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SUMMABILITY OF SUBSEQUENCES OF A DIVERGENT SEQUENCE

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ABSTRACT. In this paper we show that no regular matrix can sum all subsequences of a divergent sequence. We also show that the Cesaro matrix cannot sum almost all subsequences of a divergent sequence. These results can be viewed as generalizations of a well-known result of Steinhaus.

A well-known result due to Steinhaus states that no regular matrix can sum all sequences of 0's and 1's (see **[3]** or [**4,** Theorem 6, page 133]). At the International Conference on Summability in Jacksonville, Florida, in October, 2010, Professor Kazim Khan of Kent State University raised the question of whether the Cesaro matrix can sum almost all subsequences of a divergent sequence. The purpose of this paper is to show that the answer is no. We also show that any regular matrix cannot sum all subsequences of a divergent sequence.

Recall that an infinite matrix $A = (a_{ij})$ is *regular* if A preserves limits for convergent sequences. That is, if a sequence $x \to L$ then $Ax \to L$. The Cesaro matrix C_1 is certainly the most famous example of a regular matrix.

To show that no regular matrix can sum every subsequence of a divergent subsequence, we first need to observe that this is obviously true for an unbounded sequence, since we can extract subsequences that grow arbitrarily rapidly. Secondly, no regular matrix can sum all subsequences of a divergent sequence of 0's and 1's. This is so because *any* sequence of 0's and 1's can be obtained as a subsequence, and so Steinhaus's theorem applies. To see this, let x be a divergent sequence with range $\{0, 1\}$. To construct a subsequence y with support $S \subset \mathbf{N}$, choose an increasing sequence of integers (i_k) so that $x_{i_k} = 0$ for $i_k \notin S$ and $x_{i_k} = 1$ for $i_k \in S$. This is possible because there are infinitely many 0's and 1's in x beyond any fixed integer.

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So now let us consider an arbitrary bounded divergent sequence x. The method used to show the desired result is to prove that a sequence y can be constructed as a subsequence of x that is arbitrarily close to any fixed sequence of 0's and 1's. We can then use the method of proof used in Steinhaus's result to contradict the assumption that A is regular.

The following theorem is known as the Silverman-Toeplitz theorem and gives necessary and sufficient conditions for a matrix to be regular.

Theorem 1 ([**4,** Theorem 5, page 131]). *Let A be an infinite matrix. Then A is regular if and only if*

(i) $\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$.

(ii) $\lim_i a_{ij} = 0$ *for all* j.

(iii) $\lim_i \sum_{j=1}^{\infty} a_{ij} = 1$.

So now let A be a regular matrix with non-negative entries, and let $x = (x_i)$ be a bounded divergent sequence in the convergence domain of A, denoted c_A. We show in the following proposition that, given any sequence b of 0's and 1's, we can construct a sequence from a linear combination of subsequences of x that is within ε of b with respect to the supremum norm. So, if c_A contains all subsequences of x, then it must contain the constructed sequence, since c_A is a linear space.

Proposition 2. *Let $S \subset \mathbf{N}$, and let $b = \chi(S)$ be the characteristic function of S. Fix $\varepsilon > 0$. There exists a sequence u that is a linear combination of subsequences of x satisfying* $\operatorname{sppt}(u) = S$ *and* $\|u - b\|_\infty < \varepsilon$.

Proof. Since x is divergent and bounded, the range of x has at least two distinct limit points, L and M. Without loss of generality, we can assume that $L > M$ and that $\varepsilon \ll L - M$. We can find subsequences (y_n) and (z_n) of x such that $y_n \to L$, $z_n \to M$ and $|y_n - L| < \frac{\varepsilon(L-M)}{2}$, $|y_n - L| < \frac{\varepsilon(L-M)}{2}$ for all n. We can construct a new subsequence (w_n) as a linear combination of subsequences of (y_n) and (z_n) with $w_n = y_n$ for $n \in S$ and $w_n = z_n$ for $n \notin S$. Finally, let $u_n = \frac{1}{L-M}(w_n - z_n)$.

Clearly, $\operatorname{sppt}(u) = S$, where $\operatorname{sppt}(u)$ denotes the support of u. For $n \in S$, $|w_n - z_n - (L-M)| \le |w_n - L| + |z_n - M| < \varepsilon(L-M)$. Therefore, $|u_n - 1| < \varepsilon$ for all $n \in S$. Since we have used linear combinations of subsequences of x in the construction of $u = (u_n)$, $u \in c_A$. □

In the main result, we make use of the following special case of the basic matrix theorem (BMT). This theorem is used extensively in [**4**] and other publications to prove classical results, such as the uniform boundedness principle and the Orlicz-Pettis theorem. It can be viewed as an abstraction of the well-known "gliding hump" method employed in functional analysis, summability and measure theory.

