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ON THE RANKS OF THE JACOBIANS OF CURVES DEFINED BY JACOBI POLYNOMIALS

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ABSTRACT. We show that the degree-3 Jacobi Polynomials define two one-parameter families of elliptic curves $\mathcal{E}_r/\mathbf{Q}(r)$ and $\mathcal{F}_s/\mathbf{Q}(s)$ with positive rank outside of an explicit finite set of $(r,s) \in \mathbf{Q} \times \mathbf{Q}$. In addition we initiate a program to study the ranks of the Jacobians of these curves in higher degree.

1. Introduction. Orthogonal polynomials have played a central role in mathematics for centuries, but it is only recently that they have been studied for their arithmetic properties. Schur pioneered this field of arithmetic by studying the irreducibility and Galois-theoretic properties of certain subfamilies of the Hermite and generalized Laguerre polynomials. Schur's results have been generalized by many, including Coleman [**3**], Gow [**5**], Hajir [**6, 7**] and Sell [**11**]. In [**8**], Hajir and Wong initiated an algebro-geometric program for studying the irreducibility and Galois properties of *any* one-parameter family of polynomials. The techniques they employ are different than the earlier approaches and illustrate the far-reaching impact of this area of mathematics. In this paper we study certain arithmetic properties of the *Jacobi polynomials.*

The Jacobi polynomials are a two-parameter family of polynomials, defined as follows:

$$P_n^{(\alpha,\beta)}(x) := \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \binom{n+\alpha+\beta+j}{j} \left(\frac{x-1}{2}\right)^j.$$

These polynomials were originally discovered as solutions to the differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0,$$

and, together with the generalized Laguerre polynomials and Hermite polynomials, make up the three classical families of orthogonal polynomials.

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The arithmetic of Jacobi polynomials is less well-understood than that of Laguerre and Hermite. In particular, Jacobi polynomials yield as special cases the Chebyshev, Gegenbauer, and Legendre polynomials, the irreducibility properties of which are only partially known. An investigation of the arithmetic properties of certain one-parameter subfamilies of the Jacobi polynomials using the program outlined in [**8**] was begun in [**4**]. Jacobi polynomials have also been studied in connection with the arithmetic of elliptic curves. In [**1**], it is shown that there is a connection between the supersingular polynomial $ss_p(t)$ and certain Jacobi polynomials: $ss_p(x) \equiv J_p(x), xJ_p(x), (x-1728)J_p(x)$, or $x(x-1728)J_p(x)$ if $p \equiv 1, 5, 7, 11 \pmod{12}$, respectively, where $J_p(x)$ is an explicit Jacobi polynomial. Other applications of Jacobi polynomials to elliptic curves have been studied by Kaneko and Zagier [**9**] and Mahlburg and Ono [**10**].

In this note we examine Jacobi polynomials as models for curves giving rise to abelian varieties by taking their Jacobians. We begin by introducing a linear change of variables that decouples the (α, β)-coefficients and more clearly shows to which degree these coefficients belong. We define:

$$P_n^{\langle r,s\rangle}(x) := P_n^{(-1-n-r,r+s+1)}(2x+1) = \sum_{j=0}^{n} \binom{-1-r}{n-j}\binom{s+j}{j} x^j.$$

In this parameterization, the Jacobi polynomials enjoy the symmetry relation $P_n^{\langle r,s\rangle}(x) = (-x)^n P_n^{\langle s,r\rangle}(1/x)$, which will be used in Section 3.

Set $n = 3$ and denote by X the projective closure of the zero-set of $P_3^{\langle r,s\rangle}(x)$ so that X defines a projective surface in $\mathbf{P}^3_{\mathbf{Q}}$. Alternatively, one can view the surface $X/\mathbf{Q}$ as defining a curve over the function field $\mathbf{Q}(r)$ or $\mathbf{Q}(s)$; we will see that these give rise to elliptic curves $\mathcal{E}_r$ and $\mathcal{F}_s$. The generalized Mordell-Weil theorem over function fields of number fields applies, and so it makes sense to ask about the rank of the Mordell-Weil groups of these curves. If they have positive rank, then a theorem of Silverman [**12**, page 211] implies that the ranks of the specializations over $\mathbf{Q}$ are positive for all but finitely-many specializations. Once this fact has been established, one can try to compute the exceptional set explicitly. This is our main result.

