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K-THEORY FOR TOPOLOGICAL ALGEBROIDS

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ABSTRACT. We define K-theory spectra associated to certain topological categories and compare these spectra to analytic K-theory spectra. As an application, we show how the assembly map in the analytic Novikov conjecture can be factored through an assembly map defined through more homotopy-theoretic machinery.

1. Introduction. There are a number of ways to extend the definition of algebraic K-theory to topological ringoids where the space of objects has the discrete topology. In this article, we look at one such way; we enlarge a given ringoid to an additive category and basically use Segal's construction in [**19**]. This construction actually gives us a spectrum where the strictly positive stable homotopy groups are the algebraic K-theory groups.

It is quite easy to establish the elementary properties of these K-theory spectra, either directly or using existing machinery.

Roughly speaking, the *group completion theorem* (see for example, [**1, 9**]) tells us the homology of the space ΩBM, where M is a topological monoid. In this article, we use the group completion theorem to show that the analytic K-theory of Banach categories are obtained as special cases of the spectra we examine here.

Our final topic is assembly maps. Given a homotopy-invariant functor, an assembly map is an approximation of that functor by a generalized homology. The precise details of the definition can be found in [**24**], or [**2**] for the more complicated equivariant case. The analytic map featuring in the analytic Novikov conjecture is an example.

The abstract machinery involving assembly along with our unified picture of algebraic and analytic K-theory allow us to prove that the analytic Novikov assembly map can be factorized into a composite of an assembly map resembling the algebraic K-theory assembly map (see [**21**]) and a map arising from the completion of a certain topological

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ringoid into a Banach category. This unified picture is also used in coarse geometry, in the article [**16**].

There are other approaches to viewing analytic and algebraic K-theory as examples of the same general construction. For example, [**4**] uses a version of Quillen's Q-construction to present a unified picture of the algebraic and analytic K-theory of an algebra. However, working with topological categories in the sense described here, and obtaining the K-theory of C^*-categories as a special case is new, as is the idea of obtaining properties of analytic K-theory from corresponding results in algebraic K-theory.

2. Ringoids and additive categories. Recall (see, for example, [**12**]) that a small category $\mathcal{R}$ is called a *ringoid* if every morphism set $\mathrm{Hom}\,(a,b)_{\mathcal{R}}$ is an abelian group, and composition of morphisms

$$\mathrm{Hom}\,(b,c)_{\mathcal{R}} \times \mathrm{Hom}\,(a,b)_{\mathcal{R}} \longrightarrow \mathrm{Hom}\,(a,c)_{\mathcal{R}}$$

is bilinear with respect to group addition.

Definition 2.1. We call a ringoid $\mathcal{R}$ a *topological ringoid* if each morphism set $\mathrm{Hom}\,(a,b)_{\mathcal{R}}$ is a topological abelian group, and composition of morphisms

$$\mathrm{Hom}\,(b,c)_{\mathcal{R}} \times \mathrm{Hom}\,(a,b)_{\mathcal{R}} \longrightarrow \mathrm{Hom}\,(a,c)_{\mathcal{R}}$$

is continuous.

Note that we impose no topology on the set of objects in a topological ringoid.

Example 2.2. Let $\mathcal{V}$ be the category in which the objects are the vector spaces $\mathbf{R}^n$, and the morphism set $\mathrm{Hom}\,(\mathbf{R}^m,\mathbf{R}^n)$ consists of all $n\times m$ matrices, topologized as a subset of $\mathbf{R}^{mn}$. Composition is defined by multiplication of matrices, and addition is matrix addition.

Then $\mathcal{V}$ is a topological ringoid.

Example 2.3. Let π be a discrete groupoid, and let R be a commutative topological ring with an identity element. Then we define the *groupoid ringoid*, $R\pi$, to be the category with objects $\mathrm{Ob}\,(R\pi) = \mathrm{Ob}\,(\pi)$ and morphism sets

$$\mathrm{Hom}\,(a,b)_{R\pi} := \{x_1g_1 + \cdots + x_ng_n \mid x_i \in R, g_i \in \mathrm{Hom}\,(a,b)_\pi\}$$

Composition of morphisms is defined by the formula

$$\left(\sum_{i=1}^{m} x_ig_i\right)\left(\sum_{j=1}^{n} y_jh_j\right) = \sum_{i,j=1}^{m,n} (x_iy_j)g_ih_j.$$

The topology on the space $\mathrm{Hom}\,(a,b)_{R\pi}$ is defined by viewing it as a direct limit of a family of subspaces homeomorphic to the space R^n for some n.

It is easy to check that the category $R\pi$ is a topological R-algebroid.

In applications, the groupoid π in the above example is often the fundamental groupoid of a topological space. In this case, we ignore then the natural topology on the set of objects. The fundamental groupoid is useful when we want something containing the same information as the fundamental group, but, for reasons of functoriality, do not want to choose a specific basepoint.

Further examples of topological ringoids are provided in analysis by C^*-categories, as examined for instance in [**3, 14**], though C^*-categories also have a fair amount of additional structure.

Definition 2.4. A functor $F\colon \mathcal{R} \to \mathcal{S}$ between topological ringoids is termed a *homomorphism* of topological ringoids if each map

$$F\colon \mathrm{Hom}\,(a,b)_{\mathcal{R}} \longrightarrow \mathrm{Hom}\,(F(a),F(b))_{\mathcal{R}}$$

is a continuous group homomorphism.

Let $\mathcal{R}$ be a ringoid, and consider objects $a,b \in \mathrm{Ob}\,(\mathcal{R})$. Then an object $a \oplus b$ is called a *biproduct* of the objects a and b if it comes equipped with morphisms $i_a\colon a \to a \oplus b$, $i_b\colon b \to a \oplus b$, $p_a\colon A \oplus B \to A$, and $p_B\colon a \oplus b \to b$ satisfying the equations

$$p_ai_a = 1_a, \qquad p_bi_b = 1_b, \qquad i_ap_a + i_bp_b = 1_{a\oplus b}.$$

Observe that a biproduct is simultaneously a product and a coproduct in the category-theoretic sense, as in [**10**]. A ringoid $\mathcal{R}$ is called an *additive category* if it has a zero object (that is, an object that is simultaneously initial and terminal), and every pair of objects has a biproduct.

