A SHORT PROOF OF
A THEOREM OF COBHAM ON SUBSTITUTIONS

ETHAN M. COVEN, ANDREW DYKSTRA AND MICHELLE LEMASURIER

ABSTRACT. This paper is concerned with the lengths of constant length substitutions that generate topologically conjugate systems. We show that if the systems are infinite, then these lengths must be powers of the same integer. This result is a dynamical formulation of a special case of a 1969 theoretical computer science result of Cobham [**1**]. Our proof is rather simple.

1. Introduction. This paper is concerned with the lengths of constant length substitutions that generate topologically conjugate systems. We show that if the systems are infinite, then these lengths must be powers of the same integer. This result is a dynamical formulation of a special case (for constant length substitutions) of a 1969 theoretical computer science result by Cobham [**1**]. Our proof is rather simple.

Durand [**4**, Theorem 8] put Cobham's theorem in the setting of symbolic dynamics. For a brief discussion of the history of Cobham's theorem and how it came to symbolic dynamics via computer science and logic, see [**4**]. Included there are Eilenberg's remarks on the "highly technical" nature of Cobham's original proof [**5**], as well as comments on subsequent proofs.

Durand's proof, unlike ours, works for non-constant length substitutions, too, and hence is more complicated. Our proof is, we believe, the simplest proof yet of (this special case of) Cobham's theorem. It relies on the characterization in [**2**] of symbolic minimal systems topologically conjugate to constant length substitution minimal systems.

2. Basic concepts. A *dynamical system* is a pair (X, T), where $T : X \to X$ is a homeomorphism. Dynamical systems (X, T) and (Y, S) are considered "the same" if the actions of T on X and of S on Y are the same; only the names of the points have been changed. Formally, (X, T) and (Y, S) are *topologically conjugate* if and only if there is a

homeomorphism $\varphi : X \to Y$, called a *topological conjugacy*, such that $\varphi \circ T \equiv S \circ \varphi$. A dynamical system (X, T) is called *minimal* if and only if X contains no nonempty, closed, T-invariant subset.

A *symbolic dynamical system* is a dynamical system (X, σ), where σ is the (left) shift and X is a closed, shift-invariant subset of some $A^{\mathbf{Z}} = \prod_{-\infty}^{\infty} A$, the space of all doubly infinite sequences with entries from the finite *alphabet* A. Here A has the discrete topology and $A^{\mathbf{Z}}$ the product topology. (If $\#A \geq 2$, then $A^{\mathbf{Z}}$ is homeomorphic to the Cantor set.) The *shift* $\sigma : A^{\mathbf{Z}} \to A^{\mathbf{Z}}$ is defined by $[\sigma(x)]_i := x_{i+1}$ for every $i \in \mathbf{Z}$. To avoid notational clutter, when the domain is clear, we will use the same symbol σ to denote the shift on all $A^{\mathbf{Z}}$ and on all closed, shift-invariant subsets.

If X is a closed, σ^k-invariant subset of some $A^{\mathbf{Z}}$, then (X, σ^k) is topologically conjugate to (Y, σ), where the symbols of Y are the words of length k of X that appear starting at places that are multiples of k. Thus, (X, σ^k) is a symbolic dynamical system.

A *substitution* of constant length $L \geq 2$ is a mapping $\theta : A \to A^L$, the words of length L, where A is a finite alphabet. The most famous example is the Morse (Morse-Thue-Prouhet) substitution $0 \mapsto 01, 1 \mapsto 10$.

A substitution $\theta : A \to A^L$ maps A^2 to A^{2L} by juxtaposition: $\theta(ab) := \theta(a)\theta(b)$. In the same way, for every $k \geq 2$, θ maps A^k to A^{kL}, and $A^{\mathbf{Z}}$ to itself. The substitution θ is *primitive* if and only if there exists i such that for every $a \in A$, every member of A appears in $\theta^i(a)$. If θ is primitive, then there is a unique smallest symbolic dynamical system (X_θ, σ) such that every word $\theta^i(a)$, $a \in A$, $i \geq 1$ appears in X_θ. Then (X_θ, σ) is minimal and is called the *substitution minimal system* generated by θ.

