

COMPACT CARLESON MEASURES FROM SPARSE SEQUENCES

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ABSTRACT. In [1], Belov, Seip and the author studied the Carleson measures for certain spaces of analytic functions of which the de Branges spaces and the model subspaces of the Hardy space H^2 are prime examples. In this paper, we continue this line of research by studying the compact Carleson measures for such spaces.

1. Introduction. This paper is concerned with the compact embedding maps induced by Carleson measures in certain spaces of analytic functions. Bounded, unitary, and invertible maps were studied first in [1, 2].

Let $\Gamma = (\gamma_n)$ be a sequence of distinct complex numbers and $v = (v_n)$ a weight sequence that satisfies the admissibility condition

$$(1.1) \quad \sum_{n=1}^{\infty} \frac{v_n}{1 + |\gamma_n|^2} < \infty.$$

When (1.1) holds, we will call v an admissible weight sequence for Γ . Any such pair (Γ, v) parameterizes the space $\mathcal{H}(\Gamma, v)$ which consists of all functions

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n v_n}{z - \gamma_n}$$

for which

$$\|f\|_{\mathcal{H}(\Gamma, v)} = \|(a_n)\|_{\ell_v^2} < \infty,$$
$$\ell_v^2 = \left\{ (a_n) : \|a\|_v^2 = \sum_{n=1}^{\infty} |a_n|^2 v_n < \infty \right\}$$

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and z belongs to the set

$$(\Gamma, v)^* = \left\{ z \in \mathbf{C} : \sum_{n=1}^{\infty} \frac{v_n}{|z - \gamma_n|^2} < \infty \right\}.$$

This means that we obtain the value of a function f in $\mathcal{H}(\Gamma, v)$ at a point z in $(\Gamma, v)^*$ by computing the weighted discrete Hilbert transform:

$$(1.2) \quad (a_n) \mapsto \sum_{n=1}^{\infty} \frac{a_n v_n}{z - \gamma_n},$$

which is well defined whenever (a_n) belongs to ℓ_v^2 . The de Branges spaces, model subspaces of the Hardy space H^2 which admit Riesz bases of reproducing kernels and the Fock-type spaces studied in [3] are all examples of spaces of the type $\mathcal{H}(\Gamma, v)$. We refer to [1, 2, 5] for detailed accounts.

We say that a nonnegative measure μ on $(\Gamma, v)^*$ is a Carleson measure for $\mathcal{H}(\Gamma, v)$ if there is a positive constant C for which

$$\int_{(\Gamma, v)^*} |f(z)|^2 d\mu(z) \leq C \|f\|_{\mathcal{H}(\Gamma, v)}^2$$

holds for every f in $\mathcal{H}(\Gamma, v)$. Such measures have been described in [1] (see Theorem 1.1 below) when Γ grows at least exponentially, i.e., when

$$(1.3) \quad \inf_{n \geq 1} |\gamma_{n+1}|/|\gamma_n| > 1.$$

To state the result in [1], we first partition the complex plane \mathbf{C} in the following way. Set $\Omega_1 = \{z \in \mathbf{C} : |z| < (|\gamma_1| + |\gamma_2|)/2\}$ and then

$$\Omega_n = \left\{ z \in \mathbf{C} : \frac{|\gamma_{n-1}| + |\gamma_n|}{2} \leq |z| < \frac{|\gamma_n| + |\gamma_{n+1}|}{2} \right\} \quad \text{for } n \geq 2.$$

Theorem 1.1 [1]. *Suppose that the sequence Γ satisfies the sparse-condition (1.3) and that v is an admissible weight sequence for Γ .*

If μ is a nonnegative measure on \mathbf{C} with $\mu(\Gamma) = 0$, then the following are equivalent.

(i) The weighted Hilbert transform in (1.2) is bounded from ℓ_v^2 to $L^2(\mathbf{C}, \mu)$.

(ii) The measure μ is a Carleson measure for the space $\mathcal{H}(\Gamma, v)$.

