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COMMON CYCLIC VECTORS FOR DIAGONAL OPERATORS ON THE SPACE OF ENTIRE FUNCTIONS

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ABSTRACT. In this paper, a unicity theorem for Borel series is obtained and used to show that the collection of cyclic operators acting on the space of entire functions with non-dense eigenvalues and having the monomials z^n as eigenvectors has a dense set of common cyclic vectors.

1. Introduction. The purpose of this paper is to deduce a unicity theorem for Borel series and to use this result to show that various collections of cyclic operators having a complete set of common eigenvectors have a common dense set of cyclic vectors.

Recall that an operator (i.e, continuous linear map) $T : \mathcal{X} \to \mathcal{X}$ acting on a complete metrizable topological vector space $\mathcal{X}$ is *cyclic* if there exists a vector $x \in \mathcal{X}$ whose orbit $x, Tx, T^2x, T^3x, \ldots$ has dense linear span in $\mathcal{X}$. Any such vector x, if it exists, is called a *cyclic vector* for T. Examples include the so-called diagonal operators $D : \mathcal{H} \to \mathcal{H}$ acting on a Hilbert space $\mathcal{H}$ for which there exist an orthonormal basis $\{e_n : n \geq 0\}$ for $\mathcal{H}$ and a bounded sequence of complex numbers $\{\lambda_n : n \geq 0\}$ for which $D(e_n) = \lambda_n e_n$ for all $n \geq 0$. A diagonal operator D with eigenvalues $\{\lambda_n\}$ is cyclic if and only if its eigenvalues are distinct (see, for instance, [**31**, page 723, Lemma 1]). Moreover, a vector $x \equiv \sum_{n=0}^{\infty} a_n e_n \in \mathcal{H}$ is cyclic for D if and only if the only functional L annihilating every vector $D^k x$ in the orbit of x is the zero functional. Since every functional L on a Hilbert space $\mathcal{H} \approx \mathcal{H}^*$ is given by $L(\sum_{n=0}^{\infty} b_n e_n) \equiv \sum_{n=0}^{\infty} b_n l_n$ where $\{l_n\}$ is in ℓ^2 (and conversely), it follows that $x \equiv \sum_{n=0}^{\infty} a_n e_n$ is cyclic for D if and only if the conditions $\{l_n\} \in \ell^2$ and $0 \equiv L(D^k x) = L(\sum_{n=0}^{\infty} a_n \lambda_n^k e_n) = \sum_{n=0}^{\infty} a_n l_n \lambda_n^k$ for all $k \geq 0$ together imply that $l_n \equiv 0$ for all $n \geq 0$. It follows that a diagonal operator with eigenvalues $\{\lambda_n\}$ has a non-cyclic vector $\sum_{n=0}^{\infty} a_n e_n$ where $a_n \neq 0$ for all $n \geq 0$ if and only if the *Moment*

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condition:

$$\sum_{n=0}^{\infty} w_n \lambda_n^k \equiv 0 \quad \text{holds for all } k \geq 0,$$

where $\{w_n\}$ is a non-trivial sequence in ℓ^1 (see [**22**, page 604, Proposition 5]). It may be tempting to believe that no such sequence $\{w_n\}$ exists. However, the following example due to Wolff [**42**] from 1921 shows that this need not be the case for certain sequences $\{\lambda_n\}$ (see [**24**, page 107] for the elegant proof).

Wolff's example. Let $\{B(\lambda_n, r_n)\}$ be any sequence of disjoint open balls $B(\lambda_n, r_n) \equiv \{z \in \mathbf{C} : |z - \lambda_n| < r_n\}$ in the open unit ball $B(0,1) \equiv \{z \in \mathbf{C} : |z| < 1\}$ for which the Lebesgue area measure of $B(0,1) \backslash \cup_{n=0}^{\infty} B(\lambda_n, r_n)$ is zero. If $\lambda_n \neq 0$ for all n, then $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ where the sequence $\{w_n\} \equiv \{r_n \lambda_n\} \in \ell^1$ is not identically zero. Since every sequence in ℓ^1 factors as the product of sequences in ℓ^2, we have that $w_n = a_n l_n$ where $\{a_n\} \in \ell^2$ and $\{l_n\} \in \ell^2$. Hence, Wolff's example yields a cyclic diagonal operator D on a Hilbert space having eigenvalues $\{\lambda_n\}$ and with non-cyclic vectors $\widehat{x} \equiv \sum a_n e_n$ even though $a_n \neq 0$ for all $n \geq 0$. Moreover, the closed linear span $\mathcal{M} \equiv \bigvee \{D^k \widehat{x}\}$ of the orbit of any such non-cyclic vector $\widehat{x}$ is invariant for D but does not equal the closed linear span of the eigenvectors that it contains; that is to say, D fails spectral synthesis, in view of which, questions about cyclic vectors and invariant subspaces of diagonal operators D on a Hilbert space and analytic function theory are intimately related to the moment condition $\sum_{n=0}^{\infty} w_n \lambda_n^k \equiv 0$ holding for all $k \geq 0$.

There are numerous conditions known to be equivalent to the Moment condition $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ holding for all $k \geq 0$ whenever $\{\lambda_n\}$ is a bounded sequence of distinct complex numbers. For instance, it follows from the Fubini-Tonelli theorem that $0 \equiv \sum_{n=0}^{\infty} w_n/(z - \lambda_n) = \sum_{k=0}^{\infty} [1/z^{k+1} \sum_{n=0}^{\infty} w_n \lambda_n^k]$ whenever $|z| > \sup |\lambda_n|$. Hence, $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ if and only if $0 \equiv \sum_{n=0}^{\infty} w_n/(z - \lambda_n)$ whenever $|z| > \sup |\lambda_n|$. Moreover, $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ if and only if the Dirichlet series $g(z) \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ vanishes identically on the complex plane (since $g \equiv 0$ if and only if $0 \equiv g^{(k)}(0) = \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$). This, in turn, is equivalent to the measure

