$. Let $\Omega_k = \bigcup_i Q_i^k = \bigcup_i Q(x_i^k, r_i^k)$ be the Whitney decomposition and $b_i^k := (f - P_i^k)\eta_i^k$ if $r_i^k < 1$ and $b_i^k := f\eta_i^k$ if $r_i^k \geq 1$ as in [**14**], where η_i^k is smooth function compacted in $Q_i^{k*} = (1 + 2^{-(10+n)})Q_i^k$. Then there exist positive constants C_1, C_2 so that for $i \in \mathbf{N}$,

$$\mathcal{M}_N^0(b_i^k)(x) \leq C_1 M_N f(x)\chi_{Q_i^{k*}} + C_1 \frac{(l_i^k)^{n+N}}{(r_i^k + |x - x_i^k|)^{n+N}}\chi_{\{|x-x_i^k|<C_2\}}(x),$$

where $N \geq [n(q_\omega/p - 1)] + 1$.

From this, it is easy to see that

$$\left\| \sum_i b_i^k \right\|_{h_\omega^{p_0}} \le C 2^{k(1-p/p_0)}, \qquad \frac{n}{n+N} < p_0 < p.$$

Hence, $\sum_i b_i^k$ converges in the sense of distributions. So we have the Calderón-Zygmund decomposition $f = \sum_{k=k'}^{+\infty}(g_k + \sum_i b_i^k)$, where $k' \in \mathbf{Z}$ such that $2^{k'-1} \le \inf_{x \in \mathbf{R}^n} \mathcal{M}_N f(x) < 2^{k'}$. If $\inf_{x \in \mathbf{R}^n} \mathcal{M}_N f(x) = 0$, write $k' = -\infty$, and $\|g_k\|_{L^\infty} \le C 2^k$.

Let $f_k = g_{k+1} - g_k$. Then $\sum_i \eta_i^k b_j^{k+1} = \chi_{\Omega_k} b_j^{k+1} = b_j^{k+1}$ for all j,

$$\begin{aligned}
g^{k+1} - g^k &= \left(f - \sum_j b_j^{k+1} \right) - \left(f - \sum_i b_i^k \right) \\
&= \sum_i b_i^k - \sum_j b_j^{k+1} \\
&= \sum_i \left[b_i^k - \sum_{j \in F_1^k} b_j^{k+1} \eta_i^k + \sum_{j \in F_2^k} b_j^{k+1} \eta_i^k \right] \\
&\equiv \sum_i h_i^k,
\end{aligned}$$

where $F_1^k = \{i \in \mathbf{N} : |Q_i^k| \ge 1\}$ and $F_2^k = \{i \in \mathbf{N} : |Q_i^k| < 1\}$ and all the series converges in $\mathcal{D}'(\mathbf{R}^n)$.

Set $\operatorname{supp} h_i^k \subset Q_i^k$. If $|Q_i^k| \le 1$, it is easy to see that h_i^k satisfies all the conditions in (b). If $|Q_i^k| > 1$, then we can decompose Q_i^k into a finite number of disjoint cubes $\{Q_{i,k}^j\}_{j=1}^{N_{ki}}$, each with sidelength lying between 1 and 2. Then, $\{h_i^k \chi_{Q_{i,k}^j}\}_{j=1}^{N_{ki}}$ also satisfy all the conditions in (b). Obviously, $f = \sum_{k=k'}^{+\infty} f_k$ in the sense of distributions.

For the converse, we fix $\alpha > 0$. Next we choose k_0 so that $2^{k_0} \le \alpha < 2^{k_0+1}$. Write $f = \sum_{k=k'}^{k_0} f_k + \sum_{k=k_0+1}^{+\infty} f_k := F_1 + F_2$. Without loss of generality, we always assume that $k' = -\infty$. Then we have