Theorem 1. *The elliptic curves $\mathcal{E}_r$ and $\mathcal{F}_s$ each have Mordell-Weil group $\mathbf{Z}/3 \times \mathbf{Z}$ over $\mathbf{Q}(r)$ and $\mathbf{Q}(s)$, respectively. Moreover, the* *exceptional set of rank-0 specializations of (r,s) to $\mathbf{Q}\times\mathbf{Q}$ is exactly $S\times S$, where*

$$S=\{0,-3/4,-6/5,-3/2\}.$$

2. The elliptic curves $\mathcal{E}_r$ and $\mathcal{F}_s$. We start by viewing X as a projective curve over the function field $\mathbf{Q}(s)$ given by the zero-set of the homogeneous polynomial

$$\begin{aligned}\mathscr{P}_3(x,r,z) := &\left(\frac{1}{6}s^3+s^2+\frac{11}{6}s+1\right)x^3\\ &+\left(\left(-\frac{1}{2}s^2-\frac{3}{2}s-1\right)r+\left(-\frac{1}{2}zs^2-\frac{3}{2}zs-z\right)\right)x^2\\ &+\left(\left(\frac{1}{2}s+\frac{1}{2}\right)r^2+\left(\frac{3}{2}zs+\frac{3}{2}z\right)r\right.\\ &\left.+\left(z^2s+z^2\right)\right)x+\left(-\frac{1}{6}r^3-zr^2-\frac{11}{6}z^2r-z^3\right).\end{aligned}$$

With a view towards specialization, denote this curve by $\mathcal{F}_s$. We compute the resultants

$$R_1 := \operatorname{Res}\left(P_3^{\langle r,s\rangle}(x),\partial_x P_3^{\langle r,s\rangle}(x)\right),$$

and

$$R_2 := \operatorname{Res}\left(P_3^{\langle r,s\rangle}(x),\partial_r P_3^{\langle r,s\rangle}(x)\right).$$

Altogether:

$$\operatorname{Res}(R_1,R_2)=\frac{(s+1)^{30}(s+2)^{24}(s+3)^{14}(17s^2+39s+18)^2}{2^{36}3^{42}}\neq 0.$$

Thus $\mathcal{F}_s$ is non-singular in the affine plane over $\mathbf{Q}(s)$ and it is similarly easy to check non-singularity at infinity. Therefore $\mathcal{F}_s$ defines a smooth cubic curve with many rational points (e.g., $[0:-1:1]$), and so it is an elliptic curve. A standard change of variables [**13**, Section 3] gives us a Weierstrass model for $\mathcal{F}_s$:

$$\begin{aligned}y^2 = x^3 &+ \frac{-3^5(9s^2+50s+73)}{(s+1)^2(s+2)^4(s+3)^4}x\\ &+\frac{2\cdot 3^6(27s^4+252s^3+866s^2+1340s+827)}{(s+1)^4(s+2)^6(s+3)^6}.\end{aligned}$$

Alternatively, X defines a sextic curve over $\mathbf{Q}(r)$. This curve is smooth in the affine plane and is singular at infinity; the singular points are $[0:1:0]$ and $[1:0:0]$. Let $\pi : X \to \mathbf{P}^1_{\mathbf{Q}(r)}$ denote the projection-to-s map, the branch locus of which is given by the roots of the discriminant:

$$\operatorname{disc}(P_3^{\langle r,s\rangle}(x)) = \frac{-(r+1)(r+2)}{12}(s+1)^2(s+2)(s+3+r)(s+4+r)^2.$$

The Riemann-Hurwitz formula applied to $\pi : X \to \mathbf{P}^1_{\mathcal{Q}(r)}$ yields

$$2g(X) - 2 = (\deg \pi)(-2) + \sum_{p\in X}(e_p - 1),$$

with $\deg \pi = 3$. Plugging in the ramification indices and taking note that there is no ramification at infinity, we see that the genus of $X/\mathbf{Q}(r)$ is 1. Thus $X/\mathbf{Q}(r)$ has a resolution of singularities $\mathcal{E}_r$ which is a smooth curve of genus 1 with a rational point, hence is an elliptic curve. A Weierstrass model for $\mathcal{E}_r$ is

$$\begin{aligned}(1)\qquad y^2 &= x^3 - 3(r+1)^2(9r^2+50r+73)x \\ &\quad + 2(r+1)^2(27r^4+252r^3+866r^2+1340r+827).\end{aligned}$$

This equation is obtained by applying the change of variables in [**13**, Section 3] to the proper transform in Section 3 of this paper when $n = 3$.

