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A NOTE ON WEYL'S INEQUALITY FOR EIGHTH POWERS

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ABSTRACT. We establish a new bound for the number of solutions of a pair of symmetric diophantine equations, one quartic and one quadratic, in ten variables. This estimate is then used to deduce a modest refinement of Weyl's inequality for eighth powers, which improves on an earlier result of Robert and Sargos.

1. Introduction. Estimates for exponential sums play a prominent role in analytic number theory, and in particular the sum over kth powers defined by

$$f_k(\alpha; P) = \sum_{1 \le x \le P} e(\alpha x^k), \tag{1.1}$$

where $e(z) = e^{2\pi i z}$, is central to the study of many diophantine problems. The starting point for investigations along these lines is the celebrated work of Weyl [**18**] on uniform distribution, which leads to upper bounds for $f_k(\alpha; P)$ that depend on the nature of rational approximations to α. Specifically, Weyl's inequality states that if $|\alpha - a/q| \le q^{-2}$ for some integers a and q with $q \ge 1$ and $(a, q) = 1$ then one has

$$f_k(\alpha; P) \ll P^{1+\varepsilon}(q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}}. \tag{1.2}$$

Thus, in particular if $P \ll q \ll P^{k-1}$, then one obtains $f_k(\alpha; P) \ll P^{1-2^{1-k}+\varepsilon}$, and this provides one of the key ingredients for handling the minor arcs in the method devised by Hardy and Littlewood [**6**] for applications such as Waring's problem. The strategy for proving (1.2),

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known as Weyl differencing, involves successive squaring of $|f_k(\alpha;P)|$ to reduce the degree of the monomial x^k. After $k-1$ applications of this process, one is left with a linear polynomial, and the resulting innermost summation is geometric.

For larger k, better results can be obtained by an approach based on Vinogradov's mean value theorem [**17**]. Following refinements by Linnik [**10**], Hua [**8**], Wooley [**20, 21**], and others, one can replace 2^{1-k} in (1.2) by an exponent $\sigma(k)$ satisfying $\sigma(k)^{-1} \sim (32)k^2 \log k$ (see [**21**, Theorem 1]). For a more comprehensive account of the methods of Weyl and Vinogradov and their applications to diophantine problems, the interested reader is referred to the books of Baker [**1**], Davenport [**4**] and Vaughan [**16**].

Heath-Brown [**7**] has developed an alternative approach, which leads to superior estimates when k is of moderate size and α has a rational approximations with denominator lying in an intermediate range. Here a symmetric form of Weyl differencing is employed $k-3$ times to relate estimates for the exponential sum (1.1) to mean values associated to a system of two symmetric diagonal equations, one cubic and one linear. This approach leads to the estimate

$$f_k(\alpha;P) \ll P^{1-(8/3)2^{-k}+\varepsilon}(P^3q^{-1}+1+qP^{3-k})^{(4/3)2^{-k}}, \tag{1.3}$$

under the same conditions preceding (1.2), whenever $k \geq 6$. This conclusion is superior to Weyl's inequality whenever $P^{5/2+\delta} \ll q \ll P^{k-5/2-\delta}$ for some $\delta > 0$.

More recently, Robert and Sargos [**12**] adapted Heath-Brown's approach, but with $k-4$ symmetric differences, to relate (1.1) to mean values of the exponential sum

$$F(\beta,\gamma) = F(\beta,\gamma;P) = \sum_{1\leq x\leq P} e(\beta x^2 + \gamma x^4). \tag{1.4}$$

Define

$$I_{2s}(P) = \int_0^1 \int_0^1 |F(\beta,\gamma)|^{2s}\, d\beta\, d\gamma, \tag{1.5}$$

and observe that, by orthogonality, $I_{2s}(P)$ counts the number of solutions of the system of diophantine equations

$$\begin{aligned} x_1^4 + \cdots + x_s^4 &= y_1^4 + \cdots + y_s^4 \\ x_1^2 + \cdots + x_s^2 &= y_1^2 + \cdots + y_s^2 \end{aligned} \tag{1.6}$$

with $\mathbf{x}, \mathbf{y} \in [1, P]^s$. As a consequence of a more general mean value estimate, Robert and Sargos showed that $I_{10}(P) \ll P^{49/8+\varepsilon}$ and used this to deduce (see [**12**], Theorem 4 and Lemma 7) that

$$f_k(\alpha; P) \ll P^{1-3\cdot 2^{-k}+\varepsilon}(P^4 q^{-1} + 1 + qP^{4-k})^{(8/5)2^{-k}}, \tag{1.7}$$

under the hypotheses preceding (1.2), whenever $k \geq 8$. This conclusion is superior to Heath-Brown's estimate (1.3) whenever $P^{91/24+\delta} \ll q \ll P^{k-91/24-\delta}$ for some $\delta > 0$.

