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ULTRAMODULARITY AND COPULAS

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ABSTRACT. Ultramodular binary copulas are discussed, i.e., copulas of a random vector whose components are mutually stochastically decreasing with respect to each other. The additive generators of Archimedean ultramodular binary copulas are fully characterized. Finally, a new construction method for binary copulas based on n-ary ultramodular aggregation functions is proposed.

1. Introduction. Ultramodularity of real functions [**21**] is a stronger version of both the convexity and the supermodularity (for functions in one variable, ultramodularity and convexity are equivalent). In the context of aggregation, ultramodular aggregation functions were studied extensively in [**17**]. In statistics, where a copula describes the dependence structure of a random vector (X, Y), the ultramodularity of a copula (i.e., the convexity of all of its horizontal and vertical sections) means that the random variables X and Y are mutually stochastically decreasing with respect to each other (this concept of stochastic monotonicity was first introduced in [**27**], see also [**2, 25**]). This fact has motivated us to study the ultramodularity of binary copulas.

2. Preliminaries. We start with the concept of *n-ary aggregation functions*, i.e., monotone non-decreasing functions $A\colon [0,1]^n \to [0,1]$ which also satisfy the two boundary conditions $A(0,0,\dots,0) = 0$ and $A(1,1,\dots,1) = 1$ (see [**3, 12**]). For the arguments of such a function in n variables we shall use the notations $\mathbf{x}$ and $(x_1,\dots,x_n)$ synonymously.

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The following definition restricts the notion of ultramodularity of real functions, as considered in [**21**], to the class of aggregation functions.

Definition 2.1. An n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ is called *ultramodular* if, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^n$ with $\mathbf{x}+\mathbf{y}+\mathbf{z} \in [0,1]^n$,

$$A(\mathbf{x}+\mathbf{y}+\mathbf{z}) - A(\mathbf{x}+\mathbf{y}) \geq A(\mathbf{x}+\mathbf{z}) - A(\mathbf{x}). \tag{2.1}$$

If $\mathbf{y} \wedge \mathbf{z} = (0,0,\ldots,0)$ then $(\mathbf{x}+\mathbf{y}) \vee (\mathbf{x}+\mathbf{z}) = \mathbf{x}+\mathbf{y}+\mathbf{z}$ and $(\mathbf{x}+\mathbf{y}) \wedge (\mathbf{x}+\mathbf{z}) = \mathbf{x}$. In this case, writing $\mathbf{u} = \mathbf{x}+\mathbf{y}$ and $\mathbf{v} = \mathbf{x}+\mathbf{z}$, (2.1) turns into

$$A(\mathbf{u} \wedge \mathbf{v}) + A(\mathbf{u} \vee \mathbf{v}) \geq A(\mathbf{u}) + A(\mathbf{v}), \tag{2.2}$$

which is just the *supermodularity* of the aggregation function A. This means that each ultramodular aggregation function is necessarily supermodular.

The following characterization of supermodular functions $f\colon [0,1]^n \to [0,1]$ is due to [**4, 13**]:

Proposition 2.2. *An n-ary function $f\colon [0,1]^n \to [0,1]$ is supermodular if and only if each of its two-dimensional sections is supermodular, i.e., for each $\mathbf{x} \in [0,1]^n$ and all $i, j \in \{1,2,\ldots,n\}$ with $i \neq j$, the function $f_{\mathbf{x},i,j}\colon [0,1]^2 \to [0,1]$ given by $f_{\mathbf{x},i,j}(u,v) = f(\mathbf{y})$, where $y_i = u$, $y_j = v$ and $y_k = x_k$ for $k \in \{1,2,\ldots,n\} \setminus \{i,j\}$, is supermodular.*

The following result ([**21**, Corollary 4.1]) states the exact relationship between ultramodular and supermodular functions $f\colon [0,1]^n \to [0,1]$:

Proposition 2.3. *A function $f\colon [0,1]^n \to [0,1]$ is ultramodular if and only if f is supermodular and each of its one-dimensional sections is convex, i.e., for each $\mathbf{x} \in [0,1]^n$ and each $i \in \{1,\ldots,n\}$ the function $f_{\mathbf{x},i}\colon [0,1] \to [0,1]$ given by $f_{\mathbf{x},i}(u) = f(\mathbf{y})$, where $y_i = u$ and $y_j = x_j$ whenever $j \neq i$, is convex.*

Remark 2.4. (i) Because of Propositions 2.2 and 2.3, for an n-ary aggregation function $A\colon [0,1]^n \to [0,1]$, the following are equivalent: (a) A is ultramodular;

(b) each two-dimensional section of A is ultramodular;

(c) each two-dimensional section of A is supermodular and each one-dimensional section of A is convex.