Proposition 1. *The Mordell-Weil groups $\mathcal{F}_s(\mathbf{Q}(s))$ and $\mathcal{E}_r(\mathbf{Q}(r))$ are each isomorphic to $\mathbf{Z}/3 \times \mathbf{Z}$.*

Proof. For any elliptic curve E over a field k with Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, the change of variables $x = u^2x' + \alpha$, $y = u^3y' + \beta u^2x' + \gamma$, where $\alpha, \beta, \gamma \in k$ and $u \in k^\times$ preserves the form of the Weierstrass equation, the point at infinity, and is an isomorphism of curves. In the case of $\mathcal{F}_s$, setting $\alpha = \beta = \gamma = 0$ and $u = 3/((s+1)(s+2)(s+3))$ produces the same Weierstrass equation as $\mathcal{E}_r$. It follows that $\mathcal{F}_s$ has the same arithmetic properties as $\mathcal{E}_r$.

If k is the function field $\mathbf{Q}(t)$, a theorem of van Luijk [**14**, Proposition 6.2] states that if p is a rational prime of good reduction for E, and if $\overline{E}$ denotes the reduction modulo p, then $E(\mathbf{Q}(t))$ injects into $\overline{E}(\mathcal{F}_p(t))$. One checks that the discriminant Δ_r of $\mathcal{E}_r$ is $\Delta_r = -1728(r+1)^4(r+2)^3(r+3)^2$. Choose $p=5$ so that by (1), the Weierstrass equation for $\overline{\mathcal{E}_r}$ over $\mathcal{F}_5(r)$ is

$$Y^2 = X^3 + 3(r+1)^2(r^2+2)X - (r+1)^2(r^4+r^3+3r^2+1).$$

This curve has an $\mathcal{F}_5(r)$-rational 3-torsion point, so the Mordell-Weil group can be computed by an explicit descent-by-3-isogeny. We omit the details but instead refer the reader to [**2**] for an explicit approach to such computations. The conclusion is that

$$\overline{\mathcal{E}_r}(\mathcal{F}_5(r)) \simeq \mathbf{Z}/3 \times \mathbf{Z}.$$

To determine the torsion subgroup of $\mathcal{E}_r(\mathbf{Q}(r))$, we look for a 3-torsion point $A(r)$. In order for $[3]A(r) = \mathcal{O}$, we require $[2]A(r) = -A(r)$, and hence that $x([2]A(r)) = x(A(r))$:

$$\frac{x^4+6(r+1)^2(9r^2+50r+73)x^2-16(r+1)^2(27r^4+252r^3+866r^2+1340r+827)x}{4x^3-12(r+1)^2(9r^2+50r+73)x+8(r+1)^2(27r^4+252r^3+866r^2+1340r+827)} + \frac{9(r+1)^4(9r^2+50r+73)^2}{4x^3-12(r+1)^2(9r^2+50r+73)x+8(r+1)^2(27r^4+252r^3+866r^2+1340r+827)} = x.$$

A little algebra reveals that $A(r) = (3(r+1)^2/4, 2(r+1)(r+2))$ has order 3.

We can conclude that $\mathcal{E}_r(\mathbf{Q}(r)) \simeq \mathbf{Z}/3 \times \mathbf{Z}$ provided we exhibit a rational point that does not have order 3. A search for an arbitrary $\mathbf{Q}(r)$-rational point $B(r)$ of $\mathcal{E}_r$ of quadratic type reveals that $B(r) = ((r+1)(3r+11), 8(r+1)(r+3))$ is a point of $\mathcal{E}_r$ which is not of order 3. This proves the proposition. □

It remains to determine the exceptional set of rank-0 specializations. Due to the discriminants of the curves, we only consider specializations of r and s to the set $\mathbf{Q} \setminus \{-1,-2,-3\}$.