(iii) We have that

$$\sup_{n \geq 1} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z - \gamma_n|^2} < \infty$$

and

$$\sup_{n \geq 1} \left(\sum_{m=1}^n v_m \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} + \sum_{m=n+1}^{\infty} \frac{v_m}{|\gamma_m|^2} \sum_{m=1}^n \mu(\Omega_m) \right) < \infty.$$

The purpose of this paper is to identify those Carleson measures μ for which the embedding maps I_μ from $\mathcal{H}(\Gamma, v)$ into $L^2(\mathbf{C}, \mu)$ are compact. Whenever μ induces such an embedding, we call it a vanishing or compact Carleson measure for $\mathcal{H}(\Gamma, v)$. Compact Carleson measures appear naturally in the study of compact composition operators in various function spaces. As far as their characterization is concerned, there exists a general “folk theorem”: once the Carleson measures are described by a certain “big oh” condition, vanishing Carleson measures are then characterized by the corresponding “little oh” counterparts. This does not mean that such a “folk theorem” is always true. See [4] for a counterexample. Our result shows that it still holds in the space $\mathcal{H}(\Gamma, v)$.

Theorem 1.2. *Suppose that the sequence Γ satisfies the sparseness condition (1.3) and that v is an admissible weight sequence for Γ . A nonnegative measure μ on \mathbf{C} with $\mu(\Gamma) = 0$ is a compact Carleson measure for $\mathcal{H}(\Gamma, v)$ if and only if*

$$(1.4) \quad \lim_{n \rightarrow \infty} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z - \gamma_n|^2} = 0$$

and

$$(1.5) \quad \lim_{n \rightarrow \infty} \left(\sum_{m=1}^n v_m \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} + \sum_{m=n+1}^{\infty} \frac{v_m}{|\gamma_m|^2} \sum_{m=1}^n \mu(\Omega_m) \right) = 0.$$

Condition (1.4) of Theorem 1.2 is a condition on the local behavior of μ , while condition (1.5) deals with its global behavior. Combining the two conditions, we see that (1.4) may be replaced by a stronger global necessary condition:

$$\lim_{n \rightarrow \infty} \int_{\mathbf{C}} \frac{v_n d\mu(z)}{|z - \gamma_n|^2} = 0.$$

We consider two cases in which (1.5) is automatically fulfilled once either this condition or the original one (1.4) holds.

Corollary 1.1. *Suppose the sequence Γ satisfies the sparseness condition (1.3), that the numbers v_n grow at least exponentially and that the numbers $v_n/|\gamma_n|^2$ decay at least exponentially with n . A nonnegative measure μ on \mathbf{C} with $\mu(\Gamma) = 0$ is a compact Carleson measure for $\mathcal{H}(\Gamma, v)$ if and only if*

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z - \gamma_n|^2} = 0.$$

Corollary 1.2. *Suppose the sequence Γ satisfies the sparseness condition (1.3) and that $(v_n) \in \ell_1$. A nonnegative measure μ on \mathbf{C} with $\mu(\Gamma) = 0$ is a compact Carleson measure for $\mathcal{H}(\Gamma, v)$ if and only if*

$$\lim_{n \rightarrow \infty} \int_{\mathbf{C}} \frac{v_n d\mu(z)}{|z - \gamma_n|^2} = 0.$$

Both corollaries follow immediately from Theorem 1.2.

We note that the discrete version of Theorem 1.1 was discussed in detail in [1, 5] since it describes the Bessel sequences of normalized reproducing kernels in $\mathcal{H}(\Gamma, v)$. Since a Bessel sequence of normalized reproducing kernels corresponds to a compact measure if and only if the sequence is finite, discrete versions of Theorem 1.2 are less interesting and will not be considered in this paper.