(ii) For $n = 2$, the ultramodularity (2.1) of an aggregation function $A\colon [0,1]^2 \to [0,1]$ is equivalent to A being *P-increasing* (see [**11**]), i.e., to

$$A(u_1, v_1) + A(u_4, v_4) \geq \max\bigl(A(u_2, v_2) + A(u_3, v_3), A(u_3, v_2) + A(u_2, v_3)\bigr)$$

for all $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \in [0,1]$ satisfying $u_1 \leq u_2 \wedge u_3 \leq u_2 \vee u_3 \leq u_4$, $v_1 \leq v_2 \wedge v_3 \leq v_2 \vee v_3 \leq v_4$, $u_1 + u_4 \geq u_2 + u_3$, and $v_1 + v_4 \geq v_2 + v_3$.

Well-known examples of supermodular aggregation functions are copulas. In this paper, we will consider binary copulas where the supermodularity can be used as one of the axioms.

Definition 2.5. An aggregation function $C\colon [0,1]^2 \to [0,1]$ is called a *binary copula* (or, briefly, a *copula*) if it is supermodular and has 1 as neutral element, i.e., $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$.

Remark 2.6. (i) Given a copula $C\colon [0,1]^2 \to [0,1]$, for each $c \in [0,1]$ the horizontal section $h_c\colon [0,1] \to [0,1]$ given by $h_c(x) = C(x,c)$ obviously satisfies $h_c(0) = 0$ and $h_c(1) = c$. Then the greatest possible convex horizontal section h_c is given by $h_c(x) = c \cdot x$, corresponding to the product copula Π given by $\Pi(x,y) = x \cdot y$ (hence we have $C(x,c) \leq c \cdot x = \Pi(x,c)$). It is easy to verify that Π is an ultramodular copula, and hence Π is the greatest ultramodular copula.

(ii) As already mentioned, in statistics, ultramodular copulas describe the dependence structure of stochastically decreasing random vectors (see [**25**, Corollary 5.2.11] and Section 3 below), and thus each ultramodular copula is *negative quadrant dependent* (NQD). Note that stochastically increasing random variables are described by copulas with concave horizontal and vertical sections. These copulas can be characterized by the inequality

$$C(x+u, y+b) + C(x+v, y+a) \geq C(x, y+a+b) + C(x+u+v, y) \tag{2.3}$$

for all $x, y, u, v, a, b \in [0,1]$ with $\{x+u+v, y+a+b\} \subset [0,1]$. Putting $u = b = 0$, (2.3) implies that C is supermodular. Similarly, $u = v = 0$ yields the concavity of the vertical sections of C, and $a = b = 0$ yields the concavity of horizontal sections of C.

In [**17**] the following result extending [**9**, Theorem 5.2] was given, which will be the basic tool in a new construction method proposed in Section 4:

Theorem 2.7. *Let* $A\colon [0,1]^n \to [0,1]$ *be an aggregation function and* $k \geq 2$. *Then the following are equivalent*:

(i) A *is ultramodular.*

(ii) *If* $B_1, \ldots, B_n\colon [0,1]^k \to [0,1]$ *are monotone non-decreasing supermodular functions, then the composite* $D\colon [0,1]^k \to [0,1]$ *given by* $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ *is a supermodular function.*

Also the following result from [**17**] will be applied:

Theorem 2.8. *Let* $A\colon [0,1]^n \to [0,1]$ *and* $B_1, \ldots, B_n\colon [0,1]^k \to [0,1]$ *be ultramodular aggregation functions. Then the composite function* $D\colon [0,1]^k \to [0,1]$ *given by* $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ *is also an ultramodular aggregation function.*

3. Ultramodular binary copulas.

If an ultramodular binary aggregation function with annihilator 0 also has neutral element 1, then it necessarily is an ultramodular copula, i.e., a copula with convex horizontal and vertical sections. Clearly, the set $\mathcal{C}_u$ of all ultramodular binary copulas is convex. The greatest element of $\mathcal{C}_u$ is the product copula Π, and the smallest element of $\mathcal{C}_u$ is the lower Fréchet-Hoeffding bound W given by $W(x,y) = \max(x+y-1, 0)$.