$$M_N^0 F_1(x) \le \alpha.$$

So we need only prove

$$\omega(\{x \in \mathbf{R}^n : M_N^0 F_2(x) > \alpha\}) \le C A \alpha^{-p}.$$

Let $Q_i^k = Q(x_i^k, r_k^i)$ and $\overline{Q}_i^k = Q(x_i^k, l_k^i)$, where

$$l_i^k = \min\{2(3/2)^{(k-k_0)p/(qn)} r_i^k, 4n\}.$$

Set

$$\Omega_{k_0} = \bigcup_{k=k_0+1}^{+\infty} \bigcup_i \overline{Q}_i^k.$$

Note that $\omega(\overline{Q}_i^k) \le C(3/2)^{(k-k_0)p} \omega(Q_i^k)$ by the properties of A_q^{loc}, so

$$\omega(\Omega_{k_0}) \le C \sum_{k=k_0+1}^{+\infty} \sum_i \left(\frac{3}{2}\right)^{(k-k_0)p} \omega(Q_i^k)$$

$$\le C \sum_{k=k_0+1}^{+\infty} 2^{-(k-k_0)p} \left(\frac{3}{2}\right)^{(k-k_0)p} 2^{-k_0 p} A$$

$$\le C\alpha^{-p} A.$$

Hence, it suffices to prove

$$(3.1) \qquad \int_{\Omega_{k_0}^c} (M_N^0 F_2(x))^p \omega(x)\, dx \le C\alpha^{-p} A.$$

To prove (3.1), we need to estimate $\int_{(\overline{Q}_i^k)^c} (M_N^0 h_i^k(x))^p \omega(x) dx$. Let $x \in (\overline{Q}_i^k)^c$; it is not difficult to make the following estimate

$$M_N^0 h_i^k(x) \le C2^k |Q_k^i|^{(n+N)/n} |x - x_k^i|^{-(n+N)} \chi_{\{|x-x_k^i|<4n\}}(x),$$

where $N \ge [n(q_\omega/p - 1)] + 1$

From this, and noting that $\omega \in A_{q_\omega+\varepsilon}^{\mathrm{loc}}$ for any $\varepsilon > 0$, we get

$$\int_{(\overline{Q}_i^k)^c} (M_N^0 h_i^k(x))^p \omega(x)\, dx$$

$$\le C \int_{(3/2)^{(k-k_0)p/(qn)} r_i^k < |x-x_i^k| < 4n} 2^{kp} \frac{|Q_k^i|^{p(n+N)/n}}{|x - x_k^i|^{p(n+N)}} \omega(x)\, dx$$

$$\le C2^{kp} \left(\frac{2}{3}\right)^{l_N(k-k_0)p/(nq)} \omega(Q_k^i),$$

where $l_N = p(n + N) - n(q_\omega + \varepsilon) > 0$ if ε is small enough.

Therefore,

$$\int_{\Omega_{k_0}^c} (M_N^0 F_2(x))^p \omega(x) \, dx \leq C \sum_{k=k_0+1}^{+\infty} 2^{kp} \sum_i \left(\frac{2}{3}\right)^{l_N(k-k_0)p/(nq)} \omega(Q_i^k)$$

$$\leq C\alpha^{-p} A.$$

Thus (3.1) holds. Theorem 3.1 is proved. $\square$

4. Applications. In this section, we shall show the boundedness for truncated Riesz transforms and pseudo-differential operators on the Wh_ω^p spaces.

Let Φ be a non-negative, radial and C^∞-function on $\mathbf{R}^n$ with compact support $B(0, 2)$ and $\Phi \equiv 1$ on $B(0, 1)$. Define the truncated Riesz transforms by

$$R_j f(x) = \int_{\mathbf{R}^n} K_j(x - y) f(y) \, dy,$$

$$K_j(z) = \frac{z_j}{|z|^{n+1}} \Phi(z), \quad j = 1, \dots, n.$$

Theorem A. *Let R_j be as above. Then:*

(i) $\|R_j f\|_{L_\omega^p(\mathbf{R}^n)} \leq C_{p,\omega} \|f\|_{L_\omega^p(\mathbf{R}^n)}$ *for $1 < p < \infty$ and $\omega \in A_p^{\mathrm{loc}}$.*

(ii) $\|R_j f\|_{L_\omega^{1,\infty}(\mathbf{R}^n)} \leq C_\omega \|f\|_{L_\omega^1(\mathbf{R}^n)}$ *for $\omega \in A_1^{\mathrm{loc}}$.*

The proof is obvious; see also [**9**].