Definition 2.5. A topological ringoid that is an additive category is termed an *additive topological category.*

If $\mathcal{R}$ and $\mathcal{S}$ are additive topological categories, a topological ringoid homomorphism $F\colon \mathcal{R} \to \mathcal{S}$ is called an *additive functor* if the object $F(a \oplus b)$ is a biproduct of the objects $F(a)$ and $F(b)$ whenever $a, b \in \mathrm{Ob}\,(\mathcal{R})$.

Definition 2.6. Let $\mathcal{R}$ be a topological ringoid. Then we define the *additive completion*, $\mathcal{R}_\oplus$, to be the category in which the objects are formal sequences of the form

$$a_1 \oplus \cdots \oplus a_n, \quad a_i \in \mathrm{Ob}\,(\mathcal{R}).$$

Repetitions are allowed in such formal sequences. The empty sequence is also allowed, and labelled 0.

The morphism set $\mathrm{Hom}\,(a_1 \oplus \cdots \oplus a_m, b_1 \oplus \cdots \oplus b_n)$ is defined to be the set of matrices of the form

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,m} \end{pmatrix}, \quad x_{i,j} \in \mathrm{Hom}\,(a_j, b_i),$$

and composition of morphisms is defined by matrix multiplication.

The set of objects in the category $\mathcal{R}_\oplus$ is given the discrete topology. The morphism set $\mathrm{Hom}\,(a_1 \oplus \cdots \oplus a_m, b_1 \oplus \cdots \oplus b_n)$ is topologized as a subspace of the space $\mathcal{R}^{mn}$.

Given a topological ringoid homomorphism $F\colon \mathcal{R} \to \mathcal{S}$, there is an induced additive functor $F_\oplus\colon \mathcal{R}_\oplus \to \mathcal{S}_\oplus$ defined by writing

$$F_\oplus(a_1 \oplus \cdots a_n) = F(a_1) \oplus \cdots \oplus F(a_n), \quad a_i \in \mathrm{Ob}\,(\mathcal{R})$$

and

$$F_{\oplus}\begin{pmatrix} x_{1,1} & \cdots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,m} \end{pmatrix} = \begin{pmatrix} F(x_{1,1}) & \cdots & F(x_{1,m}) \\ \vdots & \ddots & \vdots \\ F(x_{n,1}) & \cdots & F(x_{n,m}) \end{pmatrix}$$
$$x_{i,j} \in \operatorname{Hom}(a_j, b_i).$$

The following result is easy to check.

Proposition 2.7. *The assignment $\mathcal{R} \mapsto \mathcal{R}_{\oplus}$ defines a functor from the category of topological ringoids and homomorphisms to the category of additive topological categories and additive functors.*

At times, it will be useful to look at ringoids with slightly more structure.

Definition 2.8. Let R be a commutative topological ring equipped with an identity element. Then we call a topological ringoid $\mathcal{M}$ an *topological R-algebroid* if:

- Each morphism set $\operatorname{Hom}(a,b)_{\mathcal{M}}$ is a topological R-module.
- Given an element $r \in R$, and morphisms $x \in \operatorname{Hom}(a,b)_{\mathcal{M}}$ and $y \in \operatorname{Hom}(b,c)_{\mathcal{M}}$, we have the equation

$$r(xy) = (rx)y.$$

Observe that any topological ringoid can be considered a topological $\mathbf{Z}$-algebroid. The ringoid $\mathcal{V}$ we mentioned in Example 2.2 is an example of a topological $\mathbf{R}$-algebroid.

Definition 2.9. Let R be a commutative topological ring with an identity element. A *non-unital topological algebroid*, $\mathcal{M}$, consists of a set of objects $\operatorname{Ob}(\mathcal{M})$, and a topological R-module, $\operatorname{Hom}(a,b)_{\mathcal{M}}$, for each pair of objects $a, b \in \operatorname{Ob}(\mathcal{M})$ such that the following axioms are satisfied:

- There is a continuous associative *composition law*

$$\operatorname{Hom}(b,c)_{\mathcal{M}} \times \operatorname{Hom}(a,b)_{\mathcal{M}} \longrightarrow \operatorname{Hom}(a,c)_{\mathcal{M}}.$$

• Given an element $r \in R$, and morphisms $x \in \mathrm{Hom}\,(a,b)_{\mathcal{M}}$ and $y \in \mathrm{Hom}\,(b,c)_{\mathcal{M}}$, we have the equation

$$r(xy) = (rx)y.$$

Thus, a non-unital R-algebroid is a collection of objects and morphisms similar to an R-algebroid, apart from the fact that identity endomorphisms need not exist.

In the rest of this article, when we refer to an R-algebroid, it may by non-unital. We will refer to a *unital* R-algebroid or *non-unital* R-algebroid when we wish to be more specific.

There are obvious notions of a homomorphism of R-algebroids, and of a sub-algebroid of an R-algebroid. When necessary, we will use these notions without further comment.

3. K-theory of additive categories. It is well known (see, for example, [**19, 22**]) how to define the K-theory of discrete additive categories. The same approach can be used when there is a topology involved.

Definition 3.1. Let $\mathcal{R}$ be an additive topological category. Then we define $N_n\mathcal{R}$ to be the category in which the objects are n-tuples of the form

$$(a_1, \dots, a_n), \quad a_i \in \mathrm{Ob}\,(\mathcal{R})$$

together with choices of biproduct $a_{i_1} \oplus \cdots \oplus a_{i_k}$ whenever $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$.

The morphisms and topology in the category $N_n\mathcal{R}$ are those arising from the morphisms and topology in the category $\mathcal{R}^n$.