$\mu \equiv \sum_{n=0}^{\infty} w_n \delta_{\{\lambda_n\}}$ (the sum of weighted point masses) annihilating the monomials (since $\int z^k d\mu = \sum_{n=0}^{\infty} w_n \lambda_n^k$). If the points $\{\lambda_n\}$ lie in a Jordan region Ω and accumulate only on its boundary, then $\sum_{n=0}^{\infty} w_n/(z-\lambda_n) \equiv 0$ whenever $|z| > \sup|\lambda_n|$ where $\{w_n\}$ is a non-trivial sequence in ℓ^1 if and only if $\{\lambda_n\}$ is a dominating set for Ω, that is, if and only if $\sup\{|f(z)| : z \in \Omega\} = \sup\{|f(\lambda_n)| : n \geq 0\}$ for all functions f bounded and analytic on Ω (see [**4**, page 167, Theorem 3]). If Ω is the open unit disc, then this condition is equivalent to almost every point of the unit circle (with respect to Lebesgue arc length measure) being the non-tangential limit point of $\{\lambda_n\}$. Deep connections to operator theory are provided by the work of Sarason [**28, 29**] who shows that the Borel series $\sum_{n=0}^{\infty} w_n/(z-\lambda_n) \equiv 0$ whenever $|z| > \sup|\lambda_n|$ for some non-trivial sequence $\{w_n\}$ in ℓ^1 if and only if there exists a closed invariant subspace for the diagonal operator D having eigenvalues $\{\lambda_n\}$ which is not invariant for the adjoint D^* of D. This condition, in turn, is equivalent to the weakly closed algebra generated by D and the identity operator not containing D^*. For more on the connections between Borel series and complete normal operators, please see Wermer [**40**], Scroggs [**30**] and Nikolskii [**23, 24**].

The study of Borel series has a rich and fabled history. Of particular interest has been conditions for a function analytic on a region to be representable as a Borel series, and conditions for such a representation, if one exists, to be unique. In particular, the seminal work of Leontev [**16, 17**], Korobeinik [**12–15**], Leont'eva [**18–20**] and Brown, Shields, and Zeller [**4**], amongst others, has examined the extent to which the existence of non-trivial expansions of zero by Dirichlet series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0$ on regions Ω in the complex plane imply (and, under additional conditions, is equivalent to) the ability to represent an arbitrary function $f(z)$ analytic on Ω as a Dirichlet series $f(z) = \sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ on Ω. It follows from the preceding comments that the non-uniqueness of any such representation is equivalent to the existence of the Borel series which vanish identically on Ω. In addition to Wolff's example [**42**], Denjoy [**6**] in 1924 and Leont'eva [**18, 19**] in the late 1960's gave examples of Borel series which vanish identically where the coefficients satisfy various decay rates just shy of exponential decay (see [**26**, page 26]), and in 1959, Makarov [**21**] showed that, for every sequence of complex numbers $\{\lambda_n\}$ for which $|\lambda_n| \to \infty$, there exists a sequence of complex numbers $\{w_n\}$ for which the moment condition $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ holds for all $k \geq 0$ where the coefficients $\{w_n\}$ satisfy the decay rate $0 < \sum_{n=0}^{\infty} |w_n| \cdot |\lambda_n^k| < \infty$.

There has also been particular interest regarding the converse, namely, the so-called unicity problem, which is to determine the rate at which $|w_n|$ must decrease so that $\sum_{n=0}^{\infty} w_n/(z-\lambda_n)$ does not extend analytically to a region containing $\{\lambda_n\}$. Beurling [**2**], Borel [**3**], Carleman [**5**], Gonchar [**10**] and Poincare all determined decay rates in the unicity problem in their investigations on Borel series, which were focused mainly on issues regarding quasianalyticity and analytic continuation. For more on the history of Borel series and a discussion of generalized analytic continuation, please see the recent monograph of Ross and Shapiro [**26**]. The following rather definitive unicity result was obtained by Sibilev in 1995 for the case $\{\lambda_n\}$ bounded (see [**38**, page 146, Theorem]).

Theorem 1. *Let $\{\varepsilon_n\}$ be any sequence of positive numbers which decreases monotonically to zero. Then the relations $\sum A_k/(z-\lambda_k) \equiv 0$ for $|z| > 1$ and $|A_k| \leq \text{constant} \cdot \varepsilon_k$ imply $A_k \equiv 0$ whenever $\{\lambda_n\}$ is a bounded sequence of distinct complex numbers if and only if $\sum (\log \varepsilon_k)/k^2 = \infty$.*

The main results of this paper, that certain collections of cyclic operators acting on spaces of functions analytic in a region in the complex plane have dense sets of common cyclic vectors, occur in Section 2 (see Theorems 2 and 3). In our setting, the underlying spaces are no longer Hilbert spaces, but are examples of complete locally convex topological vector spaces (see Section 2). Moreover, the eigenvalues $\{\lambda_n\}$ of the operators in question need only satisfy the condition $\limsup |\lambda_n|^{1/n} < \infty$; in particular, the collection of eigenvalues of such an operator may be unbounded. In the Appendix, we deduce a unicity theorem for the Borel series $\sum_{n=0}^{\infty} a_n/(z-\lambda_n)$ where $\{\lambda_n\}$ is any sequence of distinct complex numbers for which $\limsup |\lambda_n|^{1/n} < \infty$ and $\overline{\{\lambda_n\}} \neq \mathbf{C}$ (see Theorem 4). This result is inspired by the work of Sibilev (as well as Beurling, Korobeinik, Leontev and Makarov) and relies heavily on his techniques. It is used in Section 2 to deduce the main results of this paper, namely Theorems 2 and 3.