Theorem 3 ([**4,** Theorem 2, page 92]). *Let $M = (m_{ij})$ be an infinite matrix of real numbers that satisfies*

(1) $\lim_i m_{ij} = 0$.

(2) *For every subsequence (j_k) of integers there is a further subsequence (l_k) such that $\lim_i \sum_k m_{il_k}$ exists and forms a convergent sequence.*

Then $\lim_i m_{ij} = \lim_j m_{ij}$ and $\lim_i m_{ii} = 0$.

Lemma 4. *Let $a \in l^1$. There exists a finite set F such that $|\sum_{i\in F} a_i| \ge \frac{\|a\|_1}{3}$, and a_i is always positive or always negative for $i \in F$.*

Proof. Let $a^+ = \{i : a_i > 0\}$ and $a^- = \{i : a_i < 0\}$. Either $|\sum_{i\in a^+} a_i| \ge \frac{\|a\|_1}{2}$ or $|\sum_{i\in a^-} a_i| \ge \frac{\|a\|_1}{2}$. Assume the former inequality. Then, for any $\varepsilon > 0$, there exists a finite set $F \subset a^+$ such that $|\sum_{i\in F} a_i| > \frac{\|a\|_1}{2} - \varepsilon$. In particular, $|\sum_{i\in F} a_i| \ge \frac{\|a\|_1}{3}$. □

Theorem 5. *A matrix A that sums all subsequences of a bounded divergent sequence cannot be regular.*

Proof. By the Silverman-Toeplitz conditions, each row of A, which we denote a^i, is in l^1 and $\|a^i\|_1 \to 1$ as $i \to \infty$. So we can find a subsequence of (a^i), which for simplicity in what follows we will still denote as (a^i), such that $\|a^i\| > \frac{1}{2}$. By the preceding lemma, we can find a finite subset F_1 for which $|\sum_{j\in F_1} a_j^{i_1}| > \frac{1}{3}$, and $a_j^{i_1}$ is always positive or always negative on F_1. By the Silverman-Toeplitz theorem, the columns of A go to 0, so we can find $i_2 > i_1$ and F_2 with $\min(F_2) > \max(F_1)$ so that $|\sum_{j\in F_2} a_j^{i_2}| > \frac{1}{3}$ and $a_j^{i_2}$ is always positive or always negative on F_2. Proceeding inductively, we can then find an increasing sequence of integers (i_k) and of finite sets (F_k) of $\mathbf{N}$ so that $|\sum_{j\in F_k} a_j^{i_k}| > \frac{1}{3}$ and $a_j^{i_k}$ is always positive or always negative on F_k.

Let $S = \cup_k F_k$ and $b = \chi(S)$. As in Proposition 1.3 above, we can find $u \in c_A$, $\operatorname{sppt}(u) = S$ and $\|u-b\|_\infty < \varepsilon$. To use the BMT, let $m_{kl} = \sum_{j\in F_l} a_j^{i_k} u_j$. We show $M = (m_{kl})$ satisfies the conditions in the BMT.

Condition 1 of the basic matrix theorem is satisfied because, for each fixed l, $a_j^{i_k} \to 0$ as $k \to \infty$ for all $j \in F_l$, by assumption (ii) in Theorem 1. Since $u \in C_A$, $Au = \sum_{j=1}^\infty a_j^i u_j$ converges as $i \to \infty$. So,

$$\sum_{l=1}^{\infty}\left(\sum_{j\in F_l} a_j^{i_k} u_j\right) = \lim_{n\to\infty}\sum_{l=1}^{n}\sum_{j\in F_l} a_j^{i_k} u_j = \left(\sum_{j=1}^{\infty} a_j^{i_k} u_j\right)$$

is a convergent sequence. Therefore, the matrix M satisfies the BMT and $M_{kk} = \sum_{j\in F_k} a_j^{i_k} u_j \to 0$. This contradicts the fact that $\|a^i\|_1 \to 1$ for regular matrices. $\square$

We now show that the Cesaro matrix C_1 cannot sum almost every subsequence of a sequence of 0's and 1's. Recall that $C_1 = (c_{mn})$ is defined as

$$c_{mn} = \begin{cases} \frac{1}{m} & m \geq n \\ 0 & m < n. \end{cases}$$

We also need to define the phrase "almost every." Given a set $S \subset N$, the *density* of S is defined to be $d(S) = \frac{1}{n}\overline{\lim}_{n\to\infty}|\{i : i \in S \text{ and } i \leq n\}|$, where $|\bullet|$ denotes the cardinality of the set. A property holds for almost every subsequence of a given sequence if it holds for all the subsequences that have index sets with positive density.