Furthermore, we have the following result.

Theorem 4.1. *Let T be the truncated Riesz transforms operators, $\omega \in A_\infty^{\mathrm{loc}}$ and $0 < p \leq 1$. Then*

$$\|Tf\|_{Wh_\omega^p(\mathbf{R}^n)} \leq C_\omega \|f\|_{Wh_\omega^p(\mathbf{R}^n)}.$$

Proof. Indeed, it is sufficient to show

(4.1) $\qquad \omega(\{x \in \mathbf{R}^n : M_N^0(Tf)(x) > \alpha\}) \leq C\|f\|_{Wh_\omega^p}^p \alpha^{-p}.$

Let $f \in Wh_\omega^p$; by Theorem 3.1, we have

$$f = \sum_{k=k'}^{+\infty} f_k.$$

Without loss of generality, we always assume that $k' = -\infty$. We fix $\alpha > 0$, and choose k_0 so that $2^{k_0} \le \alpha < 2^{k_0+1}$. Write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{+\infty} f_k := F_1 + F_2.$$

For F_1, since $\omega \in A_\infty^{\mathrm{loc}}$ and thus $\omega \in A_q^{\mathrm{loc}}$ for some $1 < q < \infty$, we have

$$\|F_1\|_{L_\omega^q} \le C \sum_{k=-\infty}^{k_0} 2^k \omega(\Omega_k)^{1/q} \le C\|f\|_{Wh_\omega^p}^{p/q} \sum_{k=-\infty}^{k_0} 2^k$$
$$\le C\|f\|_{Wh_\omega^p}^{p/2} \alpha^{1-p/q}.$$

Therefore, since T is bounded on L_ω^q by Theorem A (i), we have

$$\omega(\{x \in \mathbf{R}^n : M_N^0(TF_2)(x) > \alpha\}) \le C\frac{\|TF_1\|_{L_\omega^q}^q}{\alpha^q} \le C\frac{\|F_1\|_{L_\omega^q}^q}{\alpha^q}$$
$$\le C\|f\|_{Wh_\omega^p}^p/\alpha^p.$$

So we need only prove that

$$\omega(\{x \in \mathbf{R}^n : M_N^0(TF_2)(x) > \alpha\}) \le C\|f\|_{Wh_\omega^p}^p/\alpha^p.$$

Let $Q_i^k = Q(x_i^k, r_k^i)$ and $\overline{Q}_i^k = Q(x_i^k, l_k^i)$, where

$$l_i^k = \min\{2(3/2)^{(k-k_0)p/(qn)} r_i^k, 4n\}.$$

Set

$$\Omega_{k_0} = \bigcup_{k=k_0+1}^{+\infty} \bigcup_i \overline{Q}_i^k.$$

Note that $\omega(\overline{Q}_i^k) \leq C(3/2)^{(k-k_0)p}\omega(Q_i^k)$ by the properties of A_q^{loc}, so

$$
\begin{aligned}
\omega(\Omega_{k_0}) &\leq C \sum_{k=k_0+1}^{+\infty} \sum_i \left(\frac{3}{2}\right)^{(k-k_0)p} \omega(Q_i^k) \\
&\leq C \sum_{k=k_0+1}^{+\infty} 2^{-(k-k_0)p} \left(\frac{3}{2}\right)^{(k-k_0)p} 2^{-k_0 p} \|f\|_{Wh_\omega^p}^p \\
&\leq C\|f\|_{Wh_\omega^p}^p / \alpha^p.
\end{aligned}
$$

Hence, it suffices to prove

$$
(4.2) \qquad \int_{\Omega_{k_0}^c} M_N^0(TF_2)(x)^p \omega(x)\,dx \leq C|f\|_{Wh_\omega^p}^p / \alpha^p.
$$

To prove (4.2), we need to estimate $\int_{(\bar{Q}_i^k)^c} M_N^0(Th_i^k)^p(x)\omega(x)\,dx$.