An interesting question is when a compact Carleson measure μ induces a Schatten p class embedding map I_μ from $\mathcal{H}(\Gamma, v)$ into

$L^2(\mathbf{C}, \mu)$ for $0 < p < \infty$. For $p = 2$ (the Hilbert-Schmidt class), the answer follows from the fact that

$$(1.6) \quad \|I_\mu\|_2^2 = \int_{\mathbf{C}} \|k_z\|_{\mathcal{H}(\Gamma, v)}^2 d\mu(z) < \infty,$$

where $\|I_\mu\|_2$ denotes the Hilbert-Schmidt norm of I_μ . We observe that the reproducing kernel of $\mathcal{H}(\Gamma, v)$ at a point λ in $(\Gamma, v)^*$ is explicitly given by

$$k_\lambda(z) = \sum_{n=1}^\infty \frac{v_n}{(\bar{\lambda} - \bar{\gamma}_n)(z - \gamma_n)};$$

this is a direct consequence of the definition of $\mathcal{H}(\Gamma, v)$. Then, a simple computation along with (1.3) gives that (1.6) holds if and only if

$$\sum_{m=1}^\infty \int_{\Omega_m} \frac{v_m d\mu(z)}{|z - \gamma_m|^2} < \infty$$

and

$$\sum_{m=1}^\infty v_m \sum_{j=m+1}^\infty \int_{\Omega_j} \frac{d\mu(z)}{|z|^2} + \sum_{m=1}^\infty \frac{v_m}{|\gamma_m|^2} \sum_{j=1}^{m-1} \mu(\Omega_j) < \infty.$$

For other ranges of p , a few partial results are available in [5]. In general, a comprehensive solution is yet to be found.

We close this section with a few words on notation. The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a Hilbert space, a set of complex numbers or a suitable index set. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2. Proof of Theorem 1.2. We note that, since $\mathcal{H}(\Gamma, v)$ is reflexive, μ induces a compact embedding if and only if each weakly convergent sequence in $\mathcal{H}(\Gamma, v)$ converges in norm in $L^2(\mathbf{C}, \mu)$. It is known that a weakly convergent sequence is uniformly norm bounded. In general, the converse statement does not hold. But, under an additional assumption, the following particular case of Nordgren’s result holds [6].

Lemma 2.1. *Let (f_n) be a sequence of functions in $\mathcal{H}(\Gamma, v)$. Then (f_n) converges weakly to zero (weakly null) if and only if it converges pointwise to zero and*

$$(2.1) \quad \sup_n \|f_n\|_{\mathcal{H}(\Gamma, v)} < \infty.$$

2.1. Proof of the necessity of Theorem 1.2. For simplicity, we set

$$V_n = \sum_{m=1}^{n-1} v_m \quad \text{and} \quad P_n = \sum_{m=n+1}^{\infty} v_m |\gamma_m|^{-2}.$$

We may first choose a sequence of normalized test functions

$$q_n(z) = \frac{\sqrt{v_n}}{z - \gamma_n}.$$

The sequence converges weakly to zero in $\mathcal{H}(\Gamma, v)$. This is a particular case of a much more general statement which says that any orthonormal sequence in a Hilbert space converges weakly to zero. This along with compactness of μ yields

$$0 = \lim_{n \rightarrow \infty} \int_{\mathbf{C}} |q_n(z)|^2 d\mu(z)$$

from which condition (1.4) follows.

We next consider another sequence of unit norm functions defined by

$$g_n(z) = \frac{1}{\sqrt{P_n}} \sum_{m=n+1}^{\infty} \frac{v_m}{\overline{\gamma_m}(z - \gamma_m)}.$$

If z belongs to the shell Ω_N , then $|g_n(z)| \simeq P_n^{1/2}$ whenever $n > N$, and therefore g_n converges pointwise to zero as $n \rightarrow \infty$. Thus, by the above lemma, the sequence g_n is weakly null. Taking into account the compactness of μ , we find that

$$(2.2) \quad 0 = \lim_{n \rightarrow \infty} \int_{\mathbf{C}} |g_n(z)|^2 d\mu(z) \geq \lim_{n \rightarrow \infty} P_n \sum_{m=1}^n \mu(\Omega_m).$$