Because of [**25, 26**], each associative copula is an ordinal sum [**6, 25, 26**] of Archimedean copulas. However, if for a copula C we have $C(a,a) = a$ for some $a \in \,]0,1[$, then $C(x,a) = \min(x,a)$ implies that there are nonconvex sections, i.e., C cannot be ultramodular. Therefore, each associative ultramodular copula C is a trivial ordinal sum of Archimedean copulas, i.e., C itself must be Archimedean. Recall that a binary aggregation function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous, strictly decreasing convex function $t\colon [0,1] \to [0,\infty]$ with $t(1)=0$ such that for all $(x,y) \in [0,1]$ (see [**24**])

$$C(x,y) = t^{-1}(\min(t(x)+t(y), t(0))). \tag{3.1}$$

The function t is called an additive generator of C, and it is unique up to a positive multiplicative constant.

If we want to see whether an Archimedean copula is ultramodular, i.e., has convex horizontal and vertical sections, its symmetry (as a consequence of (3.1)) and boundary conditions tell us that it suffices to check the convexity of all horizontal sections for $a \in]0,1[$.

Theorem 3.1. *Let $C\colon [0,1]^2 \to [0,1]$ be an Archimedean copula with a two times differentiable additive generator $t\colon [0,1] \to [0,\infty]$. Then C is ultramodular if and only if $1/t'$ is a convex function.*

Proof. Suppose that C is an Archimedean copula with a two times differentiable additive generator t. Then C is ultramodular if, and only if, for each $a \in]0,1[$ and $c = t(a) \in]0, t(0)[$, the section $f\colon [0,1] \to [0,1]$ given by

$$f(x) = C(x,a) = t^{-1}(\min(t(x)+c, t(0))),$$

is convex, which is equivalent to $f''(x) \geq 0$ whenever $f(x) > 0$, i.e., if for each $x \in [0,1]$ with $t(x)+c < t(0)$ we have

$$f''(x) = \frac{t''(x)\cdot(t'(t^{-1}(t(x)+c)))^2 - (t'(x))^2 \cdot t''(t^{-1}(t(x)+c))}{(t'(t^{-1}(t(x)+c)))^3} \geq 0.$$

Since t' is negative on $]0,1[$, this means

$$\frac{t''(x)}{(t'(x))^2} \leq \frac{t''(t^{-1}(t(x)+c))}{(t'(t^{-1}(t(x)+c)))^2}. \tag{3.2}$$

Put $u = t^{-1}(t(x)+c) < x$. Because of $t''(x)/(t'(x))^2 = (-1/t'(x))'$, (3.2) can be rewritten as

$$\left(\frac{1}{t'(x)}\right)' \geq \left(\frac{1}{t'(u)}\right)'. \tag{3.3}$$

Since a can be chosen arbitrarily, (3.3) holds for all $u, x \in \,]0,1[$ with $u < x$, i.e., the derivative of the function $1/t'$ is monotone non-decreasing implying that $1/t'$ is convex. □

Remark 3.2. (i) The requirement that the additive generator t of C be two times differentiable is not necessary in Theorem 3.1: define $t\colon [0,1] \to [0,\infty]$ by

$$t(x) = \begin{cases} -\log(8x/9) & \text{if } x \in [0, \frac{1}{2}], \\ -2\log(2x+1)/3 & \text{otherwise.} \end{cases}$$

Evidently, the additive generator t is not two times differentiable at $1/2$. On the other hand, the function $1/t'$ is given by $1/t'(x) = \max(-x, -(2x+1)/4)$, and it is convex. Moreover, the Archimedean copula C generated by t is ultramodular.