2. Common cyclic vectors. In this section, we use the unicity theorem appearing in the Appendix to show that various collections of cyclic operators have a common dense set of cyclic vectors. The general setting concerns collections of operators $T : \mathcal{X} \to \mathcal{X}$ acting on a complete metrizable topological vector space $\mathcal{X}$ having a common set $\{x_n\}$ of eigenvectors which are complete (that is, whose closed linear span is all of $\mathcal{X}$). For every operator T in such a collection, there exists a sequence of complex numbers $\{\lambda_n\}$ (depending on T) for which $T(x_n) = \lambda_n x_n$ for all $n \geq 0$. A necessary condition that T be cyclic is that its eigenvalues $\{\lambda_n\}$ be distinct. If $\sum_{n=0}^{\infty} a_n x_n$ is any absolutely summable series in $\mathcal{X}$, then for every functional L on $\mathcal{X}^*$, we have that $L(T^k \sum_{n=0}^{\infty} a_n x_n) = \sum_{n=0}^{\infty} a_n l_n \lambda_n^k$ where $l_n \equiv L(x_n)$ for all $n \geq 0$. The continuity of T and the nature of the complete set $\{x_n\}$ of common eigenvectors impose growth conditions on the resulting sets of eigenvalues $\{\lambda_n\}$, while the decay rate of the resulting sequences of coefficients $\{a_n l_n\}$ is determined by the coefficients $\{a_n\}$ in the expansion $\sum_{n=0}^{\infty} a_n x_n$, the set $\{x_n\}$ and the behavior of the action of functionals L in $\mathcal{X}^*$ on the set $\{x_n\}$, in view of which, the unicity theorem appearing in the Appendix, and its variations, become relevant in certain circumstances.

Herrero has shown that a cyclic operator on a Banach space has a dense set of cyclic vectors if and only if the point spectrum of its adjoint has non-empty interior (see [**11**, page 918, Theorem 1]). In particular, every cyclic diagonal operator D on a Hilbert space has a dense set of cyclic vectors (since the point spectrum $\sigma_p(D^*) = \{\overline{\lambda_n}\}$ of D^* is countable). Moreover, Shields has shown that the set of cyclic vectors of an operator on a Banach space is a $\mathcal{G}_\delta$ set (see [**37**, page 411, Proposition 40]). Hence, by the Baire category theorem, any countable collection of cyclic operators on a Banach space, the point spectrum of all of whose adjoints have non-empty interior, has a dense $\mathcal{G}_\delta$ set of common cyclic vectors. In fact, one might reasonably expect that every vector $\sum_{n=0}^{\infty} a_n e_n$ acting on a Hilbert space with $a_n \neq 0$ for all $n \geq 0$ is cyclic for every diagonal operator D on a Hilbert space having orthonormal basis $\{e_n\}$. However, we have already seen as a consequence of Wolff's example that this need not be the case for all bounded sets $\{\lambda_n\}$ of eigenvalues for D. Moreover, since each vector in a Hilbert space $\mathcal{H}$ is in the kernel of some cyclic operator D diagonalizable with respect to some orthonormal basis for $\mathcal{H}$ (depending on D), it follows that the collection of all cyclic operators diagonalizable with respect to some orthonormal basis for $\mathcal{H}$ fails to have a common cyclic vector. It is possible, however, for uncountable collections of cyclic operators to have common cyclic vectors, and even dense sets of common cyclic vectors. For instance, Wogen [**41**] in 1978 showed that the collection of co-analytic Toeplitz operators T_ϕ^* acting on the Hilbert space $H^2(\mathbf{D})$ with non-constant symbols ϕ have common cyclic vectors, and in 2004, Ross and Wogen [**27**] demonstrated that various collections of normal operators acting on a Hilbert space have common cyclic vectors.

In this section, we use the unicity theorem appearing in the Appendix to show that various collections of cyclic operators acting on a complete locally convex topological vector space $\mathcal{X}$ having a complete set of common eigenvectors with common dense sets of cyclic vectors. In particular, we examine the special case where $\mathcal{X}$ is the space of entire functions $\mathcal{H}(\mathbf{C})$, the space $\mathcal{H}(\mathbf{D})$ of functions analytic on the open unit disc $\mathbf{D}$ and the operators having as a complete set of common eigenvectors the monomials $\{z^n\}$. Details of the relevant background information presented below, and additional references, may be found, for instance, in [**7, 8, 22, 32, 33, 35**].

We let $\mathcal{H}(\mathbf{C})$ denote the space of entire functions. When endowed with the topology of uniform convergence on compacta, $\mathcal{H}(\mathbf{C})$ is an example of a complete locally convex topological vector space. If $\{\lambda_n : n \geq 0\}$ is any sequence of complex numbers, then the map D for which $D(z^n) = \lambda_n z^n$ for all $n \geq 0$ extends by linearity to an operator on all of $\mathcal{H}(\mathbf{C})$ if and only if $\limsup |\lambda_n|^{1/n} < \infty$. Any such operator D having the monomials z^n as eigenvectors with associated sequence of eigenvalues $\{\lambda_n\}$ is called a diagonal operator on $\mathcal{H}(\mathbf{C})$. A diagonal operator D with eigenvalues $\{\lambda_n\}$ is cyclic if and only if the eigenvalues are distinct (see [**32**]). A necessary condition that an entire function $g(z) \equiv \sum_{n=0}^\infty a_n z^n$ be cyclic for D is that $a_n \neq 0$ for all $n \geq 0$. The converse, however, is not true for all D. The question as to when the converse does hold is studied in greater detail in [**8, 33, 35**]. Finally, every continuous linear functional $L : \mathcal{H}(\mathbf{C}) \to \mathbf{C}$ assumes the form $L(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty a_n l_n$ where $\limsup |l_n|^{1/n} < \infty$.