On the other hand, if $\sup_n V_n < \infty$, then (1.5) holds trivially for each Carleson measure μ . We shall thus consider the case when $V_n \rightarrow \infty$ as $n \rightarrow \infty$. To this end, we consider a third sequence of unit norm functions

$$h_n(z) = \frac{1}{V_n^{1/2}} \sum_{m=1}^{n-1} \frac{v_m}{z - \gamma_m}.$$

We shall verify that h_n converges pointwise to zero. If z belongs to the annulus Ω_N , then the estimate

$$(2.3) \quad |h_n(z)| \simeq \frac{1}{V_n^{1/2}} \sum_{m=1}^{N-1} \frac{v_m}{|z|} + \frac{v_N}{V_n^{1/2}|z - \gamma_N|} + \frac{1}{V_n^{1/2}} \sum_{m=N+1}^{n-1} \frac{v_m}{|\gamma_m|}$$

holds when n is sufficiently large. It suffices to look at only the right-hand side of (2.3) because the first two terms clearly converge to zero as $n \rightarrow \infty$. Let c_n be a sequence increasing to infinity such that

$$\sum_{j=1}^{\infty} \frac{c_j v_j}{|\gamma_j|^2} < \infty.$$

Then an application of the Cauchy-Schwarz inequality gives

$$(2.4) \quad \begin{aligned} \frac{1}{V_n^{1/2}} \sum_{m=N+1}^{n-1} \frac{v_m}{|\gamma_m|} &\leq \left(\sum_{m=N+1}^{n-1} \frac{c_m v_m}{|\gamma_m|^2} \right)^{1/2} \left(\sum_{m=1}^{n-1} \frac{v_m}{c_m V_n} \right)^{1/2} \\ &\lesssim \frac{1}{V_n^{1/2}} \left(\sum_{m=1}^{n-1} \frac{v_m}{c_m} \right)^{1/2} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. It follows that, by Lemma 2.1, h_n converges weakly to zero.

To this end, if μ induces a compact embedding, we then have

$$0 = \lim_{n \rightarrow \infty} \int_{\mathbf{C}} |h_n(z)|^2 d\mu(z) \gtrsim \lim_{n \rightarrow \infty} V_n \sum_{k=n+1}^{\infty} \int_{\Omega_k} \frac{d\mu(z)}{|z|^2},$$

which, together with (2.2), gives the assertion in (1.5).

2.2. Proof of the sufficiency of Theorem 1.2. Assume conversely that conditions (1.4) and (1.5) hold, and consider a weak null sequence

$$f_n(z) = \sum_{m=1}^{\infty} \frac{a_m^n v_m}{z - \gamma_m}$$

in $\mathcal{H}(\Gamma, v)$. Then an appeal to the classical Riesz representation theorem gives that for each sequence (b_m) in ℓ_v^2 , we have

$$\sum_{m=1}^{\infty} a_m^n v_m \overline{b_m} \rightarrow 0$$

whenever $n \rightarrow \infty$. Upon in particular taking $b^{(l)} = (b_m^{(l)})$ where $b_l^{(l)} = 1$ and 0 otherwise implies

$$(2.5) \quad \lim_{n \rightarrow \infty} |a_m^n| v_m = 0$$

for each m . This consequence of the Riesz theorem will play an essential role in the remaining part of the proof.

We may first make the following splitting:

$$\begin{aligned} \sum_{l=1}^{\infty} \int_{\Omega_l} |f_n(z)|^2 d\mu(z) &\lesssim \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{1}{|z|^2} \left(\sum_{m=1}^{l-1} |a_m^n| v_m \right)^2 d\mu(z) \\ &\quad + \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{|a_l^n|^2 v_l^2}{|z - \gamma_l|^2} d\mu(z) \\ &\quad + \sum_{l=1}^{\infty} \mu(\Omega_l) \left(\sum_{m=l+1}^{\infty} \frac{|a_m^n| v_m}{|\gamma_m|} \right)^2, \end{aligned}$$

which follows from the Cauchy-Schwarz inequality and the growth condition (1.3). It suffices to show that each of the three right-hand sums converges to zero when $n \rightarrow \infty$. We first show that

$$(2.6) \quad \lim_{n \rightarrow \infty} \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{|a_l^n|^2 v_l^2}{|z - \gamma_l|^2} d\mu(z) = 0.$$