Very recently, Wooley [**23**] has obtained

$$f_k(\alpha; P) \ll P^{1+\varepsilon}(q^{-1} + P^{-1} + qP^{-k})^{1/[2k(k-1)]}, \tag{1.8}$$

which is superior to (1.2) for $k \geq 8$ and superior to both (1.3) and (1.7) for $k \geq 9$. The bound (1.8) is a consequence of the new efficient congruencing technique developed in [**23**] which, for the first time, removes the factor of $\log k$ in estimates associated to Vinogradov's mean value theorem.

The purpose of this paper is to provide a slight refinement of (1.7) when $k = 8$, and we accomplish this by establishing an improved mean value estimate that may be of independent interest. In Section 2, we prove the following using fairly classical arguments.

Theorem 1.1. *One has*

$$I_{10}(P) \ll P^{6+\varepsilon}.$$

In view of the diagonal solutions to (1.6), one clearly has $I_{10}(P) \gg P^5$. We also mention that the system (1.6) has been studied in detail in the case $s = 3$. For example, Salberger [**13**], improving on earlier work of Tsui and Wooley [**15**], showed that

$$I_6(P) = 6P^3 + O(P^{5/2+\varepsilon}).$$

Robert and Sargos [**12**], proceeding in a manner similar to Bombieri and Iwaniec [**3**], actually estimated the more general mean values

$$I_{2s}(P; \lambda) = \int_0^\lambda \int_0^1 |F(\beta, \gamma)|^{2s} d\beta\, d\gamma,$$

which may be interpreted in terms of the number of solutions of a system of diophantine inequalities. By applying a version of van der Corput's B Process, together with a third derivative estimate, they showed that

$$I_{2s}(P;\lambda) \ll \lambda P^{\mu_s+\varepsilon} + P^{2s-6+\varepsilon},$$

where $\mu_3 = 3$, $\mu_4 = 9/2$, and $\mu_5 = 49/8$. The above estimate for $I_6(P;\lambda)$ was then applied by Sargos [**14**] to obtain an exponential sum estimate based on the fifth derivative. It would appear that our new estimate for $I_{10}(P;1)$ does not provide any improvements in applications of this sort, but our result could potentially be relevant to work on simultaneous additive equations of the type studied in [**11**].

As a consequence of Theorem 1.1, we are able to further improve on Weyl's inequality for eighth powers, albeit for a restricted set of α. We prove in Section 3 that, if $k \geq 8$ and the conditions preceding (1.2) hold, then one has

$$f_k(\alpha;P) \ll P^{1-(16/5)2^{-k}+\varepsilon}(P^4q^{-1} + 1 + qP^{4-k})^{(8/5)2^{-k}}, \tag{1.9}$$

which is superior to (1.7) for all α and to (1.3) whenever $P^{11/3+\delta} \ll q \ll P^{k-11/3-\delta}$ for some $\delta > 0$. However, it transpires that Wooley's estimate (1.8) is superior to ours for all α when $k \geq 9$, and hence the new content of (1.9) may be summarized in the following modest refinement for eighth powers.

Theorem 1.2. *If $|\alpha - a/q| \leq q^{-2}$ for some integers a and q with $q \geq 1$ and $(a,q) = 1$, then one has*

$$f_8(\alpha;P) \ll P^{79/80+\varepsilon}(P^4q^{-1} + 1 + qP^{-4})^{1/160}.$$

For comparison, the result (1.7) of Robert and Sargos yields the same estimate with the exponent $79/80 = 0.9875$ replaced by $253/256 = 0.98828125$. Moreover, our estimate is superior to (1.2) whenever $P^{13/4+\delta} \ll q \ll P^{19/4-\delta}$, to (1.3) whenever $P^{11/3+\delta} \ll q \ll P^{13/3-\delta}$ and to (1.8) whenever $P^{24/7+\delta} \ll q \ll P^{32/7-\delta}$. We are not aware of any immediate applications of our new estimate to diophantine problems. In particular, the bound

$$\widetilde{G}(8) \leq 117$$

for the number of variables required to obtain the asymptotic formula in Waring's problem, recently established by Wooley [**22**], is not susceptible to improvement via Theorem 1.2. In this case, the strength of the mean value estimates stemming from [**23**] is so great that the quality of the Weyl-type inequalities becomes less significant.