(ii) Theorem 3.1 gives also a hint how to construct ultramodular Archimedean copulas: if $g\colon [0,1] \to [0,\infty]$ is a differentiable, convex, non-increasing function with $g(1) < 0$, then $t\colon [0,1] \to [0,\infty]$ given by

$$t(x) = -\int_x^1 \frac{1}{g(u)}\, du \tag{3.4}$$

is an additive generator of an ultramodular Archimedean copula.

(iii) Related results arising from the study of stochastically dependent random vectors can be found also in [**1, 5**].

Example 3.3. For each $\lambda \in \,]0,1[$, the function $g_\lambda\colon [0,1] \to [0,\infty]$ given by $g_\lambda(x) = -x^{1-\lambda}$ is differentiable, convex, non-increasing and satisfies $g_\lambda(1) < 0$. Then (3.4) yields the additive generator $t_\lambda\colon [0,1] \to [0,\infty]$ given by $t_\lambda(x) = (1 - x^\lambda)/\lambda$, and the corresponding copula C_λ is given by

$$C_\lambda(x,y) = \big(\max(x^\lambda + y^\lambda - 1, 0)\big)^{1/\lambda}.$$

Note that the limit cases for λ going to 0 and 1 are Π and W, respectively, and that $(C_\lambda)_{\lambda \in]0,1]}$ is the family of non-strict Clayton copulas (see [**25**]).

Note that some constructions involving copulas preserve the convexity of their horizontal and vertical sections. First of all, for each ultramodular copula C, also the corresponding survival copula $\widehat{C}$ (see [**25**]) given by $\widehat{C}(x,y) = x + y - 1 + C(1-x, 1-y)$ is ultramodular. Also, each W-ordinal sum (see [**10, 15**]) of ultramodular copulas is ultramodular.

Example 3.4. (i) For $\lambda = 0.5$, the survival copula $\widehat{C}_{0.5}$ of the corresponding Clayton copula is given by

$$\widehat{C}_{0.5}(x,y) = x + y - 1 + \left(\max(\sqrt{1-x} + \sqrt{1-y} - 1, 0)\right)^2;$$

it is ultramodular (but not associative).

(ii) The copula $C = W\text{-}(\langle 0, 0.5, \Pi\rangle)$ is given by

$$C(x,y) = \begin{cases} x \cdot (2y-1) & \text{if } (x,y) \in [0,0.5] \times [0.5,1], \\ W(x,y) & \text{otherwise;} \end{cases}$$

it is also ultramodular (but not associative).

Two types of flipping a copula C, leading to the flipped copulas C^- and C_- given by, respectively (see [**7, 25**]),

$$C^-(x,y) = x - C(x, 1-y), \tag{3.5}$$

$$C_-(x,y) = y - C(1-x, y), \tag{3.6}$$

turn convex sections into concave ones, and vice versa. Thus, starting from a copula C with concave horizontal and vertical sections, the flipped copulas C^- and C_- are ultramodular.

As already mentioned, copulas with concave horizontal and vertical sections describe the stochastic dependence structure of stochastically increasing random vectors. Thus, starting from an ultramodular copula C, both C^- and C_- are linked to stochastically increasing random vectors. Note that even if the ultramodular copula C is symmetric (i.e., it describes the stochastic dependence structure of exchangeable random variables), the copulas C^- and C_- are, in general, non-symmetric. However, the copula $D = (C^- + C_-)/2$ is always symmetric and possesses concave horizontal and vertical sections. Recall that, as a consequence of [**16**, Theorem 3] we have the following result for copulas with concave horizontal and vertical sections.

Theorem 3.5. *Let C be an Archimedean copula with additive generator t, let t' be the left derivative of t on $]0,1]$ and $t'(0) = t'(0^+)$.*

Then all the one-dimensional sections of C are concave if and only if $t'(0) = \infty$, t' is finite on $]0,1]$, and $1/t'$ is concave.