Let $N_\bullet\mathcal{R}$ denote the sequence of topological categories $(N_n\mathcal{R})$. Then the sequence $N_\bullet\mathcal{R}$ can be considered a simplicial topological category. We have face maps $\sigma_i\colon N_n\mathcal{R} \to N_{n-1}\mathcal{R}$ defined by writing

$$\sigma_i(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_i \oplus a_{i+1}, a_{i+2}, \dots, a_n), \quad i \neq 0$$

and

$$\sigma_0(a_1, \dots, a_n) = (a_2, \dots, a_n), \quad i \neq 0.$$

We define degeneracy maps $\tau_i\colon N_n\mathcal{R} \to N_{n+1}\mathcal{R}$ are by writing

$$\tau_i(a_1, \dots, a_n) = (a_i, \dots, a_i, 0, a_{i+1}, \dots, a_n).$$

Note that the above formulae clearly determine what happens to morphisms and choices of biproducts. It is easy to see that the category $N_\bullet\mathcal{R}$ is a simplicial additive topological category. We can therefore iterate the above construction to obtain an n-simplicial additive topological category $N_\bullet^{(n)}\mathcal{R} := N_\bullet \cdots N_\bullet\mathcal{R}$. Let us write $wN_\bullet^{(n)}\mathcal{R}$ to denote the category of isomorphisms in this category.

Definition 3.2. We define the *K-theory spectrum*, $\mathbf{K}(\mathcal{R})$, of the additive topological category $\mathcal{R}$ to be the spectrum where the space $\mathbf{K}(\mathcal{R})_n$ is the geometric realization $|wN_\bullet^{(n)}\mathcal{R}|$ whenever $n \geq 1$.

There is a map $\Sigma|wN_\bullet^{(n)}\mathcal{R}| \to |wN_\bullet^{(n+1)}\mathcal{R}|$ defined by inclusion of the 1-skeleton. The structure map

$$\mathbf{K}(\mathcal{R})_n \longrightarrow \Omega\mathbf{K}(\mathcal{R})_{n+1}$$

is defined to be the adjoint of the above inclusion.

Note that we are using here the 'thick' geometric realization from Appendix A of [**19**]. When dealing with simplicial spaces rather than just simplicial sets, the thick geometric realisation has appropriate formal properties involving homotopies and fibrations.

The paper [**22**] contains a construction of K-theory for discrete categories equipped with certain subcategories, called *categories of cofibrations* and *categories of weak equivalences*, that satisfy certain axioms. It is shown in subsection 1.8 of [**22**] that this general construction of K-theory agrees with the above construction when we are looking at a discrete additive category.

The proof we mention still works for topological categories, although a topological version of Quillen's Theorem A from [**17**] is needed. A suitable generalisation can be found, for example, in [**23**, Section 4].

Many other properties of the K-theory of additive topological categories (or, more generally, categories with cofibrations and weak equivalences) can be proven exactly as in the discrete case. In particular, the proof of the additivity theorem in [**8**] works without any modification at all, and many formal properties of K-theory can be obtained as corollaries of the additivity theorem, see [**20**] for details of the argument.

The following result can be proved in this way, as can the fibration theorem and cofinality theorem we state later on in this section.

Theorem 3.3. *Let $\mathcal{R}$ be a topological ringoid. Then the structure map*

$$\mathbf{K}(\mathcal{R})_n \longrightarrow \Omega\mathbf{K}(\mathcal{R})_{n+1}$$

is a weak homotopy equivalence whenever $n \geq 1$.

Whenever $n \geq 1$, we can define *K-theory groups*

$$K_n(\mathcal{R}) := \pi_n\mathbf{K}(\mathcal{R}).$$

It follows from the above result that these groups can be defined as ordinary homotopy groups

$$K_n(\mathcal{R}) = \pi_n\Omega|wN_\bullet\mathcal{R}_\oplus|$$

rather than as the stable homotopy groups of some spectrum.

Definition 3.4. Let $f\colon\mathcal{A} \to \mathcal{B}$ be a continuous additive functor. Let $PN_\bullet\mathcal{B}$ be the simplicial category defined by writing $(PN_\bullet\mathcal{B})_n = N_{n+1}\mathcal{B}$. Then we define the simplicial additive topological category $N_\bullet(f\colon\mathcal{A} \to \mathcal{B})$ by the pullback diagram

$$\begin{tikzcd}
N_\bullet(f\colon\mathcal{A} \to \mathcal{B}) \arrow[r] \arrow[d] & PN_\bullet\mathcal{B} \arrow[d] \\
N_\bullet\mathcal{A} \arrow[r] & N_\bullet\mathcal{B}
\end{tikzcd}$$

The simplicial category $PN_\bullet\mathcal{B}$ is clearly additive. It is straightforward to show that the pullback $N_\bullet(f\colon\mathcal{A} \to \mathcal{B})$ is also additive. We can therefore define spaces $|wN_\bullet^{(n)}(f\colon \mathcal{A} \to \mathcal{B})|$ through geometric realization, and have a K-theory spectrum

$$\mathbf{K}(f\colon \mathcal{A} \longrightarrow \mathcal{B}).$$

For obvious reasons, we will call the following result the *fibration theorem*.

Theorem 3.5. *The canonical sequence*

$$\mathbf{K}(f\colon \mathcal{A} \longrightarrow \mathcal{B}) \longrightarrow \mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{B})$$

is a fibration up to homotopy.

Let $\mathcal{A}$ be an additive subcategory of some additive category $\mathcal{B}$. Then we say that the category $\mathcal{A}$ is *strictly cofinal* in $\mathcal{B}$ if for each object $b \in \mathrm{Ob}\,(\mathcal{B})$, there is an object $a \in \mathrm{Ob}\,(\mathcal{A})$ such that the biproduct $b \oplus a$ is isomorphic to some object in the category $\mathcal{A}$.

The following result is known as the *cofinality theorem*, and is also a corollary of the additivity theorem.

Theorem 3.6. *Let $\mathcal{B}$ be an additive topological category, and let $\mathcal{A}$ be a strictly cofinal subcategory of $\mathcal{B}$. Then the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ induces a homotopy equivalence of K-theory spectra.*

4. K-theory of ringoids and moduloids.

Definition 4.1. Let $\mathcal{R}$ be a topological ringoid. Then we define the K-theory of $\mathcal{R}$ by the formula

$$\mathbf{K}(\mathcal{R}) := \mathbf{K}(\mathcal{R}_\oplus).$$

If the ringoid $\mathcal{R}$ is already additive, then it is clearly cofinal in the category $\mathcal{R}_\oplus$. Therefore, by Theorem 3.6 the above definition is consistent, at least up to homotopy, with the old definition made for additive topological categories.