Suppose $h_i^k(x)$ is supported in a cube $Q_i^k = Q(x_i^k, r_i^k)$ with $r_i^k \leq 2$.

Case 1. $2 \geq r_i^k \geq 1$. This is a trivial case. Let $Q_{ki}^* = (8n)Q_i^k$. Now

$$
\begin{aligned}
\int_{(\bar{Q}_i^k)^c} M_N^0(Th_i^k)^p(x)\omega(x)\,dx &= \int_{Q_{ki}^* \setminus \bar{Q}_i^k} M_N^0(Th_i^k)^p(x)\omega(x)\,dx \\
&\quad + \int_{\mathbf{R}^n \setminus Q_{ki}^*} M_N^0(Th_i^k)^p(x)\omega(x)\,dx \\
&= \int_{Q_{ki}^* \setminus (\bar{Q}_i^k)} M_N^0(Th_i^k)^p(x)\omega(x)\,dx.
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
\int_{Q_{ki}^* \setminus (\bar{Q}_i^k)} &M_N^0(Th_i^k)^p(x)\omega(x)\,dx \\
&\leq C2^k \int_{(3/2)^{(k-k_0)p/(qn)} r_i^k \leq |x-x_i^k| < 8n} \omega(x)\,dx \\
&\leq C2^k \omega(Q_i^k) \leq C2^k \left(\frac{2}{3}\right)^{(k-k_0)p/(qn)} \omega(Q_i^k),
\end{aligned}
$$

since $0 \leq k - k_0 \leq nq \log_{3/2}(8n)$.

Case 2. $r_i^k < 1$. By the vanishing moment of h_i^k, we have

$$M_N^0(Th_i^k)(x) \leq \frac{Cr_i^k}{|x - x_i^k|^{N+n}}\chi_{\{|x-x_i^k|<8n\}}(x) \int_{Q_i^k} |h_i^k(y)|\, dy$$

$$\leq C2^k \frac{(r_i^k)^{N+n}}{|x - x_i^k|^{N+n}}\chi_{\{|x-x_i^k|<8n\}}(x),$$

where $N \geq [n(q_\omega/p - 1)] + 1$

From this, and the fact that $\omega \in A_{q_\omega+\varepsilon}^{\text{loc}}$ for any $\varepsilon > 0$, it is not difficult to see that

$$\int_{(\bar{Q}_i^k)^c} M_N^0(Th_i^k)^p(x)\omega(x)\, dx$$

$$\leq C2^{kp}(r_i^k)^{(N+n)p}$$

$$\times \int_{(3/2)^{(k-k_0)p/(qn)}r_i^k \leq |x-x_i^k| \leq 4n} \frac{\omega(x)}{|x - x_i^k|^{(N+n)p}}\, dx$$

$$\leq C2^{kp}\left(\frac{2}{3}\right)^{l_N(k-k_0)p/(qn)} \omega(Q_i^k),$$

where $l_N = (N + n)p - n(q_\omega + \varepsilon) > 0$ if ε is small enough. Hence,

$$\int_{\Omega_{k_0}^c} M_N^0(Th_i^k)^p(x)\omega(x)\, dx$$

$$\leq C\sum_{k=k_0+1}^{+\infty} 2^{kp} \sum_i \left(\frac{2}{3}\right)^{l_N(k-k_0)p/(qn)} \omega(Q_i^k)$$

$$\leq C\alpha^{-p}\|f\|_{Wh_\omega^p}^p.$$

Thus, (4.2) holds. Theorem 4.1 is proved. □

Next we show that the pseudo-differential operators are bounded on $Wh_\omega^p(\mathbf{R}^n)$, where the weight ω is in the weight class $A_p(\varphi)$ which is contained in A_p^{loc} for $1 \leq p < \infty$. Let us first introduce some definitions.

Let m be a real number. Following [14], a symbol in $S_{1,\delta}^m$ is a smooth function $\sigma(x,\xi)$ defined on $\mathbf{R}^n \times \mathbf{R}^n$ such that for all multi-indices α and β the following estimate holds:

$$|D_x^\alpha D_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|+\delta|\alpha|},$$

where $C_{\alpha,\beta} > 0$ is independent of x and ξ.