2. The tenth moment estimate. Our goal in this section is to establish Theorem 1.1, and we begin by employing a strategy reminiscent of Hua [**9**, Chapter 5]. By symmetric Weyl differencing, as in Heath-Brown [**7**], we have

$$|F(\beta,\gamma)|^2 = \sum_{|h|<P/2} \sum_{z\in\mathcal{I}(h)} e(4zh\beta + (8z^3h + 8zh^3)\gamma),$$

where $\mathcal{I}(h)$ is a subinterval of $[1, P]$. An application of Cauchy's inequality followed by a second difference then yields

$$\begin{aligned} |F(\beta,\gamma)|^4 &\le P \sum_{|h|<P/2} \Bigg| \sum_{z\in\mathcal{I}(h)} e(4zh\beta + (8z^3h + 8zh^3)\gamma)\Bigg|^2 \\ &= P \sum_{|h|,|g|<P/2} \sum_{z\in\mathcal{I}(h,g)} e(8hg\beta + 16hg(3z^2 + g^2 + h^2)\gamma), \end{aligned}$$

where $\mathcal{I}(h, g)$ is a subinterval of $[1, P]$.

It therefore follows that $I_{10}(P) \le P\mathcal{V}(P)$, where $\mathcal{V}(P)$ denotes the number of integral solutions of the system

$$\begin{aligned} 16hg(3z^2 + g^2 + h^2) &= \sum_{i=1}^{3}(x_i^4 - y_i^4) \\ 8hg &= \sum_{i=1}^{3}(x_i^2 - y_i^2) \end{aligned} \tag{2.1}$$

with

$$|h|, |g| < P/2, \quad 1 \le z \le P, \quad \text{and} \quad \mathbf{x}, \mathbf{y} \in [1, P]^3. \tag{2.2}$$

Then one has

$$I_{10}(P) \le P(\mathcal{V}_0(P) + \mathcal{V}_1(P)), \tag{2.3}$$

where $\mathcal{V}_0(P)$ denotes the number of solutions of (2.1) satisfying (2.2) with $hg = 0$, and where $\mathcal{V}_1(P)$ denotes the number of solutions with $hg \neq 0$. First consider a solution counted by $\mathcal{V}_0(P)$. After fixing one of the $O(P^2)$ possible choices for h, g and z, it follows from Wooley [**19**, Theorem 4.1], that the number of possibilities for $\mathbf{x}$ and $\mathbf{y}$ is $O(P^{3+\varepsilon})$, and we therefore conclude that

$$\mathcal{V}_0(P) \ll P^{5+\varepsilon}. \tag{2.4}$$

We now consider solutions counted by $\mathcal{V}_1(P)$. We find it convenient to introduce the notation

$$S_j(\mathbf{x},\mathbf{y}) = y_1^j + y_2^j + y_3^j - x_2^j - x_3^j \quad (j = 2, 4)$$

and to further classify solutions according to whether

$$S_2(\mathbf{x},\mathbf{y})^2 - S_4(\mathbf{x},\mathbf{y}) = 0. \tag{2.5}$$

Let $\mathcal{V}_2(P)$ denote the number of solutions counted by $\mathcal{V}_1(P)$ for which (2.5) does not hold, and write $\mathcal{V}_3(P)$ for the number of solutions in which (2.5) does hold. We first consider a solution counted by $\mathcal{V}_2(P)$. Given any of the $O(P^5)$ choices for x_2, x_3, y_1, y_2 and y_3 not satisfying (2.5), the second equation in (2.1) yields

$$x_1^2 = 8hg + S_2(\mathbf{x},\mathbf{y}), \tag{2.6}$$

and, upon substituting this into the first equation of (2.1), we discover that the variables h, g and z must satisfy

$$16hg(3z^2 + g^2 + h^2 - 4hg - S_2(\mathbf{x},\mathbf{y})) = S_2(\mathbf{x},\mathbf{y})^2 - S_4(\mathbf{x},\mathbf{y}). \tag{2.7}$$

It follows from a standard estimate for the divisor function that h and g are now determined to $O(P^\varepsilon)$, and z is then determined to $O(1)$ as a solution of a non-trivial polynomial equation. We therefore have $\mathcal{V}_2(P) \ll P^{5+\varepsilon}$ so, in view of (2.3) and (2.4), the theorem will follow upon establishing the estimate

$$\mathcal{V}_3(P) \ll P^{5+\varepsilon}. \tag{2.8}$$

To establish (2.8), we relate (2.5) to the representation of integers by binary quadratic forms. It is easy to see that (2.5) implies

$$x_2^4 + x_3^4 + x_2^2 x_3^2 - A(\mathbf{y})(x_2^2 + x_3^2) + B(\mathbf{y}) = 0, \tag{2.9}$$

where

$$A(\mathbf{y}) = y_1^2 + y_2^2 + y_3^2 \quad \text{and} \quad B(\mathbf{y}) = y_1^2 y_2^2 + y_1^2 y_3^2 + y_2^2 y_3^2.$$