From (1.4), for each small $\varepsilon > 0$, there exists an N for which

$$\int_{\Omega_l} \frac{v_l}{|z - \gamma_l|^2} d\mu(z) < \varepsilon$$

when $l > N$. It follows that

$$\begin{aligned}
 (2.7) \quad \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{|a_l^n|^2 v_l^2}{|z - \gamma_l|^2} d\mu(z) &\lesssim \sum_{l=1}^N |a_l^n|^2 v_l \int_{\Omega_l} \frac{v_l}{|z - \gamma_l|^2} d\mu(z) \\
 &\quad + \varepsilon \sum_{l=N+1}^{\infty} |a_l^n|^2 v_l \\
 &\lesssim \sum_{l=1}^N |a_l^n|^2 v_l + \varepsilon;
 \end{aligned}$$

here we used (1.4) and (2.1). Taking the limit $n \rightarrow \infty$ in (2.7) and invoking (2.5) leads to the desired conclusion (2.6).

It remains to prove

$$(2.8) \quad \lim_{n \rightarrow \infty} \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{1}{|z|^2} \left(\sum_{m=1}^{l-1} |a_m^n| v_m \right)^2 d\mu(z) = 0$$

and

$$(2.9) \quad \lim_{n \rightarrow \infty} \sum_{l=1}^{\infty} \mu(\Omega_l) \left(\sum_{m=l+1}^{\infty} \frac{|a_m^n| v_m}{|\gamma_m|} \right)^2 = 0.$$

For these limits, we only have to modify the arguments used to establish the estimates in (3.2) and (3.3) in [1]. We set

$$\tau_l = \left(\int_{\Omega_l} |z|^{-2} d\mu(z) \right)^{1/2}$$

and first show that (2.8) holds. By duality, we have

$$\begin{aligned}
 \left(\sum_{l=1}^{\infty} \tau_l^2 \left(\sum_{m=1}^{l-1} |a_m^n| v_m \right)^2 \right)^{1/2} &= \sup_{\|c_l\|_l^2=1} \sum_{l=1}^{\infty} \tau_l |c_l| \sum_{m=1}^{l-1} |a_m^n| v_m \\
 &\leq \sup_{\|c_l\|_l^2=1} \sum_{m=1}^{\infty} |a_m^n| v_m \sum_{l=m+1}^{\infty} \tau_l |c_l|.
 \end{aligned}$$

The Cauchy-Schwarz inequality applied to the last sum gives

$$(2.10) \quad \left(\sum_{l=m+1}^{\infty} \tau_l |c_l| \right)^2 \leq \sum_{l=m+1}^{\infty} \tau_l^2 V_l^{1/2} \sum_{j=m+1}^{\infty} |c_j|^2 V_j^{-1/2}.$$

By (1.5), we observe that for each $\varepsilon > 0$, there exists N_1 for which $m \geq N_1$,

$$\sum_{l:2^k V_m < V_l \leq 2^{k+1} V_m} \tau_l^2 V_l^{1/2} \lesssim \frac{\varepsilon}{2^{k/2} V_{m+1}^{1/2}}$$

for $k \geq 0$ and $m \geq N_1$. Summing these inequalities for $m \geq N_1$, we get

$$(2.11) \quad \sum_{l=m+1}^{\infty} \tau_l^2 V_l^{1/2} \lesssim \frac{\varepsilon}{V_{m+1}^{1/2}}.$$

Combining (2.10) with (2.11), we find

$$(2.12) \quad \begin{aligned} \sum_{m=1}^{\infty} v_m \left(\sum_{l=m+1}^{\infty} \tau_l |c_l| \right)^2 &= \sum_{m=1}^{N_1} v_m \left(\sum_{l=m+1}^{\infty} \tau_l |c_l| \right)^2 \\ &\quad + \sum_{m=N_1+1}^{\infty} v_m \left(\sum_{l=m+1}^{\infty} \tau_l |c_l| \right)^2 \\ &\lesssim \sum_{m=1}^{N_1} \frac{v_m}{V_{m+1}} \sum_{j=m+1}^{\infty} |c_j|^2 V_j^{-1/2} \\ &\quad + \varepsilon \underbrace{\sum_{m=N_1+1}^{\infty} \frac{v_m}{V_{m+1}} \sum_{j=m+1}^{\infty} |c_j|^2 V_j^{-1/2}}_{\Delta} \\ &\lesssim \sum_{m=1}^{N_1} \frac{v_m}{V_{m+1}} \sum_{j=m+1}^{\infty} |c_j|^2 V_j^{-1/2} + \varepsilon. \end{aligned}$$