The operator T given by

$$Tf(x) = \int_{\mathbf{R}^n} \sigma(x,\xi) e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\,d\xi$$

is called a pseudo-differential operator with symbol $\sigma(x,\xi) \in S^m_{1,\delta}$, where f is a Schwartz function and $\widehat{f}$ denotes the Fourier transform of f.

In the rest of this section, we let $\varphi(t) = (1+t)^{\alpha_0}$ with $\alpha_0 > 0$.

A weight will always mean a positive function which is locally integrable. We say that a weight ω belongs to the class $A_p(\varphi)$ for $1 < p < \infty$, if there is a constant C such that for all cubes $Q = Q(x,r)$ with center x and sidelength r

$$\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega(y)\,dy\right)\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega^{-1/(p-1)}(y)\,dy\right)^{p-1} \leq C.$$

We also say that a nonnegative function ω satisfies the $A_1(\varphi)$ condition if there exists a constant C for all cubes Q

$$M_\varphi(\omega)(x) \leq C\omega(x), \quad \text{almost everywhere } x \in \mathbf{R}^n,$$

where

$$M_\varphi f(x) = \sup_{x \in Q} \frac{1}{\varphi(|Q|)|Q|}\int_Q |f(y)|\,dy.$$

Since $\varphi(|Q|) \geq 1$, so $A_p(\mathbf{R}^n) \subset A_p(\varphi)$ for $1 \leq p < \infty$, where $A_p(\mathbf{R}^n)$ denote the classical Muckenhoupt weights; see [**5**].

Remark. It is easy to see that if $\omega \in A_p(\varphi)$, then $\omega(x)\,dx$ may not be a doubling measure. In fact, let $\alpha_0 > 0$ and $0 \leq \gamma < \alpha_0$; it is easy to check that $\omega(x) = (1+|x|\log(1+|x|))^{-(n+\gamma)} \notin A_\infty(\mathbf{R}^n)$ and $\omega(x)\,dx$ is not a doubling measure, but $\omega(x) = (1+|x|\log(1+|x|))^{-(n+\gamma)} \in A_1(\varphi)$ provided that $\varphi(r) = (1 + r^{1/n})^\alpha$.

Obviously, $A_p(\varphi) \subset A_p^{\mathrm{loc}}$ for $1 \leq p < \infty$. Next we give some properties for weights $\omega \in A_\infty(\varphi) = \bigcup_{p \geq 1} A_p(\varphi)$.

Lemma 4.1. *For any cube $Q \subset \mathbf{R}^n$, then*

(i) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}(\varphi) \subset A_{p_2}(\varphi)$.*

(ii) $\omega \in A_p(\varphi)$ if and only if $\omega^{-1/(p-1)} \in A_{p'}(\varphi)$, where $1/p + 1/p' = 1$.

(iii) If $\omega \in A_p(\varphi)$ for $1 \le p < \infty$, then for any measurable set $E \subset Q$,

$$\frac{|E|}{\varphi(|Q|)|Q|} \le C \left(\frac{\omega(E)}{\omega(Q)} \right)^{1/p}.$$

Theorem B. Let T be the $S_{1,0}^0$ pseudo-differential operators. Then:

(i) $\|Tf\|_{L_\omega^p(\mathbf{R}^n)} \le C_{p,\omega} \|f\|_{L_\omega^p(\mathbf{R}^n)}$ for $1 < p < \infty$ and $\omega \in A_p(\varphi)$.

(ii) $\|Tf\|_{L_\omega^{1,\infty}(\mathbf{R}^n)} \le C_\omega \|f\|_{L_\omega^1(\mathbf{R}^n)}$ for $\omega \in A_1(\varphi)$.

(iii) $\|Tf\|_{h_\omega^p(\mathbf{R}^n)} \le C_\omega \|f\|_{h_\omega^p(\mathbf{R}^n)}$, for $\omega \in A_\infty(\varphi)$ and $0 < p \le 1$.