Given a commutative topological ring R with an identity element, a unital R-algebroid $\mathcal{M}$ is certainly a topological ringoid, so we can define the K-theory $\mathbf{K}(\mathcal{M})$ by the above procedure. We need to do slightly more in the non-unital case, however.

Definition 4.2. Let $\mathcal{M}$ be a non-unital algebroid over a commutative topological ring with identity, R. Then we define the *unitization* $\mathcal{M}^+$ to be the category with the same objects as the algebroid $\mathcal{M}$, and with morphism sets

$$\mathrm{Hom}\,(a,b)_{\mathcal{M}^+} = \begin{cases} \mathrm{Hom}\,(a,b)_{\mathcal{M}} & a \neq b \\ \mathrm{Hom}\,(a,a) \oplus R & a = b. \end{cases}$$

The topology on the morphism sets is either the original topology or the product topology, depending on which of the above cases we are considering. Composition of morphisms

$$\mathrm{Hom}\,(b,c)_{\mathcal{M}^+} \times \mathrm{Hom}\,(a,b)_{\mathcal{M}^+} \longrightarrow \mathrm{Hom}\,(a,c)_{\mathcal{M}^+}$$

is defined by the formula

$$(x+\lambda)(y+\mu) = xy + \lambda x + \mu y + \lambda\mu,$$

where $x \in \mathrm{Hom}\,(b,c)_{\mathcal{M}}$, $y \in \mathrm{Hom}\,(a,b)_{\mathcal{M}}$, and

$$\lambda \in \begin{cases} R & b = c \\ \{0\} & b \neq c, \end{cases} \qquad \mu \in \begin{cases} R & a = b \\ \{0\} & a \neq b. \end{cases}$$

It is easy to check that the unitization $\mathcal{M}^+$ is a unital R-algebroid.

Let us write $R_{\mathcal{M}}$ to denote the unital R-algebroid with the same set of objects as the algebroid $\mathcal{M}$, and morphism sets

$$\mathrm{Hom}\,(a,b)_{R_{\mathcal{M}}} = \begin{cases} R & a = b \\ \{0\} & a \neq b, \end{cases}$$

equipped with the obvious multiplication and topology. There is a natural homomorphism $\pi\colon \mathcal{M}^+ \to R_{\mathcal{M}}$ defined to be the identity map on the set of objects, and by the formula

$$\pi(x+\lambda)=\lambda, \quad x \in \operatorname{Hom}(a,b)_{\mathcal{M}},\ \lambda \in R$$

on each set of objects.

Definition 4.3. We define the K-theory spectrum of a non-unital algebroid $\mathcal{M}$ by the formula

$$\mathbf{K}(\mathcal{M}) = \mathbf{K}(\pi\colon \mathcal{M}^+ \longrightarrow R_{\mathcal{M}}).$$

According to Theorem 3.5, the spectrum $\mathbf{K}(\mathcal{M})$ is the homotopy fiber of the map

$$\pi_\star\colon \mathbf{K}(\mathcal{M}^+) \longrightarrow \mathbf{K}(R_{\mathcal{M}}).$$

Before we go any further with our explorations, we need to prove that the above definition is consistent with the old definition when the R-algebroid $\mathcal{M}$ is unital.

Proposition 4.4. *Let $\mathcal{M}$ be a unital R-algebroid. Let $\pi'\colon \mathcal{M}\oplus R_{\mathcal{M}} \to R_{\mathcal{M}}$ be the obvious quotient map. Then there is a natural isomorphism $\alpha\colon \mathcal{M}\oplus R_{\mathcal{M}} \to \mathcal{M}^+$ fitting into a commutative diagram*

$$\begin{tikzcd}
\mathcal{M}\oplus R_{\mathcal{M}} \arrow[r, "\pi'"] \arrow[d] & R_{\mathcal{M}} \arrow[d, equal] \\
\mathcal{M}^+ \arrow[r, "\pi"] & R_{\mathcal{M}}
\end{tikzcd}$$

Proof. Let us write $e_a \in \operatorname{Hom}(a,a)_{\mathcal{M}}$ to denote an identity morphism in the algebroid $\mathcal{M}$. We can define a natural isomorphism $\alpha\colon \mathcal{M}\oplus R_{\mathcal{M}} \to \mathcal{M}^+$ by writing $\alpha(a)=a$ for each object a, and

$$\alpha(x,\lambda) = x - \lambda e_a + \lambda$$

whenever $(x,\lambda) \in \operatorname{Hom}(a,b)_{\mathcal{M}\oplus R_{\mathcal{M}}}$.

The inverse of the homomorphism α is defined by the formula

$$\alpha^{-1}(y+\mu) = (y+\mu e_a, \mu)$$

whenever $y+\mu \in \operatorname{Hom}(a,b)_{\mathcal{M}^+}$. ◻

Corollary 4.5. *Let $\mathcal{M}$ be a unital algebroid. Then the K-theory spectra $\mathbf{K}(\mathcal{M})$ and $\mathbf{K}(\pi\colon \mathcal{M}^+ \to R_{\mathcal{M}})$ are naturally homotopy-equivalent.*

Proof. By the above proposition and Theorem 3.5, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{K}(\mathcal{M}) \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{M} \oplus R_{\mathcal{M}}) \arrow[r] \arrow[d] & \mathbf{K}(R_{\mathcal{M}}) \arrow[d, equal] \\
\mathbf{K}(\pi\colon \mathcal{M}^+ \to R_{\mathcal{M}}) \arrow[r] & \mathbf{K}(\mathcal{M}^+) \arrow[r] & \mathbf{K}(R_{\mathcal{M}})
\end{tikzcd}$$

where the rows are homotopy fibrations. The desired result follows easily. □

5. Quotients and exact sequences.

Definition 5.1. Let R be a commutative topological ring with identity, and let $\mathcal{M}$ be an R-algebroid. Let $\mathcal{J}$ be a sub-algebroid with the same set of objects as the algebroid $\mathcal{M}$. Then we call the sub-algebroid $\mathcal{J}$ an *ideal* in the algebroid $\mathcal{M}$ if the product of a morphism from the algebroid $\mathcal{J}$ and a morphism from the algebroid $\mathcal{M}$ is always a morphism in the algebroid $\mathcal{J}$.