Proposition 2. *Let* $B(r) = ((r+1)(3r+11), 8(r+1)(r+3))$, *and let* $r_0 \in \mathbf{Q} \setminus \{0, -3/4, -1, -6/5, -3/2, -2, -3\}$. *Then* $B(r_0)$ *is a point of infinite order on* $\mathcal{E}_{r_0}/\mathbf{Q}$.

Proof. For any $r_0 \in \mathbf{Q} \setminus \{-1,-2,-3\}$, the specialization $\mathcal{E}_{r_0}$ is an elliptic curve defined over $\mathbf{Q}$. We need to show that the only r_0 for which $B(r_0)$ is a torsion point are $r_0 = 0, -3/4, -6/5, -3/2$. By Mazur's theorem, the torsion subgroup of an elliptic curve over $\mathbf{Q}$ is precisely one of the following:

$$\begin{array}{ll} \mathbf{Z}/N: & 1 \leq N \leq 10, \text{ or } N = 12; \\ \mathbf{Z}/2 \times \mathbf{Z}/2N: & 1 \leq N \leq 4. \end{array}$$

We use this information in the following way. If, say, $B(r_0)$ were a 2-torsion point, then $y(B(r_0)) = 0$, i.e., $8(r_0+1)(r_0+3) = 0$. This is impossible since the specializations $r_0 = -1, -3$ are singular. The following table contains information for similar arguments for m-torsion points. In particular, the $\mathbf{Q}$-solutions to the equations in the third column are the only possibilities for rank-0 specializations:

m	Geometric Condition	Algebraic Condition
2	$y(B(r)) = 0$	$8(r+1)(r+3) = 0$
3	$x([2]B(r)) = x(B(r))$	$(3r-13)(r+1) = (3r+11)(r+1)$
4	$y([2]B(r)) = 0$	$16(r+1)(4r+3) = 0$
5	$x([4]B(r)) = x(B(r))$	$\frac{48r^4+248r^3+635r^2+750r+315}{(4r+3)^2} = 3r^2+14r+11$
6	$y([3]B(r)) = 0$	$\frac{32}{27}(r+3)(7r^2+15r+9) = 0$
7	$x([6]B(r)) = x(B(r))$	$\frac{1579r^6+12204r^5+38844r^4+62478r^3+51516r^2+19440r+2187}{9(7r^2+15r+9)^2} = 3r^2+14r+11$
8	$y([4]B(r)) = 0$	$\frac{-16(r+1)(4r^4+165r^3+594r^2+756r+324)}{(4r+3)^3} = 0$
9	$x([8]B(r)) = x(B(r))$	$[(r+1)(768r^{11} - 282880r^{10} - 2164512r^9 - 3793824r^8 + 16329843r^7 + 99899487r^6 + 246635280r^5 + 359761500r^4 + 334611000r^3 + 197144928r^2 + 67709520r + 10392624)]/(4r+3)^3(4r^4+165r^3 + 594r^2+756r+324)^2 = (3r+11)(r+1)$
10	$y([5]B(r)) = 0$	$\frac{-8(r+1)(r+3)(512r^6+5670r^5+22842r^4+45387r^3+48114r^2}{(2r^2-6r-9)^3} + \frac{26244r+5832)}{(2r^2-6r-9)^3} = 0$
12	$y([6]B(r)) = 0$	$\frac{-4(r+3)(4r+3)(5r+6)(208r^6+1044r^5+2673r^4+5967r^3+9963r^2}{8(7r^2+15r+9)^3} + \frac{8748r+2916)}{8(7r^2+15r+9)^3} = 0$

The rational solutions are those that belong to the set $\{-1, -2, -3, -3/4, -3/2, -6/5, 0\}$, which proves the proposition. □

Corollary 1. *The rank-*0 *specializations and Mordell-Weil groups of* $\mathcal{E}_r$ *are precisely the following*:

r_0	$\mathcal{E}_{r_0}$	$\mathcal{E}_{r_0}(\mathbf{Q})$
0	$y^2 = x^3 - 219x + 1654$	$\mathbf{Z}/9$
$-3/4$	$y^2 = x^3 - (1947/256)x + 54107/2048$	$\mathbf{Z}/12$
$-6/5$	$y^2 = x^3 - (1947/625)x + 108214/15625$	$\mathbf{Z}/12$
$-3/2$	$y^2 = x^3 - (219/16)x + 827/32$	$\mathbf{Z}/9$