We can now prove the following:

Proposition 6. *C_1 cannot sum almost every subsequence of a divergent sequence of 0's and 1's.*

Proof. Let b denote the fixed divergent sequence of 0's and 1's. The goal is to produce a subsequence that has positive density and for which the averages of the partial sums of the subsequence oscillate, and so the subsequence cannot be C_1-summable. Without loss of generality, assume $d(b^1) \geq \frac{1}{2}$, where $b^1 = \operatorname{sppt}(b)$ and $b^0 = \{i : b_i = 0\}$. Then there exists an integer n_1 such that $\frac{1}{n_1}|\{i : i \in b^1 \text{ and } i \leq n_1\}| \geq \frac{1}{3}$. Let $F_1 = \{i : i \in b^1 \text{ and } i \leq n_1\}$. Choose $n_2 > n_1$ such that $|b^0 \cap \{n_1+1, \ldots, n_2\}| = |F_1|$, and let $F_2 = b^0 \cap \{n_1+1, \ldots, n_2\}$. To construct F_3, choose n_3 such that $\frac{1}{n_3}|\{i : i \in b^1 \text{ and } i \leq n_3\}| \geq \frac{1}{3}$ and

$$\frac{1}{|\cup_{j=1}^{3} F_j|} \sum_{i \in \cup_{j=1}^{3} F_j} b_i < \frac{1}{4}.$$

Continuing in this fashion, we can construct a sequence (F_n) of finite sets of integers for which $S = \cup_n F_n = (i_j)$ has density at least $\frac{1}{3}$, but for which the sequence $(\frac{1}{k}\sum_{j=1}^{k} b_{i_j})$ oscillates between values of at least $\frac{1}{3}$ and no more than $\frac{1}{4}$ and so is not C_1-summable. $\square$

We now show the more general result: that C_1 cannot sum almost every subsequence of a bounded divergent sequence, denoted by (x_n).

Proposition 7. *C_1 cannot sum almost every subsequence of a bounded divergent sequence, denoted by (x_n).*

Proof. For any $\varepsilon > 0$, we can choose a limit point A of (x_n) such that $S = \{n : |x_n - A| < \varepsilon\}$ has positive density. Let $S = (m_k)$. Choose another limit point B of (x_n) such that $|A - B| > 3\varepsilon$. Such a B must exist for some $\varepsilon > 0$, since, if not, *all* limit points of (x_n) would be within 6ε of each other for any $\varepsilon > 0$, which would imply that (x_n) converges.

We can choose a subsequence (x_{n_k}) of (x_n) for which $|x_{n_k} - B| < \varepsilon$, so there is a gap of at least ε between the values of (x_{m_k}) and (x_{n_k}). Let $T = (n_k)$. Without loss of generality, we can assume $A > B$. We can now proceed as in the previous proposition to construct a subsequence of (x_n) that is not C_1-summable and has an index set with positive density.

Let $d(S) = D > 0$. Then there exists an integer n_1 such that

$$\frac{1}{n_1} |\{i : i \in S \text{ and } i \le n_1\}| \ge \frac{D}{2}.$$

Let $F_1 = \{i : i \in S \text{ and } i \le n_1\}$. Note that $\frac{1}{|F_1|} \sum_{i \in F_1} x_i > A - \varepsilon$. Choose $n_2 > n_1$ such that

$$\frac{1}{|F_1 \cup T \cap \{n_1+1,\dots,n_2\}|} \sum \{x_i : i \in F_1 \cup T \cap \{n_1+1,\dots,n_2\}\} < B + \frac{4\varepsilon}{3},$$

which is possible if we add on enough terms from T, since $x_i < B + \varepsilon$ for $i \in T$. Let $F_2 = T \cap \{n_1+1,\dots,n_2\}$. Then, choose $n_3 > n_2$ such that

$$\frac{1}{n_3} |\{i : i \in S \text{ and } n_2 < i \le n_3\}| \ge \frac{D}{2}$$

and

$$\frac{1}{|F_1 \cup F_2 \cup S \cap \{n_2+1,\dots,n_3\}|} \sum \{x_i : i \in F_1 \cup F_2 \cup S \cap \{n_2+1,\dots,n_3\}\} > A - \frac{4\varepsilon}{3}.$$

Continuing in this fashion, we can construct a sequence (F_n) of finite sets of integers for which $W = \cup F_n = (i_j)$ has density at least $\frac{D}{2}$ and for which the sequence $(\frac{1}{k}\sum_{j=1}^{k} x_{i_j})$ oscillates between values greater than $A - \frac{4}{3}\varepsilon$ and less than $B + \frac{4}{3}\varepsilon$. Since $A - B \ge 3\varepsilon$, this shows that (x_{i_j}) is not C_1-summable. $\square$

It seems likely that this proposition could be generalized for any regular matrix, but we do not have a proof of this.

Addendum. Professor Cihan Orhan of Ankara, Turkey, has recently informed the author that Theorem 5 of this paper is included in results of C.R. Buck. Please see [**1**] and [**2**].

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