Definition 5.2. Let $\mathcal{J}$ be an ideal in an R-algebroid $\mathcal{M}$. Then we define the *quotient* $\mathcal{M}/\mathcal{J}$ to be the R-algebroid with the same objects as the algebroid $\mathcal{M}$, with morphism sets

$$\mathrm{Hom}\,(a,b)_{\mathcal{M}/\mathcal{J}} := \mathrm{Hom}\,(a,b)_{\mathcal{M}/\mathcal{J}}/\mathrm{Hom}\,(a,b)_{\mathcal{J}}.$$

It is straightforward to check that the above definition makes sense, and does indeed define an R-algebroid as inherently claimed. There is a canonical inclusion map $i\colon \mathcal{J} \to \mathcal{M}$ and quotient map $j\colon \mathcal{M} \to \mathcal{M}/\mathcal{J}$.

Theorem 5.3. *The sequence*

$$\mathbf{K}(\mathcal{J}) \xrightarrow{i_\star} \mathbf{K}(\mathcal{M}) \xrightarrow{j_\star} \mathbf{K}(\mathcal{M}/\mathcal{J})$$

is a fibration up to homotopy.

Proof. Suppose that the algebroid $\mathcal{M}$ is unital. Recall that the spectrum $\mathbf{K}(\mathcal{J})$ is defined by iterating the simplicial additive category $N_\bullet(\pi\colon \mathcal{J}^+ \to R_\mathcal{M})$ defined through the pull-back diagram

$$\begin{tikzcd} N_\bullet(\pi\colon \mathcal{J}^+ \to R_\mathcal{M}) \arrow[r] \arrow[d] & PN_\bullet\mathcal{R}_\mathcal{M} \arrow[d] \\ N_\bullet\mathcal{J}^+ \arrow[r] & N_\bullet R_\mathcal{M}. \end{tikzcd}$$

Since the algebroid $\mathcal{M}$ is unitial, there is an obvious inclusion $\mathcal{J}^+ \hookrightarrow \mathcal{M}$ and a commutative diagram

$$\begin{tikzcd} N_\bullet(\pi\colon \mathcal{J}^+ \to R_\mathcal{M}) \arrow[r] \arrow[d] & PN_\bullet\mathcal{R}_\mathcal{M} \arrow[d] \\ N_\bullet\mathcal{J}^+ \arrow[r] \arrow[d] & N_\bullet R_\mathcal{M} \arrow[d] \\ N_\bullet\mathcal{M} \arrow[r] & \mathcal{M}/\mathcal{J}. \end{tikzcd}$$

The outer square of this diagram is still a pull-back square. It follows that we have an isomorphism

$$N_\bullet(\pi\colon \mathcal{J}^+ \longrightarrow R_\mathcal{M}) \cong N_\bullet(j\colon \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{J}).$$

It follows from Theorem 3.5 that

$$\mathbf{K}(\mathcal{J}) \xrightarrow{i_\star} \mathbf{K}(\mathcal{M}) \xrightarrow{j_\star} \mathbf{K}(\mathcal{M}/\mathcal{J})$$

is a fibration up to homotopy.

Now, suppose that the algebroid $\mathcal{M}$ is not unital. Then by the above argument we have a commutative diagram

$$\begin{tikzcd} \mathbf{K}(\mathcal{J}) \arrow[r] \arrow[d, equal] & \mathbf{K}(\mathcal{M}) \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{M}/\mathcal{J}) \arrow[d] \\ \mathbf{K}(\mathcal{J}) \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{M}^+) \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{M}^+/\mathcal{J}) \arrow[d] \\ * \arrow[r] & \mathbf{K}(R_\mathcal{M}) \arrow[r, equal] & \mathbf{K}(R_\mathcal{M}) \end{tikzcd}$$

where the columns and bottom two rows are fibrations up to homotopy. A standard diagram chase tells us that the top row is also a fibration up to homotopy, and we are done. □

6. Banach categories and group completion. Recall that a topological ringoid $\mathcal{A}$ is called a *Banach category* if each morphism set $\mathrm{Hom}(a,b)_{\mathcal{A}}$ is a Banach space, and the inequality

$$\|xy\| \leq \|x\|\|y\|$$

holds whenever x and y are composable morphisms. Since a Banach category $\mathcal{A}$ is already a topological ringoid, we can use our earlier definitions to form the K-theory spectrum $\mathbf{K}(\mathcal{A})$.

The papers [**5, 13**] define K-theory for C^*-categories, of which Banach categories are a more general case. However, the definitions in either of these papers could easily be generalized to Banach categories. This generalization amounts to the following.

Definition 6.1. Let $\mathcal{A}$ be a Banach category. Fix an ordering on the set of objects $\mathrm{Ob}\,(\mathcal{A})$. For each object $a \in \mathrm{Ob}\,(\mathcal{A}_\oplus)$, let $\mathrm{GL}\,(A)$ denote the space of invertible endomorphisms $x \in \mathrm{Hom}\,(a,a)_{\mathcal{A}_\oplus}$.

We define the topological group $\mathrm{GL}_\infty(\mathcal{A})$ to be the direct limit

$$\bigcup_{a_1 \leq \cdots \leq a_n} \mathrm{GL}\,(a_1 \oplus \cdots \oplus a_n)$$

under the inclusions

$$\mathrm{GL}\,(a_1 \oplus \cdots \oplus a_{k-1} \oplus a_{k+1} \oplus \cdots \oplus a_n) \hookrightarrow \mathrm{GL}\,(a_1 \oplus \cdots \oplus a_n)$$

defined in the obvious way by looking at matrices.

The space $\mathrm{GL}_\infty(\mathrm{A})$ is not functorial. However, it can if necessary be replaced by a homotopy-equivalent space that is functorial. We do not need to do this here; see [**13**] for details.

Definition 6.2. We define the *analytic K-theory groups* of the Banach category $\mathcal{A}$ to be the homotopy groups

$$K_n(\mathcal{A})^{\mathrm{Analytic}} := \pi_n \mathrm{BGL}_\infty(\mathcal{A})$$

when $n \geq 1$.