Example 3.6. (i) Similarly as in Example 3.3, for each $\lambda \in]0,\infty[$, consider the concave function $g_\lambda\colon [0,1] \to [0,\infty]$ given by $g_\lambda(x) = -x^{\lambda+1}$. It is related via $g_\lambda = 1/t'_\lambda$ to the additive generator $t_\lambda\colon [0,1] \to [0,\infty]$ given by $t_\lambda(x) = (x^{-\lambda} - 1)/\lambda$, and the corresponding copula C_λ is given by

$$C_\lambda(x,y) = \big(\max(x^{-\lambda} + y^{-\lambda} - 1, 0)\big)^{-1/\lambda},$$

which is exactly a strict Clayton copula (see [**25**]). Then both $(C_\lambda)^-$ and $(C_\lambda)_-$ are ultramodular (and not associative):

$$\begin{aligned}(C_\lambda)^-(x,y) &= x - \big(x^{-\lambda} + (1-y)^{-\lambda} - 1\big)^{-1/\lambda},\\ (C_\lambda)_-(x,y) &= y - \big((1-x)^{-\lambda} + y^{-\lambda} - 1\big)^{-1/\lambda};\end{aligned}$$

observe that $(C_\lambda)_-(x,y) = (C_\lambda)^-(y,x)$.

(ii) The only Archimedean copulas with the property that also the flipped copulas given by (3.5) and (3.6) are Archimedean are the Frank copulas $(F_\lambda)_{\lambda\in]0,\infty[}$ given by

$$F_\lambda(x,y) = \begin{cases} x\cdot y & \text{if } \lambda = 1,\\ \log_\lambda(1 + [(\lambda^x - 1)\cdot(\lambda^y - 1)]/\lambda - 1) & \text{otherwise.}\end{cases}$$

Observe that $(F_\lambda)^- = (F_\lambda)_- = F_{1/\lambda}$, and that F_λ is ultramodular if and only if $\lambda \in [1,\infty[$.

Recently, an interesting class of copulas which are invariant under univariate conditioning was introduced in [**8**], compare also [**23**]. For each continuous, convex and strictly decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ (i.e., for each additive generator of an Archimedean copula, see (3.1)) define the function $C_f\colon [0,1]^2 \to [0,1]$ by

$$C_f(x,y) = \begin{cases} 0 & \text{if } x = 0,\\ x\cdot f^{-1}(\min((f(y)/x), f(0))) & \text{otherwise.}\end{cases} \tag{3.7}$$

It was shown in [**8**, Proposition 3.1] that C_f is a copula, and f was called a *horizontal generator* of C_f. In full analogy, we can use f as a *vertical generator* to construct the copula $C^f\colon [0,1]^2 \to [0,1]$ via

$$(3.8) \qquad C^f(x,y) = \begin{cases} 0 & \text{if } y = 0, \\ y \cdot f^{-1}(\min((f(x)/y), f(0))) & \text{otherwise.} \end{cases}$$

Note also that we have $C_{c\cdot f} = C_f$ and $C^{c\cdot f} = C^f$ for each positive constant c. Using similar reasoning as in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.7. *Let $f\colon [0,1] \to [0,\infty]$ be a two times differentiable horizontal or vertical generator. If $1/f'$ is a convex function, then C_f and C^f are ultramodular.*

Because of the similarity between Theorems 3.1 and 3.7, we can give some examples based on the same functions as those in Example 3.3.

Example 3.8. For each $\lambda \in]0,1[$, define the function $f_\lambda\colon [0,1] \to [0,\infty]$ by $f_\lambda(x) = 1 - x^\lambda$. Then all the requirements of Theorem 3.7 are fulfilled and, therefore, the (non-symmetric) copulas C_{f_λ} and C^{f_λ} given by

$$C_{f_\lambda}(x,y) = \max\left(x^{\lambda-1} \cdot y^\lambda + x^\lambda - x^{\lambda-1}, 0\right)^{1/\lambda},$$
$$C^{f_\lambda}(x,y) = \max\left(x^{\lambda} \cdot y^{\lambda-1} + y^\lambda - y^{\lambda-1}, 0\right)^{1/\lambda},$$

are ultramodular. Observe that $C_{f_1} = C^{f_1} = W$ and that, taking into account $f_0 = \lim_{\lambda\searrow 0} f_\lambda = -\log$, we have $C_{f_0} = \lim_{\lambda\searrow 0} C_{f_\lambda}$, where C_{f_0} is given by

$$C_{f_0}(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x \cdot y^{1/x} & \text{otherwise.} \end{cases}$$