We now set $X_2 = x_2^2$ and $X_3 = x_3^2$, and make the change of variable $X_1 = X_2 + (1/2)X_3$. Then (2.9) becomes

$$X_1^2 + \frac{3}{4}X_3^2 - A(\mathbf{y})(X_1 + \frac{1}{2}X_3) + B(\mathbf{y}) = 0. \tag{2.10}$$

Next we complete the square to obtain

$$(X_1 - \tfrac{1}{2}A(\mathbf{y}))^2 + \tfrac{3}{4}(X_3 - \tfrac{1}{3}A(\mathbf{y}))^2 = \tfrac{1}{3}A(\mathbf{y})^2 - B(\mathbf{y}),$$

from which it follows easily that

$$\begin{aligned} 3(2X_1 - A(\mathbf{y}))^2 &+ (3X_3 - A(\mathbf{y}))^2 \\ &= 2(y_1^2 - y_2^2)^2 + 2(y_1^2 - y_3^2)^2 + 2(y_2^2 - y_3^2)^2. \end{aligned} \tag{2.11}$$

Clearly, the right-hand side of (2.11) is zero if any only if $y_1 = y_2 = y_3$. In this case, we trivially have $O(P)$ choices for $\mathbf{y}$ and $O(P^2)$ choices for X_1 and X_3. Moreover, for a choice of $\mathbf{y}$ with $y_1 y_2 y_3 \neq 0$, we know from Estermann [**5**] that the number of possibilities for X_1 and X_3 is $O(P^\varepsilon)$. Thus, in any case, the number of solutions to (2.11) in the variables X_1, X_3, y_1, y_2 and y_3 is $O(P^{3+\varepsilon})$, and this also determines x_2 and x_3. The values of h and g may be assigned in $O(P^2)$ ways, and the values of x_1 and z are now determined to $O(1)$ by (2.6) and (2.7). This establishes (2.8) and hence completes the proof of Theorem 1.1.

3. Weyl's inequality. With our estimate for $I_{10}(P)$ in hand, the deduction of Theorem 1.2 proceeds exactly as in [**12**, Section 10]. We provide some details for the sake of completeness. Let $k \geq 8$, and set

$$K = 2^k \quad \text{and} \quad H = \tfrac{2}{3}k!P^{k-4}.$$

First of all, by applying the argument of the proof of [**7**, Lemma 1], one obtains
(3.1)

$$|f_k(\alpha;P)|^{K/16} \ll P^{K/16-1} + P^{K/16-k+3+\varepsilon} \sum_{h=1}^{H} \left| \sum_{n=1}^{N_h} e(a_h n^4 + b_h n^2) \right|,$$

where $a_h = \alpha h$, where the b_h are real numbers depending on α, and where the N_h are integers satisfying $1 \le N_h \le P$. By Hölder's inequality, one has

$$\left(\sum_{h=1}^{H} \left| \sum_{n=1}^{N_h} e(a_h n^4 + b_h n^2) \right| \right)^{2s} \ll H^{2s-2} \left| \sum_{h,\mathbf{m}} \xi_h r_h(\mathbf{m}) e(a_h m_4 + b_h m_2) \right|^2, \tag{3.2}$$

where the ξ_h are complex numbers with $|\xi_h| = 1$ and where $r_h(\mathbf{m})$ denotes the number of solutions of the system

$$\begin{aligned} m_4 &= n_1^4 + \cdots + n_s^4 \\ m_2 &= n_1^2 + \cdots + n_s^2 \end{aligned}$$

with $1 \le n_i \le N_h$.

It follows from the double large sieve (see Bombieri and Iwaniec [**2**, Lemma 2.4]) that

$$\left| \sum_{h,\mathbf{m}} \xi_h r_h(\mathbf{m}) e(a_h m_4 + b_h m_2) \right|^2 \ll (1+P^4)(1+P^2)\mathcal{N}(P) I_{2s}(P), \tag{3.3}$$

where

$$\mathcal{N}(P) = \operatorname{card}\{\mathbf{h} \in [1,H]^2 : \|a_{h_1} - a_{h_2}\| \le P^{-4} \text{ and } \|b_{h_1} - b_{h_2}\| \le P^{-2}\}.$$