Theorem 2. *The collection of cyclic diagonal operators acting on the space of entire functions $\mathcal{H}(\mathbf{C})$, each of whose set of eigenvalues in not dense in $\mathbf{C}$, has a dense set of common cyclic vectors.*

Proof. We show that the collection of cyclic diagonal operators acting on $\mathcal{H}(\mathbf{C})$ has the set $\mathcal{C} \equiv \{\sum_{n=0}^{\infty} c_n z^n : 0 < |c_n| < 1/e^{n^4} \text{for all } n\}$ as a set of common cyclic vectors. By means of contradiction, suppose that this is not the case. Hence, there exists an entire function $g(z) \equiv \sum_{n=0}^{\infty} c_n z^n$ in $\mathcal{C}$ which is not cyclic for some diagonal operator D acting on $\mathcal{H}(\mathbf{C})$. Let $\{\lambda_n\}$ denote the sequence of eigenvalues for D. Since D is cyclic, the eigenvalues are distinct, and since D is continuous, $\limsup |\lambda_n|^{1/n} < \infty$. If $\{\lambda_n\}$ is bounded, then g is cyclic for D by [**22**, page 607, Lemma 7]. So we may assume without loss of generality that $\{\lambda_n\}$ is unbounded. Since g is not a cyclic vector for D, the closed linear span of the vectors $\{D^k g : k \geq 0\}$ is not all of $\mathcal{H}(\mathbf{C})$. Hence, there exists a non-zero functional L in $\mathcal{H}^*(\mathbf{C})$ for which $0 \equiv L(D^k g)$ for all $k \geq 0$. So there exists a sequence of complex numbers $\{l_n\}$ for which $L(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n l_n$ for every entire function $\sum_{n=0}^{\infty} a_n z^n$ where $\equiv \limsup |l_n|^{1/n} < \infty$. We have then that $0 \equiv L(D^k g) = L(D^k(\sum_{n=0}^{\infty} c_n z^n)) = L(\sum_{n=0}^{\infty} c_n \lambda_n^k z^n) = \sum_{n=0}^{\infty} c_n l_n \lambda_n^k$ for all non-negative integers k. Since $0 \equiv \limsup |l_n|^{1/n} < \infty$ and $|a_n| \leq 1/e^{n^4}$ for all $n \geq 0$, it follows that $|c_n l_n| < 1/e^{n^3}$ for all n sufficiently large, and so $a_n c_n \equiv 0$ for all non-negative integers n by Theorem 4. Since $a_n \neq 0$ for all $n \geq 0$, we have that $l_n \equiv 0$ for all n, and so L is the zero functional, a contradiction. The result follows since the set $\mathcal{C}$ is easily seen to be dense in $\mathcal{H}(\mathbf{C})$. $\square$

A similar result holds for the set of cyclic diagonal operators acting on $\mathcal{H}(\mathbf{D})$, the space of functions analytic on the unit disc. The proof, being similar to that of Theorem 2, is omitted.

Theorem 3. *The collection of cyclic diagonal operators acting on $\mathcal{H}(\mathbf{D})$, each of whose set of eigenvalues in not dense in $\mathbf{C}$, has a dense set of common cyclic vectors.*

The example at the end of the Appendix demonstrates that the diagonal operator D having eigenvalues $\{\lambda_n\} = \mathbf{Z} \times i\mathbf{Z}$ is continuous and cyclic on both $\mathcal{H}(\mathbf{C})$ and on $\mathcal{H}(\mathbf{D})$ (see [**7**, Lemma 1] and [**22**, Proposition 1]), admits spectral synthetic on $\mathcal{H}(\mathbf{C})$ (see [**35**, Theorem 5]), but fails spectral synthetic on $\mathcal{H}(\mathbf{D})$ (see [**35**, Theorem 2]), in view of which, every vector $\sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}\mathbf{C})$ for which $a_n \neq 0$ for all $n \geq 0$ is cyclic for D as an operator acting on $\mathcal{H}(\mathbf{C})$ (see [**7**, Theorem

5]), but there exists a vector $\sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}(\mathbf{D})$ with $a_n \neq 0$ for all $n \geq 0$ which is not cyclic for D as an operator acting on $\mathcal{H}(\mathbf{D})$ (see [**35**, Theorem 2]).

Theorem 4 can also be used to conclude that certain collections of cyclic operators acting on a complete metrizable topological vector space which have a spanning set of common eigenvalues have a dense set of common cyclic vectors (see [**34**]). It applies equally well to discontinuous linear maps, including, for example, powers of the Laplacian defined on various domains (see [**36**]).

APPENDIX

A.1. A unicity theorem. In this section, we show the following unicity theorem for sequences of complex numbers $\{\lambda_n\}$ for which $\limsup |\lambda_n|^{1/n} < \infty$, a growth rate specifically chosen to obtain the cyclicity results of Section 2. This result, and its numerous variations, are similar to those appearing throughout the literature; see, for example, the work of Anderson, Khavinson, Shapiro [**1**], Beurling [**2**], Korobeinik [**12–15**], Leontev [**16, 17**], Makarov [**21**] and Sibilev [**39**], amongst others.

Theorem 4. *Let* $\{\lambda_n : 0 \leq n < \infty\}$ *be any sequence of distinct complex numbers for which* $\limsup |\lambda_n|^{1/n} < \infty$ *and* $\overline{\{\lambda_n\}} \neq \mathbf{C}$*, and let* $\{a_n : 0 \leq n < \infty\}$ *be any sequence of complex numbers for which* $|a_n| < 1/e^{n^3}$ *for all* n *sufficiently large. If* $0 \equiv \sum_0^{\infty} a_n \lambda_n^k$ *for all* $k \geq 0$*, then* $a_n \equiv 0$ *for all* $n \geq 0$.

Throughout the remainder of this section, we let $\{\lambda_n : 0 \leq n < \infty\}$ denote a sequence of distinct complex numbers for which $\limsup |\lambda_n|^{1/n} < \infty$ and $\overline{\{\lambda_n\}} \neq \mathbf{C}$, and let $\{a_n : 0 \leq n < \infty\}$ denote any sequence of complex numbers for which $0 \equiv \sum_0^{\infty} a_n \lambda_n^k$ for all $k \geq 0$ with $|a_n| < 1/e^{n^3}$ for all n sufficiently large. In order to prove Theorem 4, we will assume, by means of contradiction, that the complex numbers a_n are not all zero and obtain a contradiction.