The *suspension* of a Banach category, $\mathcal{A}$, is defined to be the Banach category $\Sigma\mathcal{A}$ with the same objects as $\mathcal{A}$, where the morphism set $\mathrm{Hom}\,(a,b)_{\mathcal{A}}$ is defined to be the set of all continuous maps $f\colon [0,1] \to \mathcal{A}$ such that $f(0) = f(1) = 0$. Addition and composition of morphisms are pointwise, and the norm on the space $\mathrm{Hom}\,(a,b)_{\mathcal{A}}$ is defined by the formula

$$\|f\| = \sup\{\|f(t)\| \mid t \in [0,1]\}.$$

The following result is obvious.

Proposition 6.3. *Let $\mathcal{A}$ be a Banach category. Then the spaces $\mathrm{GL}_\infty(\Sigma\mathcal{A})$ and $\Omega\mathrm{GL}_\infty(\mathcal{A})$ are naturally homotopy-equivalent.*

In contrast, the *algebraic K-theory groups* of a topological ringoid $\mathcal{R}$ are defined by writing

$$K_n(\mathcal{A}) := \pi_n \mathbf{K}(\mathcal{A}) = \pi_n \Omega |wN_\bullet \mathcal{A}_\oplus|$$

when $n \geq 1$.

The main purpose of this section is to prove the fact that the algebraic and analytic K-theory groups of a Banach category are isomorphic.

The argument required is an application of the group completion theorem. The version given in [**9**] suffices for our purposes.

Theorem 6.4. *Let M be a topological monoid, and let $H_\star(M)$ be the associated Pontryagin ring built from the induced product on singular homology. Suppose that the monoid $\pi = \pi_0(M)$ is in the center of the ring $H_\star(M)$. Then the ring homomorphism $i_\star\colon H_\star(M)[\pi^{-1}] \to H_\star(\Omega BM)$ induced by the canonical map $i\colon M \to \Omega BM$ induces an isomorphism*

$$H_\star(M)[\pi^{-1}] \to H_\star(\Omega BM).$$

Along with the group completion theorem, we need a sufficiently general version of Whitehead's theorem. The following result is proved in [**7**].

Theorem 6.5. *Let X and Y be path-connected H-spaces. Suppose we have a map of H-spaces $f\colon X \to Y$ such that the induced maps of* *homology groups $f_\star\colon H_p(X) \to H_p(Y)$ are all isomorphisms. Then the map f is a weak homotopy equivalence.*

Before applying the above two results, observe that we can simplify our calculations slightly if we observe that there are isomorphisms

$$K_n(\mathcal{A})^{\text{Analytic}} \cong \pi_n \mathrm{BGL}_\infty(Sk(\mathcal{A}))$$

and

$$K_n(\mathcal{A}) \cong \pi_n \left(\Omega B \amalg_{a\in Sk(\mathcal{A}_\oplus)} BGL(a)\right)$$

where $Sk(\mathcal{C})$ denotes the skeleton of a category $\mathcal{C}$. Roughly speaking, our plan is to prove that the spaces $\mathrm{BGL}_\infty(Sk(\mathcal{A}))$ and $\Omega B \amalg_{a\in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a)$ are weakly homotopy-equivalent.

In order to apply the group completion theorem, we want to define a topological monoid structure on the space

$$M := \amalg_{a\in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a).$$

Any two objects $a, b \in Sk(\mathcal{A}_\oplus)$ have a unique biproduct $a \oplus b$. Let $i_a\colon a \to a\oplus b$, $i_b\colon b \to a\oplus b$, $p_a\colon a\oplus b \to a$ and $p_b\colon a\oplus b \to b$ be the canonical morphisms associated to this biproduct, and define maps $j_a\colon GL(a) \to GL(a\oplus b)$ and $j_b\colon GL(b) \to GL(a\oplus b)$ by the formulae

$$j_a(x) = i_a x p_a + i_b p_b, \qquad j_b(y) = i_a p_a + i_b y p_b,$$

respectively.

After delooping, we have induced continuous maps $Bj_a\colon BGL(a) \to BGL(a\oplus b)$ and $Bj_b\colon BGL(b) \to BGL(a\oplus b)$. The space M is therefore a topological monoid, with operation

$$x \oplus y = Bj_a(x) + Bj_b(y).$$

Observe that the set $\pi = \mathrm{Ob}\,(Sk(\mathcal{A}_\oplus))$ can be identified with the set of path-components $\pi_0(M)$. Let G be the Grothendieck completion of the monoid π. Then there is a homomorphism of topological monoids

$$\alpha\colon M \to \mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G$$

defined by taking a point $x \in \mathrm{BGL}\,(a)$ to the pair $(i(x), a)$ where $i\colon \mathrm{BGL}\,(a) \to \mathrm{BGL}_\infty(\mathcal{A})$ is the canonical inclusion.

Lemma 6.6. *The above homomorphism induces a ring isomorphism*

$$\alpha_\star\colon H_\star(M)[\pi^{-1}] \longrightarrow H_\star(BGL_\infty(Sk(\mathcal{A}) \times G)).$$

We defer the proof of the above lemma. Together with the group completion theorem, this lemma is the key to our main result.

Theorem 6.7. *The spaces* $\mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G$ *and* $\Omega B \amalg_{a \in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a)$ *are weakly homotopy-equivalent.*

Proof. The monoid π clearly lies is the center of the Pontryagin ring $H_\star(M)$. The map α induces a map $\Omega BM \to \Omega B(\mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G)$ of H-spaces. But the space $\mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G$ is already a topological group, meaning it is homotopy-equivalent to the H-space $\Omega B(\mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G)$. Hence, by the above lemma and the group completion theorem, the canonical map of H-spaces

$$\Omega B \amalg_{a \in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a) \longrightarrow \mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G$$

induces an isomorphism at the level of homology.

Now, let X_0 denote the path-component of the H-space

$$\Omega B \amalg_{a \in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a)$$

containing the unit. Then we have a map of path-connected H-spaces

$$X_0 \longrightarrow \mathrm{BGL}_\infty(Sk(\mathcal{A}))$$

that induces an isomorphism at the level of homology. Hence, by THEOREM 6.5, the map $X_0 \to \mathrm{BGL}_\infty(\mathrm{Sk}\,(\mathcal{A}))$ is a homotopy-equivalence.