Lemma 7.3 and Theorem B can be found in [**13, 14**]. The following lemma was proved in [**6**].

Lemma 4.2. Let T be the $S_{1,0}^0$ pseudo-differential operators. If $\varphi \in \mathcal{D}$, then $T_t f = \varphi_t * Tf$ has a symbol σ_t which satisfies $D_x^\beta D_\xi^\alpha \sigma_t(x,\xi) \le C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|}$ and a kernel $K_t(x,z) = FT_\xi \sigma_t(x,\xi)$ which satisfies $|D_x^\beta D_z^\alpha K_t(x,z)| \le C_{\alpha,\beta}|z|^{-n-|\alpha|}$, where $C_{\alpha,\beta}$ is independent of t if $0 < t < 1$.

Theorem 4.2. Let T be the $S_{1,0}^0$ pseudodifferential operators. Then

$$\|Tf\|_{Wh_\omega^p(\mathbf{R}^n)} \le C_{p,\omega} \|f\|_{Wh_\omega^p(\mathbf{R}^n)}$$

for $\omega \in A_\infty(\varphi)$ and $0 < p \le 1$.

Proof. As in the proof in Theorem 4.1, we fix $\alpha > 0$ and choose integer k_0 so that $2^{k_0} \le \alpha < 2^{k_0+1}$. Write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{+\infty} f_k := F_1 + F_2.$$

For F_1, Since $\omega \in A_z \infty^{\mathrm{loc}}$ and thus $\omega \in A_q(\varphi)$ for some $1 < q < \infty$, we have

$$\|F_1\|_{L_\omega^q} \le C \sum_{k=-\infty}^{k_0} 2^k \omega(\Omega_k)^{1/q} \le C\|f\|_{Wh_\omega^p}^{p/q} \sum_{k=-\infty}^{k_0} 2^k$$

$$\le C\|f\|_{Wh_\omega^p}^{p/2} \alpha^{1-p/q}.$$

Therefore, since T is bounded on L^q_ω by Theorem B (i), we have

$$\omega(\{x \in \mathbf{R}^n : M^0_N(TF_2)(x) > \alpha\}) \le C \frac{\|TF_1\|^q_{L^q_\omega}}{\alpha^q} \le C \frac{\|F_1\|^q_{L^q_\omega}}{\alpha^q}$$
$$\le C\|f\|^p_{Wh^p_\omega}/\alpha^p.$$

So we need only prove

$$\omega(\{x \in \mathbf{R}^n : M^0_N(TF_2)(x) > \alpha\}) \le C\|f\|^p_{Wh^p_\omega(\mathbf{R}^n)}\alpha^{-p}.$$

Let $Q^k_i = Q(x^k_i, r^i_k)$ and $\overline{Q}^k_i = Q(x^k_i, l^i_k)$, where

$$l^k_i = 2(3/2)^{(k-k_0)p/(qn(1+\alpha_0))}r^k_i.$$

Set

$$\Omega_{k_0} = \bigcup_{k=k_0+1}^{+\infty} \bigcup_i \bar{Q}^k_i.$$

Note that $\omega(\overline{Q}^k_i) \le C(3/2)^{(k-k_0)p/(1+\alpha_0)}\omega(Q^k_i)$ by Lemma 4.1 (iii), so

$$\omega(\Omega_{k_0}) \le C \sum_{k=k_0+1}^{+\infty} \sum_i \left(\frac{3}{2}\right)^{(k-k_0)p/(1+\alpha_0)} \omega(Q^k_i)$$
$$\le C \sum_{k=k_0+1}^{+\infty} 2^{-(k-k_0)p} \left(\frac{3}{2}\right)^{(k-k_0)p/(1+\alpha_0)} 2^{-k_0 p}\|f\|^p_{Wh^p_\omega(\mathbf{R}^n)}$$
$$\le C\alpha^{-p}\|f\|^p_{Wh^p_\omega(\mathbf{R}^n)}.$$

Hence, it suffices to prove

$$(4.3) \qquad \int_{\Omega^c_{k_0}} M^0_N(TF_2)^p(x))\omega(x)\,dx \le C\alpha^{-p}\|f\|^p_{Wh^p_\omega(\mathbf{R}^n)}.$$