For any complex number $\lambda \in \mathbf{C} \setminus \overline{\{\lambda_n\}}$, we have that $\inf\{|\lambda - \lambda_n| : n \geq 0\} \equiv \gamma > 0$. Moreover, it follows by induction from the condition $0 \equiv \sum_{n=0}^{\infty} a_n \lambda_n^k$ for all $k \geq 0$ that $0 \equiv \sum_{n=0}^{\infty} a_n (\lambda_n - \lambda)^k$ for all $k \geq 0$.

Hence, we may assume without loss of generality that $|\lambda_n| \geq \gamma > 0$ for all $n \geq 0$. For notational convenience, we define

$$f_n(z) \equiv \sum_{j=0}^{n} \frac{a_j}{z-\lambda_j} \quad \text{and} \quad F_n(z) \equiv f_n\left(\frac{1}{z}\right)$$

for all positive integers n. These functions converge almost everywhere with respect to Lebesgue area measure m on the complex plane to

$$f(z) \equiv \sum_{j=0}^{\infty} \frac{a_j}{z-\lambda_j} \quad \text{and} \quad F(z) \equiv f\left(\frac{1}{z}\right) = z\sum_{j=0}^{\infty} \frac{a_j}{1-z\lambda_j},$$

respectively (see [**2**]). We proceed with a series of lemmas.

Lemma 1. *Let A be any positive constant. Then*

$$|F_n(z)| \leq \frac{1}{B^{n+2}} \cdot \frac{1}{B^{An(n+1)/2}}\left\{\sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2} + \frac{B}{2^{An}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2}\right\}$$

whenever $|z| > 2B^{n+2}$.

Proof. Since $B_0 \equiv \limsup |\lambda_n|^{1/n} < \infty$, we have that $|\lambda_n| \leq (1+B_0)^n$ for all n sufficiently large. Hence, there exists a constant $B > 1$ for which $|\lambda_n| \leq B^{n+1}$ for all $n \geq 0$. Since $0 \equiv \sum_0^\infty a_n \lambda_n^k$ for all $k \geq 0$, it follows for $|z| > B^{n+1}$ by the Fubini-Tonelli theorem that

$$\begin{aligned} f_n(z) &= \sum_{j=0}^{n} \frac{a_j}{z-\lambda_j} = \sum_{j=0}^{n} \frac{1}{z} \cdot \frac{a_j}{1-\lambda_j/z} \\ &= \sum_{j=0}^{n} \frac{1}{z} \cdot a_j \sum_{p=0}^{\infty} \left(\frac{\lambda_j}{z}\right)^p \\ &= \sum_{p=0}^{\infty} \frac{1}{z^{p+1}} \sum_{j=0}^{n} a_j \lambda_j^p \end{aligned}$$

$$= -\sum_{p=0}^{\infty} \frac{1}{z^{p+1}} \sum_{j=n+1}^{\infty} a_j \lambda_j^p.$$

If $M \equiv [An]$ is the greatest integer of An, then again, since $0 \equiv \sum_0^\infty a_n \lambda_n^k$ for all $k \geq 0$, it follows for $|z| > B^{n+1}$ that

$$f_n(z) = -\sum_{p=0}^{M} \frac{1}{z^{p+1}} \sum_{j=n+1}^{\infty} a_j \lambda_j^p + \sum_{p=M+1}^{\infty} \frac{1}{z^{p+1}} \sum_{j=0}^{n} a_j \lambda_j^p.$$

For $p \leq M = [An] \leq An$ and $j \geq n+1$,

$$\begin{aligned} &B^{p(j+1)} B^{An(n+1)/2} / (B^{p(n+2)} B^{Aj(j-1)/2}) \\ &\qquad\qquad \leq B^{An(j-n-1)+A[n(n+1)-j(j-1)]/2}, \end{aligned}$$

which has a maximum of 1 occurring when $j = n+1$. Hence, for $|z| > 2B^{n+2}$, we have that

$$\begin{aligned} &\left| -\sum_{p=0}^{M} \frac{1}{z^{p+1}} \sum_{j=n+1}^{\infty} a_j \lambda_j^p \right| \\ &\qquad \leq \sum_{p=0}^{M} \frac{1}{|z|^{p+1}} \sum_{j=n+1}^{\infty} |a_j| \cdot B^{p(j+1)} \\ &\qquad = \sum_{p=0}^{M} \frac{1}{|z|^{p+1}} \sum_{j=n+1}^{\infty} |a_j| \cdot B^{Aj(j-1)/2} \\ &\qquad \cdot \frac{B^{p(n+2)}}{B^{An(n+1)/2}} \cdot \frac{B^{p(j+1)} B^{An(n+1)/2}}{B^{p(n+2)} B^{Aj(j-1)/2}} \\ &\qquad \leq \frac{1}{|z|} \sum_{p=0}^{M} \frac{1}{|z|^p} \sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2} \frac{B^{p(n+2)}}{B^{An(n+1)/2}} \\ &\qquad \leq \frac{1}{|z|} \sum_{p=0}^{M} \left(\frac{B^{n+2}}{|z|} \right)^p \frac{1}{B^{An(n+1)/2}} \sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2} \\ &\qquad \leq \frac{1}{B^{n+2}} \cdot \frac{1}{B^{An(n+1)/2}} \cdot \sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2}. \end{aligned}$$