Each path-component of the space $\Omega B \amalg_{a \in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a)$ is homotopy-equivalent to the space X_0. Looking at each path-component in turn, it follows that we have a homotopy-equivalence

$$\Omega B \amalg_{a \in Sk(\mathcal{A}_\oplus)} \mathrm{BGL}\,(a) \longrightarrow \mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G,$$

as required. ∎

Hence, by the definitions of the relevant K-theory groups, we obtain the isomorphism

$$K_n(\mathcal{A}) \cong K_n(\mathcal{A})^{\mathrm{A}nalytic}$$

that we were looking for.

Of course, we still need to prove Lemma 6.6. We need two technical results about analytic K-theory. The first of these results is a special case of Theorem 3.1 in [**13**]. See also [**11**, Appendix].

Proposition 6.8. *Let $(X_i, \phi_{ij})_{i\in I}$ be a directed family of topological spaces such that each map $\phi_{ij}\colon X_i \to X_j$ is injective, each space X_i has the T_1-separation property, that is to say, that finite subsets are closed, and for each element $j \in I$ there are only finitely many elements $i \in I$ such that $i \leq j$.*

Let X be the direct limit of this family, and let $\phi_i\colon X_i \to X$ be the canonical map associated to each space X_i. Suppose that $K \subseteq X$ is a compact set. Then $K \subseteq \phi_i[X_i]$ for some space X_i.

The second technical proposition follows from commutativity of the analytic K-theory groups, a result again proved in [**13**].

Proposition 6.9. *The topological monoid $\mathrm{BGL}_\infty(Sk(\mathcal{A}))$ is homotopy-commutative.*

Let σ be a permutation of the set $\{1, \ldots, k\}$. It follows from the above results that any map $X \to \mathrm{BGL}\,(a_1 \oplus \cdots \oplus a_k)$ is homotopic to a map $X \to \mathrm{BGL}\,(\sigma(a_1) \oplus \cdots \oplus \sigma(a_k))$. With this observation, we can proceed to the proof of our lemma.

Lemma 6.10. *The homomorphism $\alpha\colon M \to \mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G$ induces a ring isomorphism*

$$\alpha_\star\colon H_\star(M)[\pi^{-1}] \longrightarrow H_\star(\mathrm{BGL}_\infty(Sk(\mathcal{A}) \times G)).$$

Proof. Let $[\sigma] \in H_k(\mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G)$ be a singular homology class, generated by a continuous map

$$\sigma\colon \Delta^k \longrightarrow \mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G.$$

By Proposition 6.8, we know that $\operatorname{im}\sigma \subseteq \mathrm{BGL}\,(a) \times \{g\}$ for some object $a \in \mathrm{Ob}\,(Sk(\mathcal{A})_\oplus)$ and element $g \in G$. By Proposition 6.9, we can assume that $a \in \mathrm{Ob}\,(Sk(\mathcal{A}_\oplus))$. Let us write $g = a \oplus b \oplus c^{-1}$ where $b, c \in \pi$. Then certainly

$$\operatorname{im}\sigma \subseteq \mathrm{BGL}\,(a \oplus b) \times \{a \oplus b \oplus c^{-1}\}.$$

Hence, by definition of the map α, the homology class $[\sigma]$ lies in the image of the homomorphism

$$\alpha_\star \colon H_\star(m)[\pi^{-1}] \longrightarrow H_\star(\mathrm{BGL}_\infty(Sk(\mathcal{A})) \times G).$$

Thus, the homomorphism $\alpha_\star$ is surjective. Injectivity can be proven similarly. $\square$

7. Assembly. The assembly map in algebraic K-theory originally appeared in [**6**]. A version of this map, adapted to give a homotopy-theoretic description of the analytic assembly map, was used in [**21**].

In this section we will describe the K-theory assembly map in terms of the tools developed here, and give a new proof that the homotopy-theoretic description of the analytic assembly map agrees with the usual definition.

Let π be a discrete groupoid, and let R be a commutative topological ring with an identity element. Recall from Example 2.3 that we define the groupoid ringoid, $R\pi$, to be the category with objects $\mathrm{Ob}\,(R\pi) = \mathrm{Ob}\,(\pi)$ and morphism sets

$$\mathrm{Hom}\,(a,b)_{R\pi} := \{x_1 g_1 + \cdots + x_n g_n \mid x_i \in R, g_i \in \mathrm{Hom}\,(a,b)_\pi\}.$$

Proposition 7.1. *Let R be a commutative topological ring with an identity element. Then the category $R\pi$ is isomorphic to the tensor product $\mathbf{Z}\pi \otimes_{\mathbf{Z}} R$.*

Proof. We can define an isomorphism $\theta\colon R\pi \to \pi\mathbf{Z} \otimes_{\mathbf{Z}} R$ by writing $\theta(a) = a$ for each object a, and

$$\theta(x_1 g_1 + \cdots + x_n g_n) = g_1 \otimes x_1 + \cdots + g_n \otimes x_n$$

for all elements $x_i \in R$ and $g_i \in \mathrm{Hom}\,(a,b)_\pi$. $\square$

Now, we have an inclusion

$$\Sigma|w\mathbf{Z}\pi_{\oplus}| \hookrightarrow |wN_{\bullet}\mathbf{Z}\pi_{\oplus}|$$

defined by looking at the 1-skeleton, and therefore, taking adjoints, a natural map

$$|w\mathbf{Z}\pi_{\oplus}| \longrightarrow \Omega|wN_{\bullet}\mathbf{Z}\pi_{\oplus}|.$$

Composing this map with the inclusion $B\pi \hookrightarrow \Sigma|w\mathbf{Z}\pi_{\oplus}|$, we have a natural map

$$B\pi_{+} \longrightarrow \Omega|wN_{\bullet}\mathbf{Z}\pi_{\oplus}|.$$

Here we write X_{+} to denote a topological space with an added disjoint basepoint. If we do this, we can deem the above map to be basepoint-preserving. We can compose the above map with the exterior product to obtain a natural map

$$B\pi_{+} \wedge \mathbf{K}(R) \longrightarrow \mathbf{K}(\mathbf{Z}\pi) \wedge \mathbf{K}(R) \longrightarrow \mathbf{K}(R\pi).$$

Definition 7.2. The above map is called the *K-theory assembly map* associated to the groupoid π and topological ring R.