However, Theorem 3.7 provides only a sufficient condition for the ultramodularity of copulas (in contrast to Theorem 3.1 where a necessary and sufficient condition is given); indeed, if $f\colon [0,1] \to [0,1]$ is given by $f(x) = (1/x) - 1$, then f is two times differentiable and the copulas C_f

and C^f, given by

$$C_f(x,y) = \frac{x^2 y}{1-y+xy}, \qquad C^f(x,y) = \frac{xy^2}{1-x+xy},$$

are both ultramodular, but $1/f'$ is not convex (in fact, it is concave).

4. Constructing copulas. Theorem 2.8 has important implications for the construction of binary copulas.

Theorem 4.1. *Let $A\colon [0,1]^n \to [0,1]$ be a continuous ultramodular aggregation function. Let $C_1, \ldots, C_n\colon [0,1]^2 \to [0,1]$ be copulas, and assume that the continuous monotone non-decreasing functions $f_1, \ldots, f_n, g_1, \ldots, g_n\colon [0,1] \to [0,1]$ satisfy $f_i(1) = g_i(1) = 1$ for each $i \in \{1, \ldots, n\}$ and $A(f_1(0), \ldots, f_n(0)) = A(g_1(0), \ldots, g_n(0)) = 0$. Define $\xi, \eta\colon [0,1] \to [0,1]$ by*

$$\xi(x) = \sup\{u \in [0,1] \mid A(f_1(u), \ldots, f_n(u)) \le x\},$$
$$\eta(x) = \sup\{u \in [0,1] \mid A(g_1(u), \ldots, g_n(u)) \le x\}.$$

Then the function $C\colon [0,1]^2 \to [0,1]$ given by

$$(4.1)\quad C(x,y) = A\left(C_1(f_1 \circ \xi(x), g_1 \circ \eta(y)), \ldots, C_n(f_n \circ \xi(x), g_n \circ \eta(y))\right)$$

is a copula.

Proof. Defining the functions $f, g\colon [0,1] \to [0,1]$ by

$$f(x) = A(f_1(x), \ldots, f_n(x)), \qquad g(x) = A(g_1(x), \ldots, g_n(x))$$

it is clear that f and g are surjective monotone non-decreasing functions (and, therefore, continuous), while ξ and η are the (upper) pseudo-inverses of f and g (see [**18, 26**]). Consequently, $f \circ \xi = g \circ \eta = \mathrm{id}_{[0,1]}$, proving that 1 is the neutral element of C. Because of Theorem 2.8, C is also ultramodular and, therefore, a copula. □

Theorem 4.1 can be used to construct non-symmetric copulas from symmetric ones, some of which have already been considered in the literature.

Example 4.2. (i) If we put $n = 2$, $A = C_1 = \Pi$ and define, for $\alpha, \beta \in [0,1]$, the functions f_1, f_2, g_1, g_2 by $f_1(x) = x^{1-\alpha}$, $f_2(x) = x^{\alpha}$, $g_1(x) = x^{1-\beta}$, and $g_2(x) = x^{\beta}$, then for each copula C_2 the construction in (4.1) yields the copula $C_{\alpha,\beta}$ given by

$$C_{\alpha,\beta} = x^{1-\alpha} \cdot y^{1-\beta} \cdot C_2(x^{\alpha}, y^{\beta}). \tag{4.2}$$

Note that $C_{\alpha,\beta}$ was shown to be a copula in [**14**] and that a generalization of (4.2) based on the n-ary product Π was given in [**20**].