Finally, it is clear that, for every $n \geq 2$, θ and θ^n generate the same substitution minimal system.

3. Cobham's theorem.

Cobham's theorem. *The lengths of primitive, constant length substitutions that generate topologically conjugate infinite substitution minimal systems are powers of the same integer.*

Proof. Suppose that θ and ζ are primitive, constant length substitutions that generate topologically conjugate infinite substitution minimal systems (X_θ, σ) and (X_ζ, σ). In [**3**], Dekking proved that the set of prime divisors of the length of a primitive, constant length substitution that generates an infinite substitution minimal system is a topological conjugacy invariant of that substitution minimal system. We show that if the lengths of θ and ζ are not powers of the same integer, then both (X_θ, σ) and (X_ζ, σ) are topologically conjugate to a substitution minimal system generated by a substitution whose length has fewer prime factors than do the lengths of θ and ζ, contradicting Dekking's theorem.

So let the lengths of θ and ζ be

$$p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \quad \text{and} \quad p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

where the p's are distinct primes and the m's and n's are positive.

Let J be such that

(1) $$\frac{m_J}{n_J} \leq \frac{m_i}{n_i} \quad \text{for all } i.$$

If these lengths are not powers of the same integer, then

(2) $$\frac{m_J}{n_J} < \frac{m_i}{n_i} \quad \text{for some } i.$$

Replace θ by θ^{n_J} and ζ by ζ^{m_J}. The powers generate the same substitution minimal systems as do θ and ζ. It follows from (1) and (2) that the lengths M and N of θ^{n_J} and ζ^{m_J} satisfy $M = RN$, where $R > 1$ and has fewer prime factors than do M and N. By the lemma below, there is a substitution minimal system generated by a primitive substitution of constant length R that is topologically conjugate to (X_θ, σ) and (X_ζ, σ). This contradicts Dekking's theorem. $\square$

Lemma. *Suppose that θ and ζ are primitive substitutions of constant lengths M and N that generate topologically conjugate infinite substitution minimal systems (X_θ, σ) and (X_ζ, σ). If $M = RN$, where $R > 1$, then there is a primitive substitution of constant length R that generates a substitution minimal system that is topologically conjugate to (X_θ, σ) and (X_ζ, σ).*

Proof. Replace X_ζ by its image in X_θ under the topological conjugacy from (X_ζ, σ) to (X_θ, σ). Since the condition of the theorem on page 1450 [**2**] is preserved by topological conjugacy, there are proper σ^M- and σ^N-invariant subsets X_M and X_N of X_θ such that (X_M, σ^M) and (X_N, σ^N) both are topologically conjugate to (X_θ, σ).

Since

$$\bigcup_{i=0}^{N-1} \sigma^i [X_M \cup \sigma^N(X_M) \cup \cdots \cup \sigma^{(R-1)N}(X_M)] = X_\theta,$$

it follows from the minimality of (X_N, σ^N) that one of the members of the (large) union is X_N. Thus, X_N has a proper subset, $\sigma^i(X_M)$ for some i, such that (X_N, σ^N) is topologically conjugate to $(\sigma^i(X_M), (\sigma^N)^R)$.

Therefore, by the theorem in [**2**], (X_N, σ^N) is topologically conjugate to a substitution minimal system generated by a primitive substitution of constant length R. Hence, so is (X_θ, σ). □

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DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CT 06459
Email address: ecoven@wesleyan.edu

DEPARTMENT OF MATHEMATICS, HAMILTON COLLEGE, CLINTON, NY 13323
Email address: adykstra@hamilton.edu

DEPARTMENT OF MATHEMATICS, HAMILTON COLLEGE, CLINTON, NY 13323
Email address: mlemasur@hamilton.edu