For $p \geq M = [An] \geq An - 1$ and $j \leq n$,

$$\frac{B^{p(j+1)} B^{An(n+1)/2}}{B^{p(n+2)} B^{Aj(j-1)/2}} \leq B^{(An-1)(j-n-1)+A[n(n+1)-j(j-1)]/2},$$

which has a maximum of B occurring when $j = n$. Hence, for $|z| > 2B^{n+2}$, we have that

$$\begin{aligned}
\Big| \sum_{p=M+1}^{\infty} & \frac{1}{z^{p+1}} \sum_{j=0}^{n} a_j \lambda_j^p \Big| \\
&\leq \sum_{p=M+1}^{\infty} \frac{1}{|z|^{p+1}} \sum_{j=0}^{n} |a_j| B^{p(j+1)} \\
&= \frac{1}{|z|} \sum_{p=M+1}^{\infty} \frac{1}{|z|^p} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \\
&\quad \cdot \frac{B^{p(n+2)}}{B^{An(n+1)/2}} \cdot \frac{B^{p(j+1)} B^{An(n+1)/2}}{B^{p(n+2)} B^{Aj(j-1)/2}} \\
&\leq \frac{B}{|z|} \sum_{p=M+1}^{\infty} \left(\frac{B^{n+2}}{|z|} \right)^p \frac{1}{B^{An(n+1)/2}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \\
&\leq \frac{B}{B^{n+2}} \cdot \frac{1}{2^{M+1}} \cdot \frac{1}{B^{An(n+1)/2}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \\
&\leq \frac{1}{B^{n+2}} \cdot \frac{1}{B^{An(n+1)/2}} \cdot \frac{B}{2^{An}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2}.
\end{aligned}$$

Since

$$f_n(z) = -\sum_{p=0}^{M} \frac{1}{z^{p+1}} \sum_{j=n+1}^{\infty} a_j \lambda_j^p + \sum_{p=M+1}^{\infty} \frac{1}{z^{p+1}} \sum_{j=0}^{n} a_j \lambda_j^p,$$

we have that

$$\begin{aligned}
|f_n(z)| \leq & \frac{1}{B^{n+2}} \\
& \cdot \frac{1}{B^{An(n+1)/2}} \left\{ \sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2} + \frac{B}{2^{An}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \right\}
\end{aligned}$$

whenever $|z| > 2B^{n+2}$. The result follows since $F_n(z) = f_n(1/z)$.

Lemma 2. *Let A be any positive constant. Then*

$$|F_n(z)| \le \frac{2e^{-b_n}}{[B^{n+2} \cdot B^{An(n+1)/2}]}$$

whenever $|z| > 2B^{n+2}$ for all n sufficiently large where $b_n \equiv \beta n$ for all non-negative integers n and $\beta \equiv \min[1; (A\log 2)/2]$.

Proof. Since $|a_j| < 1/e^{j^3}$ for all j sufficiently large, $\sum_{j=n+1}^{\infty} |a_j| \cdot B^{Aj(j-1)/2} < e^{-n}$ for all n sufficiently large. Moreover, $|a_j|B^{Aj(j-1)/2}$ converges to zero as j tends to infinity, and so has a maximum α. Thus,

$$\frac{B}{2^{An}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \le \frac{\alpha B(n+1)}{2^{An}},$$

and so

$$\sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2} + \frac{B}{2^{An}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \le e^{-n} + \alpha B(n+1)/2^{An} \le e^{-n} + 1/[2^{An/2}] \le 2e^{-b_n},$$

for all n sufficiently large. Thus, the set

$$\{z : |F_n(z)| < 2e^{-b_n}/[B^{n+2} \cdot B^{An(n+1)/2}]\}$$

contains the set

$$\Big\{z : |F_n(z)| < \frac{1}{B^{n+2}} \cdot \frac{1}{B^{An(n+1)/2}} \Big\{ \sum_{j=n+1}^{\infty} |a_j| B^{Aj(j-1)/2} + \frac{B}{2^{An}} \sum_{j=0}^{n} |a_j| B^{Aj(j-1)/2} \Big\}\Big\}$$

for all n sufficiently large, and the result follows from Lemma 1.

Lemma 3. *Let α and ε be positive constants, and let $K = B(0, R)$ be the open ball in the complex plane of radius $R > 0$. Then*

$$m(\{z \in K : |F_n(z)| < \varepsilon\}) \leq m(\{z \in K : |F_{n-1}(z)| < \varepsilon + \alpha\}) + \frac{|a_n|^2 R^2 \pi}{\alpha^2 |\lambda_n|^2}.$$

Proof. Since

$$F_n(z) = f_n(1/z) = \sum_{j=0}^{n} \frac{a_j}{1/z - \lambda_j} = -z \sum_{j=0}^{n} \frac{a_j/\lambda_j}{z - 1/\lambda_j},$$

for all z in $\{z \in K : |F_n(z)| < e\} \setminus \{z \in K : |F_{n-1}(z)| < \varepsilon + \alpha\}$, we have that

$$\varepsilon + \alpha \leq |F_{n-1}(z)| \leq |F_n(z)| + |a_n/\lambda_n| \cdot R/|z - 1/\lambda_n| < \varepsilon + |a_n/\lambda_n| \cdot R/|z - 1/\lambda_n|.$$

Hence, $|z - 1/\lambda_n| < |a_n|R/(\alpha|\lambda_n|)$, and the result follows. ◻

The proof of the following lemma, being similar to that of Lemma 6 in [**22**, page 26], is omitted.

Lemma 4. *Let $K = B(0, R)$ be the open ball in the complex plane of radius R. If $F(z) \neq 0$ m almost everywhere on* $\mathbf{C}$, *then there exist positive constants ε and δ for which* $\limsup m(\{z \in K : |F_n(z)| < \varepsilon\}) < (\pi - \delta)R^2$.

For convenience, we define

$$S_j \equiv m(\{z \in K : |F_j(z)| < e^{-b_j/2}/B^{Aj(j+1)/2}\})$$

for all non-negative integers j where here $K = B(0, R)$ denotes the open ball in the complex plane of radius R and $b_j \equiv \beta j$ where $\beta \equiv \min[1; (A \log 2)/2]$.

Lemma 5. *Let $K = B(0, R)$ be the open ball in the complex plane of radius $R > 0$. If $F(z) \neq 0$ m almost everywhere on $\mathbf{C}$, then there exists a constant $c > 0$ (independent of j and of A) for which*

$$S_{j-1} \geq S_j \cdot e^{[b_j - b_{j-1} + 4cA \log Bj]/(8c(j+1))}$$

for all j sufficiently large.