The double sum in (2.12) or Δ is bounded by an absolute constant C because

$$(2.13) \quad \sum_{m=1}^{N_1} \frac{v_m}{V_{m+1}} \sum_{j=m+1}^{\infty} |c_j|^2 V_j^{-1/2} \leq \sum_{j=1}^{\infty} |c_j|^2 V_j^{-1/2} \sum_{m=1}^{j-1} \frac{v_m}{V_{m+1}}$$

when we change the order of the summation and so

$$V_j^{-1/2} \sum_{m=1}^{j-1} \frac{v_m}{V_{m+1}} \leq V_j^{-1/2} \int_0^{V_j} t^{-1/2} dt = 2.$$

To obtain (2.8), we see that

$$\begin{aligned} \sum_{m=1}^{\infty} |a_m^n|^2 v_m \sum_{m=1}^{\infty} \left(\sum_{l=m+1}^{\infty} \tau_l |c_l| \right)^2 &\lesssim C \sum_{m=1}^{N_1} |a_m^n|^2 v_m + \varepsilon \sum_{m=N_1+1}^n |a_m^n|^2 v_m \\ &\lesssim \sum_{m=1}^{N_1} |a_m^n|^2 v_m \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ which follows from (2.5).

To prove (2.9), we note that the Cauchy-Schwarz inequality gives

$$\sum_{l=1}^{\infty} \mu(\Omega_l) \left(\sum_{m=l+1}^{\infty} \frac{|a_m^n| v_m}{|\gamma_m|} \right)^2 \leq \sum_{m=l+1}^{\infty} |a_m^n|^2 v_m P_{m-1}^{1/2} \sum_{j=l+1}^{\infty} \frac{v_j}{P_{j-1}^{1/2} |\gamma_j|^2}.$$

Since

$$\sum_{j=l+1}^{\infty} \frac{v_j}{P_{j-1}^{1/2} |\gamma_j|^2} \leq \int_0^{P_l} x^{-1/2} dx \leq 2P_l^{1/2}$$

it follows that

$$\sum_{l=1}^{\infty} \mu(\Omega_l) \left(\sum_{m=l+1}^{\infty} \frac{|a_m^n| v_m}{|\gamma_m|} \right)^2 \lesssim \sum_{l=1}^{\infty} \mu(\Omega_l) P_l^{1/2} \sum_{m=l+1}^{\infty} |a_m^n|^2 v_m P_{m-1}^{1/2},$$

which becomes

$$\sum_{l=1}^{\infty} \mu(\Omega_l) \left(\sum_{m=l+1}^{\infty} \frac{|a_m^n| v_m}{|\gamma_m|} \right)^2 \lesssim \sum_{m=1}^{\infty} |a_m^n|^2 v_m P_{m-1}^{1/2} \sum_{l=1}^{m-1} \mu(\Omega_l) P_l^{1/2}$$

when we change the order of summation. By (1.5), for each $\varepsilon > 0$, there again exists an N_2 for which for $m \geq N_2$ it follows that

$$\sum_{l: 2^k P_{m-1} \leq P_l \leq 2^{k+1} P_{m-1}} \mu(\Omega_l) P_l^{1/2} \lesssim \frac{\varepsilon}{P_{m-1}^{1/2} 2^{k/2}}.$$

Summing these inequalities with respect to k gives

$$\sum_{l=1}^{m-1} \mu(\Omega_l) P_l^{1/2} \lesssim \frac{\varepsilon}{P_{m-1}^{1/2}},$$

and we get

$$\begin{aligned} \sum_{m=1}^{\infty} |a_m^n|^2 v_m P_{m-1}^{1/2} \sum_{l=1}^{m-1} \mu(\Omega_l) P_l^{1/2} &\lesssim \sum_{m=1}^{N_2} |a_m^n|^2 v_m + \varepsilon \sum_{m=N_2+1}^{\infty} |a_m^n|^2 v_m \\ &\lesssim \sum_{m=1}^{N_2} |a_m^n|^2 v_m + \varepsilon \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ and completes the proof. \square

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