On substituting (3.3) into (3.2), we obtain

$$\sum_{h=1}^{H} \left| \sum_{n=1}^{N_h} e(a_h n^4 + b_h n^2) \right| \ll H^{1-1/s} P^{3/s} \big(\mathcal{N}(P) I_{2s}(P)\big)^{1/(2s)}.$$

We now let $s = 5$, insert this into (3.1), and apply Theorem 1.1 to get
(3.4)

$$|f_k(\alpha;P)|^{K/16} \ll P^{K/16-1} + P^{K/16-k+3+\varepsilon}H^{4/5}P^{3/5}\big(\mathcal{N}(P)I_{10}(P)\big)^{1/10} \\ \ll P^{K/16-1} + P^{K/16+1-k/5+\varepsilon}\mathcal{N}(P)^{1/10}.$$

Finally, by [**7**, Lemma 6], one has

$$\mathcal{N}(P) \ll H(1+qP^{-4})(1+q^{-1}P^{k-4}) \ll P^{2k-12}(1+qP^{4-k}+q^{-1}P^4)$$

whenever $|\alpha - a/q| \le q^{-2}$ for some integers a and q with $q \ge 1$ and $(a,q) = 1$. Hence, one deduces from (3.4) that

$$(3.5)\quad |f_k(\alpha;P)| \ll P^{1-16/K} + P^{1-16/(5K)+\varepsilon}(q^{-1}P^4+1+qP^{4-k})^{8/(5K)},$$

and Theorem 1.2 now follows on taking $k = 8$.

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REFERENCES

1. R.C. Baker, *Diophantine inequalities*, Clarendon Press, Oxford, 1986.

2. E. Bombieri and H. Iwaniec, *On the order of* $\zeta(\frac{1}{2}+it)$, Ann. Scuola Norm. Sup. Pisa **13** (1986), 449–472.

3. ———, *Some mean-value theorems for exponential sums*, Ann. Scuola Norm. Sup. Pisa **13** (1986), 473–486.

4. H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, 2nd edition, Cambridge University Press, Cambridge, 2005.

5. T. Estermann, *Einige Sätze über quadratfreie Zahlen*, Math. Ann. **105** (1931), 653–662.

6. G.H. Hardy and J.E. Littlewood, *Some problems of partitio numerorum* I: *A new solution of Waring's problem*, Göttingen Nachrichten (1920), 33–54.

7. D.R. Heath-Brown, *Weyl's inequality, Hua's inequality, and Waring's problem*, J. Lond. Math. Soc. **38** (1988), 216–230.

8. L.-K. Hua, *An improvement of Vinogradov's mean-value theorem and several applications*, Quart. J. Math. Oxford **20** (1949), 48–61.

9. L.-K. Hua, *Additive theory of prime numbers*, American Mathematical Socity, Providence, RI, 1965.

10. Yu.V. Linnik, *On Weyl's sums*, Rec. Math. **12** (1943), 28–39.

11. S.T. Parsell, *Pairs of additive equations of small degree*, Acta Arith. **104** (2002), 345–402.

12. O. Robert and P. Sargos, *Un théorème de moyenne pour les sommes d'exponentielles*, Publ. Inst. Math. (Beograd) **67** (2000), 14–30.

13. P. Salberger, *Rational points of bounded height on threefolds*, Clay Math. Proc. **7** (2007), 207–216.

14. P. Sargos, *Un creitére de las dérivée cinquième pour les sommes d'exponentielles*, Bull. Lond. Math. Soc. **32** (2000), 398–402.

15. W.Y. Tsui and T.D. Wooley, *The paucity problem for simultaneous quadratic and biquadratic equations*, Math Proc. Cambr. Philos. Soc. **126** (1999), 209–221.

16. R.C. Vaughan, *The Hardy-Littlewood method*, 2nd edition, Cambridge University Press, Cambridge, 1997.

17. I.M. Vinogradov, *New estimates for Weyl sums*, Dokl. Akad. Nauk **8** (1935), 195–198.

18. H. Weyl, *Über die Gleichverteilung von Zahlen mod Eins*, Math. Ann. **77** (1916), 313–352.

19. T.D. Wooley, *On simultaneous additive equations* II, J. reine angew. Math. **419** (1991), 141–198.

20. ———, *On Vinogradov's mean value theorem*, Mathematika **39** (1992), 379–399.

21. ———, *New estimates for Weyl sums*, Quart. J. Math. Oxford **46** (1995), 119–127.

22. ———, *The asymptotic formula in Waring's problem*, Inter. Math. Res. Not. **7** (2012), 1485–1504.

23. ———, *Vinogradov's mean value theorem via efficient congruencing*, Ann. Math. **175** (2012), 1575–1627.

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