Proof. It follows from an estimate on rational functions due to Beurling (see [**2**] or [**38**, page 153, Lemma 6]) using Lemma 4 that there exists a constant $c > 0$ (independent of j and of A) for which

$$\begin{aligned} m(\{z \in K : |F_j(z)| < e^{[-b_j/2 + (b_j - b_{j-1})/4]}/B^{Aj(j-1)/2}\}) \\ \geq m(\{z \in K : |F_j(z)| < e^{-b_j/2}/B^{Aj(j+1)/2}\}) \\ \cdot e^{[b_j - b_{j-1} + 4cA \log Bj]/(4c(j+1))} \end{aligned}$$

for all sufficiently large j. Hence, together with Lemma 3, we have that

$$\begin{aligned} S_{j-1} &\geq m(\{z \in K : |F_j(z)| < e^{[-b_j/2 + (b_j - b_{j-1})/4]}/B^{Aj(j-1)/2}\}) \\ &\quad - \frac{\pi R^2 |a_j|^2 B^{Aj(j-1)} e^{b_{j-1}/2}}{4|\lambda_j|^2 \{e^{-b_{j-1}/4} - e^{-b_j/4}\}^2} \\ &\geq S_j e^{[b_j - b_{j-1} + 4cA \log Bj]/(4c(j+1))} \\ &\quad - \frac{\pi R^2 |a_j|^2 B^{Aj(j-1)} e^{b_{j-1}/2}}{4|\lambda_j|^2 \{e^{-b_{j-1}/4} - e^{-b_j/4}\}^2}. \end{aligned}$$

We now show that

$$\begin{aligned} S_j e^{[b_j - b_{j-1} + 4cA \log Bj]/(4c(j+1))} - \frac{\pi R^2 |a_j|^2 B^{Aj(j-1)} e^{b_{j-1}/2}}{4|\lambda_j|^2 \{e^{-b_{j-1}/4} - e^{-b_j/4}\}^2} \\ \geq S_j e^{[b_j - b_{j-1} + 4cA \log Bj]/(8c(j+1))}, \end{aligned}$$

or equivalently that

$$\begin{aligned} S_j \{e^{[b_j - b_{j-1} + 4cA \log Bj]/(4cj+1)} \\ - e^{[b_j - b_{j-1} + 4cA \log Bj]/(8c(j+1))}\} \\ \geq \frac{\pi R^2 |a_j|^2 B^{Aj(j-1)} e^{b_{j-1}/2}}{4|\lambda_j|^2 \{e^{-b_{j-1}/4} - e^{-b_j/4}\}^2}. \end{aligned}$$

By the Mean Value theorem, $e^b - e^a \geq (b-a)e^a$ whenever $a \leq b$, and so $e^{-b_{j-1}/4} - e^{-b_j/4} \geq (1/4)(b_j - b_{j-1})e^{-b_j/4}$ and

$$\begin{aligned} e^{[b_j - b_{j-1} + 4cA\log Bj]/(4c(j+1))} - e^{[b_j - b_{j-1} + 4cA\log Bj]/(8c(j+1))} & \\ \geq \frac{A\log Bj}{2(j+1)} e^{[b_j - b_{j-1} + 4cA\log Bj]/(8c(j+1))} & \\ \geq \frac{A\log B}{4c} e^{A\log B/4}. & \end{aligned}$$

Moreover, $2e^{-b_j}/[B^{j+2}B^{Aj(j+1)/2}] \leq e^{-b_j}/B^{Aj(j+1)/2}$, and so by Lemma 2,

$$S_j \geq m\left(\left\{z \in K : |F_j(z)| < \frac{2e^{-b_j}}{B^{Aj(j+1)/2}}\right\}\right) \geq \frac{\pi}{[4B^{2(j+2)}]}.$$

Hence, it suffices to show that

$$\frac{\pi}{4B^{2(j+2)}} \cdot \frac{A\log B}{4c} e^{A\log B/4} \geq 4^2\pi R^2 |a_j|^2 \frac{B^{Aj(j-1)}e^{b_{j-1}/2}e^{b_j/2}}{|\lambda_j|^2(b_j - b_{j-1})^2}$$

for all j sufficiently large. However, this inequality holds for all j sufficiently large since $|\lambda_j| \geq \gamma$ and $b_j = \beta j$ for all $j \geq 0$, and $|a_j| < 1/e^{j^3}$ and for all j sufficiently large. $\square$

Proof of Theorem 4. Let $K = B(0,R)$ be the open ball in the complex plane of radius $R > 0$, and suppose that $F \neq 0$ m almost everywhere on $\mathbf{C}$. Since

$$\frac{2e^{-b_j}}{B^{j+2}B^{Aj(j+1)/2}} \leq \frac{e^{-b_j/2}}{B^{Aj(j+1)/2}},$$

we have by Lemma 2 that there exists a positive integer J_1 such that

$$\frac{\pi}{4} \leq B^{2(j+2)} m\left(\left\{z \in K : |F_j(z)| < \frac{2e^{-b_j}}{B^{j+2}B^{Aj(j+1)/2}}\right\}\right) \leq B^{2(j+2)} S_j$$

for all $j > J_1$. By Lemma 5, there exists a positive integer J_2 such that

$$\begin{aligned} S_j &\leq S_{j-1} e^{-(b_j - b_{j-1})/[8c(j+1)]} e^{-4cA\log Bj/[8c(j+1)]} \\ &= S_{j-1} e^{-\beta/[8c(j+1)]} e^{-A\log Bj/[2(j+1)]} \end{aligned}$$

for all $j > J_2$. If $J \equiv \max(J_1, J_2)$, then

$$\frac{\pi}{4} \le B^{2(J+k+2)} S_J e^{-\beta/8c} \sum_{i=J}^{J+k} \frac{1}{1+i} e^{-A \log B/2} \sum_{j=J}^{J+k} \frac{i}{1+i} \longrightarrow 0$$

as $k \to \infty$ whenever $A > 8$, a contradiction. Hence, $F = 0$, m almost everywhere on $\mathbf{C}$. That is, $0 = F(z) = f(1/z) = -z \sum_0^\infty (a_j/\lambda_j)/(z - 1/\lambda_j) = z\widehat{\mu}(z)$ where $\widehat{\mu}$ is the Cauchy transform of the finite measure $\mu \equiv \sum_0^\infty (a_j/\lambda_j)\delta_{\lambda_j^{-1}}$. Hence, $a_j \equiv 0$ for all $j \ge 0$ (see [**9**]). The result follows. □