(ii) If we put $n = 2$, $A = W$, $C_1 = C_2 = M$ defined by $M(x,y) = \min(x,y)$ and define the four functions f_1, f_2, g_1, g_2 by $f_1(x) = g_2(x) = (x+2)/3$ and $f_2(x) = g_1(x) = (2x+1)/3$, then the construction in (4.1) yields the copula C given by

$$C(x,y) = \frac{1}{3} \cdot \max\left(\min(x+1, 2y) + \min(2x, y+1) - 1, 0\right).$$

(iii) If C_1 and C_2 are arbitrary copulas and $f, g\colon [0,1] \to [0,1]$ are 1-Lipschitz monotone non-decreasing functions with $f(1) = g(1) = 1$, then $A\colon [0,1]^2 \to [0,1]$ given by

$$A(x,y) = W(C_1(f(x), g(y)), C_2(x+1-f(x), y+1-g(y)))$$

is a copula. If, e.g., $C_1 = C_2 = \Pi$ and $f(x) = (1+4x)/5$ and $g(x) = (2+x)/3$ then $A(x,y) = \max([(2x+3)(2y+3)]/10, 0)$. Observe that A is an Archimedean copula, its additive generator t being given by $t(x) = -\log(2x+3)/5$. In fact, each Archimedean copula whose additive generator t, for some $c \in \,]0,1]$, is given by $t(x) = -\log(c \cdot x + 1 - c)$ can be obtained in this way, and all these copulas are ultramodular.

(iv) Let $A\colon [0,1]^n \to [0,1]$ be the n-ary extension of a binary ultramodular Archimedean copula with additive generator $t\colon [0,1] \to [0,\infty]$, i.e.,

$$A(\mathbf{x}) = t^{-1}\left(\min\left(t(0), \sum_{i=1}^{n} t(x_i)\right)\right).$$

Assume that $C_1 = \cdots = C_n = D$, D being an Archimedean copula with additive generator $v\colon [0,1] \to [0,\infty]$, and $f_1 = \cdots = f_n = g_1 = \cdots = g_n = \mathrm{id}_{[0,1]}$. Then the construction in (4.1) yields an Archimedean copula C whose additive generator $\varrho\colon [0,1] \to [0,\infty]$ is given by $\varrho(x) = v(t^{-1}(t(x)/n))$. If the additive generator t is two times differentiable (implying that $1/t'$ is convex because of Theorem 3.1), then it is not difficult to check that the function $\zeta\colon [0,1] \to [0,1]$ given by $\zeta(x) = t^{-1}(t(x)/n)$ is concave, strictly increasing and satisfies $\zeta(1) = 1$. Therefore, C is a ζ-transform of D, i.e.,

$$C(x, y) = \zeta^{-1}(\max(\zeta(0), D(\zeta(x), \zeta(y))));$$

note that the concavity of ζ is sufficient to show that C is a copula [**19**].

Note that, because of Theorem 2.8, we can also consider the following construction of ultramodular binary copulas: if $A\colon [0,1]^n \to [0,1]$ is an idempotent ultramodular aggregation function and if $C_1, C_2, \dots, C_n$ are ultramodular binary copulas, then it is not difficult to check that the function $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x, y) = A\left(C_1(x, y), C_2(x, y), \dots, C_n(x, y)\right) \tag{4.3}$$

is an ultramodular copula. However, we conjecture that the only idempotent ultramodular aggregation functions are just the weighted arithmetic means, in which case (4.3) would be a standard convex combination of the copulas $C_1, C_2, \dots, C_n$ (a construction which obviously preserves ultramodularity).

5. Concluding remarks. We have studied ultramodular binary copulas which describe the mutual stochastical decreasingness of two random variables. Moreover, based on ultramodular n-ary aggregation functions, we have introduced a new construction method for binary copulas. Note that binary copulas are closely linked to the convexity of one-dimensional functions (e.g., additive generators of Archimedean copulas and their pseudo-inverses [**18**] are convex). Copulas of higher dimensions describe the stochastic dependence structure of k-dimensional random vectors with $k > 2$, and they are linked to a stronger form of convexity of one-dimensional functions. For example, in the case of k-dimensional Archimedean copulas (see [**22, 28**]), the pseudo-inverse of the corresponding additive generator has a derivative of order $k - 2$ which becomes a convex function when it is multiplied by $(-1)^k$. Following the idea of ultramodular functions given in (2.1), in a next step we will propose and study stronger versions of ultramodularity related to that, leading to the stronger forms of convexity mentioned above in the case of functions in one variable. We expect that this approach will be of help in constructing copulas of higher dimensions; so far, only few such methods are known in the literature.

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