In order to compare definitions of assembly maps, we need an abstract characterization. One suitable characterization can be found in [**24**].

Definition 7.3. Let F be a covariant functor from the category of spaces to the stable category of spectra. Then we call the functor F

- *homotopy-invariant* if it preserves homotopy-equivalences.
- *excisive* if it preserves coproducts up to homotopy equivalence.

Let $+$ denote the one point topological space.

Theorem 7.4. *Let F be a homotopy-invariant functor from the category of spaces to the stable category of spectra. Then there is a homotopy-invariant excisive functor $F^{\%}$ and a natural transformation $\alpha\colon F^{\%} \to F$ such that the map $\alpha\colon F^{\%}(+) \to F(+)$ induces an isomorphism at the level of stable homotopy groups.*

Further, the functor $F^{\%}$ and natural transformation α are unique up to stable equivalence.

We call the natural transformation $\alpha\colon F^{\%} \to F$ the *assembly map* associated to the functor F.

Theorem 7.5. *Let X be a topological space, and let $\pi(X)$ be the fundamental groupoid of X, equipped with the discrete topologicity. Write*

$$F(X) = \mathbf{K}(R\pi(X)), \qquad F^{\%}(X) = X_+ \wedge \mathbf{K}(R).$$

Then there is a natural transformation $\alpha\colon F^{\%} \to F$ such that the map

$$\alpha\colon F^{\%}(B\pi) \longrightarrow F(\pi)$$

is homotopy-equivalent to the K-theory assembly map for a discrete groupoid π.

Further, suppose we have a natural transformation $\alpha'\colon F^{\%} \to F$ such that the map $\alpha'\colon F^{\%}(+) \to F(+)$ is a weak homotopy-equivalence. Then the natural transformation α' is weakly homotopic to the map α.

Proof. Let $\pi^\tau(X)$ denote the fundamental groupoid of the space X, equipped with the topologicity defined by considering the groupoid $\pi^\tau(X)$ to be a quotient of the space of Moore paths

$$\{\gamma\colon [0,s] \to X \mid s \geq 0\}$$

equipped with the compact open topology. Then the identity map defines a continuous functor $\pi(X) \to \pi^\tau(X)$. It is proved in May [**18**, Appendix] that the induced map of classifying spaces

$$f\colon B\pi(X) \longrightarrow B\pi^\tau(X)$$

is a homotopy-equivalence. Hence, the induced map of spectra $f_\star\colon F^{\%}(B\pi(X)) \to F^{\%}(B\pi^\tau(X))$ has a formal inverse $f^\star$ in the category of spectra.

There is a natural inclusion $X \to B\pi^\tau(X)$ defined by mapping a point $x \in X$ to the point corresponding to the identity 1_x. At the level of spectra we obtain a natural map β by forming the composition

$$F^{\%}(X) \longrightarrow F^{\%}(B\pi^\tau(X)) \xrightarrow{f^\star} F^{\%}(B\pi(X)).$$

Composing the map β with the K-theory assembly map, we obtain a natural map $\alpha\colon F^{\%}(X) \to F(X)$, which is a homotopy-equivalence when the space X is a single point. Let π be a discrete groupoid, and let $X = B\pi$. Then the map α is homotopic to the K-theory assembly map.

Finally, note that it is easy to see that the functor F is homotopy-invariant, and the functor $F^{\%}$ is homotopy-invariant and excisive. Therefore, the map α is an assembly map in the sense of the above theorem. Hence, if we have a natural transformation $\alpha'\colon F^{\%} \to F$ such that the map $\alpha'\colon F^{\%}(+) \to F(+)$ is a weak homotopy-equivalence, then the natural transformation α' is weakly homotopic to the map α. □

We will end this section by looking at the assembly map appearing in the analytic Novikov conjecture, which we shall refer to as the *analytic assembly map*. To fit this map into our present framework, recall the following result from [**15**], which provides all we need to know about the analytic assembly map for the present purposes.

Theorem 7.6. *Let $F^{\%}$ be a homotopy-invariant and excisive functor from the category of spaces to the stable category of spectra. Let $\alpha\colon F^{\%}(X) \to \mathbf{K}C^*_{\max}\pi(X)$ be a natural map of spectra that is a stable equivalence when the space X is a single point. Let G be a discrete group. Then the map*

$$\alpha\colon F^{\%}(BG) \longrightarrow \mathbf{K}C^*_{\max}\pi(BG) \cong \mathbf{K}C^*_{\max}(G)$$

is a description of the analytic assembly map for the group G at the level of spectra.

Here, for a discrete groupoid π, the category $C^*\pi$ is a certain Banach category obtained by completing the topological ringoid $\mathbf{C}\pi$, see [**14**] for further details. There is a canonical inclusion $\mathbf{C}\pi \hookrightarrow C^*\pi$, and so a natural map $\mathbf{KC}\pi \to \mathbf{K}C^*_{\max}\pi$.

Theorem 7.7. *Let G be a discrete group. The composition of the K-theory assembly map and the above map*

$$BG_+ \wedge \mathbf{K}(\mathbf{C}) \longrightarrow \mathbf{KC}G \longrightarrow \mathbf{K}C^*_{\max}G$$

is a description of the analytic assembly map for the group G at the level of spectra.

Proof. In the proof of Theorem 7.5, we have already commented that the functor $X \mapsto X_+ \wedge \mathbf{K}(\mathbf{C})$ is excisive. We have also seen that the K-theory assembly map is natural. It follows that the composition

$$X_+ \wedge \mathbf{K}(\mathbf{C}) \longrightarrow \mathbf{KC}\pi(X) \longrightarrow \mathbf{K}C^*_{\max}\pi(X)$$

is natural. This map is the identity map and so is certainly a homotopy-equivalence, when the space X is a single point. The desired result now follows from Theorem 7.6. $\square$

A similar result holds when working with the real numbers instead of the complex numbers; the proof is identical.

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