The hypothesis in Theorem 4 that $|a_n| < 1/e^{n^3}$ for all $n \ge 0$ can be weakened considerably. Moreover, various analogues of Theorem 4 can be obtained by specifying a growth rate on the complex numbers $\{\lambda_n\}$ and choosing an appropriate decay rate on the a_n.

We conclude this section by showing that the moment condition $0 \equiv \sum_{n=0}^\infty w_n \lambda_n^k$ can hold for all $k \ge 0$ for coefficients $\{w_n\}$ which decay exponentially if the complex numbers $\{\lambda_n\}$ are unbounded. This example is in the spirit of those produced by Wolff, Denjoy and Leont'eva mentioned previously and was brought to the author's attention by Eremenko. The complex numbers $\{\lambda_n\}$ in this example are obtained by enumerating the integer lattice points $\mathbf{Z} \times i\mathbf{Z} \equiv \{m + in : m, n \in \mathbf{Z}\}$ beginning on the positive real line and moving counterclockwise around larger and larger squares. That is,

$$\begin{aligned}
&\lambda_0 = 0;\ \lambda_1 = 1;\ \lambda_2 = 1+i;\ \lambda_3 = i;\ \lambda_4 = -1+i;\\
&\lambda_5 = -1;\ \lambda_6 = -1-i;\ \lambda_7 = -i;\ \lambda_8 = 1-i;\\
&\lambda_9 = 2; \ldots \lambda_{24} = 2-i;\ \lambda_{25} = 3; \ldots .
\end{aligned}$$

There are a total of $(2k+1)^2$ integer lattice points either on or inside the kth square, and so exactly $8k$ points on the kth square. Hence, λ_j is on the kth square whenever $1 + 4k(k-1) \le j < 1 + 4k(k+1)$. In this case, either $|\mathrm{Re}\,(\lambda_j)| = k$ or $|\mathrm{Im}\,(\lambda_j)| = k$ and $k^2 \le |\lambda_j|^2 \le 2k^2$. It follows that $\{\lambda_n/n : n \ge 1\}$ is bounded and that $\limsup |\lambda_n|^{1/n} \le 1$.

Example. Let $\{\lambda_n\}$ be the enumeration of the integer lattice points $\mathbf{Z} \times i\mathbf{Z}$ obtained as above. Then there exist complex coefficients

$\{w_n\}$, not all zero, for which $0 = \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geq 0$ where $\limsup |w_n|^{1/n} < 1$.

Proof. For each positive integer r, we let S_r denote the square passing through the points $\pm(r+1/2)$ and $\pm(r+1/2)i$ and having horizontal and vertical sides. For any complex number λ on any square S_r, we have that the distance $d(\lambda)$ from λ to any point in $\mathbf{Z}\times i\mathbf{Z}$ is at least $1/2$. We denote the Weierstrass sigma function having zeros at the integer lattice points $\mathbf{Z}\times i\mathbf{Z}$ by

$$\sigma(z) \equiv z\Pi'\left(1 - \frac{z}{m+in}\right)\exp\left\{\frac{z}{m+in} + \frac{z^2}{2(m+in)^2}\right\}$$

(where here the prime indicates that the product is taken over all integers m and n, but omitting the origin where $m = 0 = n$). Since $|\sigma(\lambda)| \geq 0.2d(\lambda)e^{\pi|\lambda|^2/2}$ (see [**25**, page 157, equation (1) and page 161, Corollary 1.1]), it follows that $\lim_{r\to\infty}\int_{S_r} e^{\lambda z}/\sigma(\lambda)\,d\lambda = 0$ for every complex number z. Moreover,

$$\int_{S_r} \frac{e^{\lambda z}}{\sigma(\lambda)}\,d\lambda = \sum_{\{j:\lambda_j\in S_r^\circ\}} \frac{e^{\lambda_j z}}{\sigma'(\lambda_j)}$$

by the Residue theorem, and so

$$\sum_{j=0}^{\infty} \frac{e^{\lambda_j z}}{\sigma'(\lambda_j)} \equiv 0$$

on the complex plane. That is, $\sum_{j=0}^{\infty} w_j e^{\lambda_j z} \equiv 0$ where $w_j \equiv 1/\sigma'(\lambda_j)$. $\square$

We now estimate $|w_j|$. Let $m+in$ be any integer lattice point. Since $|\sigma(z)| \geq 0.2d(z)e^{\pi|z|^2/2}$, we have that $\sigma(z)$ maps the open ball $B(m+in, 1/4)$ of radius $1/4$ and center $m+in$ onto $B(0,\alpha)$ by the Inverse Function theorem (see [**9**, page 234]) where here $\alpha \equiv 0.05e^{\pi(|m+in|-1/4)^2/2}$. It follows from Schwarz's lemma applied to $4(\sigma^{-1}(\alpha z) - (m+in))$ that $|\sigma'(\lambda_j)| \geq 0.2e^{\pi(|\lambda_j|-1/4)^2/2}$ for all points λ_j in $\mathbf{Z}\times i\mathbf{Z}$. Since $k^2 \leq |\lambda_j|^2 \leq 2k^2$ whenever $1+4k(k-1) \leq j < 1+4k(k+1)$, it follows that $\limsup |w_j|^{1/j} < 1$.

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