Proof. If $r_0 \neq -1, -2, -3, -3/4, -3/2, -6/5, 0$ then $\mathcal{E}_{r_0}$ is an elliptic curve of positive rank over $\mathcal{Q}$. Hence, it suffices to check the Mordell-Weil groups of $\mathcal{E}_{r_0}$ for $r_0 \in \{-3/4, -3/2, -6/5, 0\}$ which are as given in the table. [Note: $\mathcal{E}_0$ and $\mathcal{E}_{-3/2}$ are isomorphic as curves, as are $\mathcal{E}_{-3/4}$ and $\mathcal{E}_{-6/5}$.] □

Corollary 2. *The Mordell-Weil group* $\mathcal{F}_{s_0}(\mathbf{Q})$ *has positive rank for all specializations*

$$s_0 \in \mathbf{Q} \setminus \{0, -3/4, -1, -6/5, -3/2, -2, -3\}.$$

Proof. There is an isomorphism of curves $\mathcal{E}_t \to \mathcal{F}_t$ defined over $\mathbf{Q}(t)$:

$$(x, y) \longmapsto \left(\left[\frac{3}{(t+1)(t+2)(t+3)} \right]^2 x, \left[\frac{3}{(t+1)(t+2)(t+3)} \right]^3 y \right).$$

Via this isomorphism, the specializations are as follows:

s_0	$\mathcal{F}_{s_0}$	$\mathcal{F}_{s_0}(\mathbf{Q})$
0	$y^2 = x^3 - (219/16)x + 827/32$	$\mathbf{Z}/9$
$-3/4$	$y^2 = x^3 - (42532864/16875)x + 1815529652224/11390625$	$\mathbf{Z}/12$
$-6/5$	$y^2 = x^3 - (253515625/6912)x + 13209716796875/1492992$	$\mathbf{Z}/12$
$-3/2$	$y^2 = x^3 - 56064x + 6774784$	$\mathbf{Z}/9$

□

3. Future work. For higher-degree Jacobi polynomials, we can similarly define the curves $\mathscr{E}_r$ and $\mathscr{F}_s$ as the zero-sets of the polynomials viewed over the function fields $\mathbf{Q}(r)$ and $\mathbf{Q}(s)$, respectively. The curve $\mathscr{F}_s$ has degree n, is smooth in every degree, and hence has genus $(n-1)(n-2)/2$. On the other hand, the curve $\mathscr{E}_r$ has two singularities $p := [1:0:0]$ and $q := [0:1:0]$ in every degree. Since $\mathscr{E}_r$ has degree $2n$, it has genus $(2n-1)(2n-2)/2 - (\delta_p + \delta_q)$, where δ_p, δ_q are the delta-invariants of the singularities. The following two lemmas show that $\mathscr{E}_r$ has genus $(n-1)(n-2)/2$ also.

Recall that an "ordinary m-fold point" is a singular point of multiplicity m with m distinct smooth branches, i.e., m different tangent directions. For example, the origin is an ordinary double point for the nodal cubic $y^2 = x^3 + x^2$. Such a singular point (for a curve in $\mathcal{C}^2$) is resolved upon blowing up the origin once.

Lemma 1. *The point $p \in \mathscr{E}_r$ is an ordinary n-fold point. Hence,*

$$\delta_p = \frac{1}{2} n(n-1).$$

Proof. We consider the local equation for $\mathscr{E}_r$ in affine coordinates centered at p :

$$\mathscr{Q}_n(1, s, z) = \sum_{j=0}^{n} \binom{-1-r}{n-j} \binom{s+jz}{j} z^{2n-2j}.$$

It is clear that every monomial in the jth term has order $2n - j$, $j = 0, \ldots, n$. Therefore, the point p has multiplicity n, and the order n piece of $\mathscr{Q}_n(1, s, z)$ is

$$\binom{s+nz}{n} = \frac{1}{n!}(s+nz)(s+(n-1)z)\cdots(s+z).$$

From here, it is clear that the singularity has n distinct smooth branches. The formula for the delta-invariant is trivial. □