To prove (4.3), we need to estimate $\int_{(\overline{Q}^k_i)^c}(M^0_N h^k_i(x))^p\omega(x)\,dx$. Let $x \in (\overline{Q}^k_i)^c$; by Lemma 4.2, it is not difficult to make the following estimate:

$$(4.4) \qquad M^0_N h^k_i(x) \le C2^k|Q^k_i|^{(n+N)/n}|x - x^i_k|^{-(n+N)}, \quad \text{if } |Q^k_i| < 1,$$

where $N = [n(q/p)(1 + \alpha_0)] + 1$ and

$$(4.5) \qquad M_N^0 h_i^k(x) \le C_M 2^k |Q_i^k| |x - x_k^i|^{-M}, \quad \text{if } 1 \le |Q_i^k| \le 2$$

holds for any $M > 0$.

If $|Q_i^k| < 1$, let $l_{k,k_0} = (3/2)^{(k-k_0)p/(n(1+\alpha_0))}$, from (4.3) and by the properties of $A_1(\varphi)$, let $l_{k,k_0} = (3/2)^{(k-k_0)p/(n(1+\alpha_0))}$. We get

$$\int_{(\bar{Q}_i^k)^c} (M_N^0 h_i^k(x))^p \omega(x)\, dx$$

$$\le C \int_{(3/2)^{(k-k_0)p/(n(1+\alpha_0))} r_i^k < |x - x_i^k|} 2^{kp} \frac{|Q_k^i|^{p(n+N)/n}}{|x - x_k^i|^{p(n+N)}} \omega(x)\, dx$$

$$\le C 2^{kp} |Q_i^k|^{p(N+1)/n}$$

$$\times \sum_{j=1}^{\infty} (l_{k,k_0} 2^j r_i^k)^{-p(n+N)} \int_{|x - x_i^k| < l_{k,k_0} 2^j r_i^k} \omega(x)\, dx$$

$$\le C 2^{kp} \omega(Q_i^k) \sum_{j=0}^{j_0} (l_{k,k_0} 2^j)^{-p(N+n)+qn}$$

$$+ C 2^{kp} \omega(Q_i^k) \sum_{k=j_0}^{\infty} (l_{k,k_0} 2^j)^{-p(N+n)} (l_{k,k_0} 2^j r_i^k)^{qn(1+\alpha_0)}$$

$$\le C 2^{kp} (2/3)^{h_N(k-k_0)p/(n(1+\alpha_0))} \omega(Q_k^i);$$

here integer j_0 satisfies $2^{j_0} \le l_{k,k_0} r_i^k \le 2^{j_0+1}$ and $h_N = p(N + n) - qn(1 + \alpha_0) > 0$.

If $1 \le |Q_i^k| \le 2$, from (4.4) and by the properties of $A_q(\varphi)$, we obtain

$$\int_{(\bar{Q}_i^k)^c} (M_N^0 h_i^k(x))^p \omega(x)\, dx$$

$$\le C \int_{(3/2)^{(k-k_0)p/(n(1+\alpha_0))} r_i^k < |x - x_i^k|} 2^{kp} \frac{|Q_k^i|^p}{|x - x_k^i|^{Mp}} \omega(x)\, dx$$

$$\le C 2^{kp} \left(\frac{2}{3}\right)^{(Mp-qn(1+\alpha_0))(k-k_0)p/(n(1+\alpha_0))} \omega(Q_k^i),$$

taking $M \ge N + 1 + n$.

Hence,

$$\int_{\Omega_{k_0}^c} M_N^0(TF_2)^p(x)\omega(x)\,dx$$

$$\leq C \sum_{k=k_0+1}^{+\infty} 2^{kp} \sum_i \left(\frac{2}{3}\right)^{h_N(k-k_0)p/(n(1+\alpha_0))} \omega(Q_i^k)$$

$$\leq C\alpha^{-p}\|f\|_{Wh_\omega^p}^p.$$

Thus (4.3) holds. The proof is finished. □

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