Lemma 2. *The δ-invariant of q is given by $\delta_q = n(n-1)$.*

Proof. We will show that, although $q = [0:1:0]$ is not an ordinary singular point, it does have multiplicity n, and upon blowing up the origin once, the proper transform of $\mathscr{E}_r$ has just one singular point, q_1, which is an ordinary n-fold point there. Once this has been shown, it follows that the delta-invariant is

$$\delta_q = \frac{1}{2}[n(n-1) + n(n-1)] = n(n-1).$$

The local equation for $\mathscr{E}_r$ in affine coordinates centered at q is

$$\mathscr{Q}_n(x,1,z) = \sum_{j=0}^{n} \binom{-1-r}{n-j}\binom{1+jz}{j} x^j z^{2n-2j}.$$

The lowest order term from the jth summand is $\binom{-1-r}{n-j} x^j z^{2n-2j}$, and hence q has multiplicity n. Now we blow up the origin in the plane with coordinates x and z. The blowup is covered by two affine charts, but in only one of these does the proper transform of $\mathscr{E}_r$ intersect the exceptional $\mathbf{P}^1$. Let this chart have coordinates (t,z), where $t = x/z$.

We have

$$\begin{aligned}
&\mathscr{Q}_n(tz,1,z)\\
&\quad= \binom{-1-r}{n} z^{2n}\\
&\qquad+ \sum_{j=1}^{n} \binom{-1-r}{n-j} \frac{1}{j!}(1+jz)(1+(j-1)z)\cdots(1+z)t^j z^{2n-j}\\
&\quad= z^n \Bigg(\binom{-1-r}{n} z^n\\
&\qquad+ \sum_{j=1}^{n} \binom{-1-r}{n-j} \frac{1}{j!}(1+jz)(1+(j-1)z)\cdots(1+z)t^j z^{n-j} \Bigg).
\end{aligned}$$

Therefore, the proper transform of $\mathscr{E}_r$ denoted $\mathscr{E}_r^T$, is defined by

$$\begin{aligned}
(2)\quad &\binom{-1-r}{n} z^n\\
&+ \sum_{j=1}^{n} \binom{-1-r}{n-j} \frac{1}{j!}(1+jz)(1+(j-1)z)\cdots(1+z)t^j z^{n-j} = 0.
\end{aligned}$$

The exceptional curve E is given by $z = 0$, so it is easy to see that $\mathscr{E}_r^T$ intersects E in only one point: $(t, z) = (0, 0)$. This is the only possible singular point on $\mathscr{E}_r^T$, and we claim it is an ordinary n-fold point. Clearly, the lowest order terms in the defining equation (2) are

$$\binom{-1-r}{n} z^n + \sum_{j=1}^{n} \binom{-1-r}{n-j} \frac{1}{j!} t^j z^{n-j}.$$

This polynomial has non-zero discriminant, so the formula for the delta-invariant follows. □

By the two lemmas, we have

$$\begin{aligned} \operatorname{genus}(\mathscr{E}_r) &= \frac{(2n-1)(2n-2)}{2} - (\delta_p + \delta_q) \\ &= \frac{(2n-1)(2n-2)}{2} - \frac{3}{2} n(n-1) \\ &= \binom{n-1}{2}. \end{aligned}$$

Therefore, the genera of the curves $\mathscr{E}_r$ and $\mathscr{F}_s$ are equal.

A natural question is whether our main result can be generalized to higher-degree curves and their Jacobians. The symmetry relation $P_n^{\langle r,s\rangle}(x) = (-x)^n P_n^{\langle s,r\rangle}(1/x)$ gives rise to the birationality of $\mathscr{E}_r$ and $\mathscr{F}_s$. Indeed, relabeling coordinates so that the affine models of $\mathscr{E}_t$ and $\mathscr{F}_t$ are given by $P_n^{\langle t,y\rangle}(x)$ and $P_n^{\langle y,t\rangle}(x)$, respectively, the transformation $(x, y) \mapsto (1/x, y)$ is rational (with rational inverse) on a Zariski-open subset. Hence, the curves are birationally equivalent. In other words, the fact that $\mathscr{E}_r$ and $\mathscr{F}_s$ are geometrically similar is not restricted to low-degree polynomials. Thus, the curves $\mathscr{E}_r$ and $\mathscr{F}_s$ each have many rational points and embed into their Jacobians, which are abelian varieties of dimension $(n-1)(n-2)/2$. It would be interesting to determine whether or not these abelian varieties have positive rank for all $n > 3$.

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