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EBERLEIN COMPACTNESS

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ABSTRACT. For finite measure space $(X, \mathcal{A}, \mu)$, a Banach space E with E' its dual, and a relatively countably compact $Q \subset (L_1(E), \sigma(L_1(E), L_\infty(E')))$, entirely different proofs are given of the results that (i) $\overline{Q}$ is Eberlein compact, (ii) the closed convex hull of $\overline{Q}$ in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ is also compact and (iii) the closed convex hull of $\overline{Q}$ in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ and in $(L_1(E), \|\cdot\|_1)$ are the same.

1. Introduction and notation. In this paper $(X, \mathcal{A}, \mu)$ is a finite measure space, E a Banach space and E' the dual Banach space. For locally convex spaces, the notations and results of [**7**] will be used; for a locally convex space F, with F' its dual and $x \in F$, $f \in F'$, $f(x)$ will also be denoted by $\langle f, x\rangle$ or $\langle x, f\rangle$. All vector spaces are taken over R, the set of real numbers. For measures, the results and notations of [**3**] will be used. For a Banach space F, $L_1(F) = L_1(\mu, F)$ and $L_\infty(F) = L_\infty(\mu, F)$ will have the usual meanings. If $F = R$, $L_1(\mu, F)$, $L_\infty(\mu, F)$ will be denoted by L_1, L_∞, respectively. A compact Hausdorff space is called *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space; a subset A of a Hausdorff topological space X is called *relatively countably compact* if every sequence in A has a cluster points in X ([**6**]). For a topological space Z, $C(Z)$ will denote the set of all continuous real-valued functions on Z and for a subset $A \subset Z$, $\overline{A}$ will denote the closure of A in Z.

In [**1, 2**], it is proved that a compact $Q \subset (L_1(E), \sigma(L_1(E), L_\infty(E')))$ is Eberlein compact and the closed convex hull of Q in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ is also compact; it is further proved that the closed convex hull of Q in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ and $(L_1(E), \|\cdot\|_1)$ are the same. In this paper, starting with a relatively countably compact Q in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ and denoting its closure and its the

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closed convex hull in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ by $\overline{Q}$ and W, respectively, we give entirely different proofs that $\overline{Q}$ is Eberlein compact, W is compact and W is also the closed convex hull of $\overline{Q}$ in $(L_1(E), \|\cdot\|_1)$.

First we set some notation straight. Relative to the measure space $(X, \mathcal{A}, \mu)$, we denote by B and S the compact unit balls of L_∞ and E' with $\sigma(L_\infty, L_1)$ and $\sigma(E', E)$ topologies. An element $f \in L_1(E)$ can also be considered an element of $C(B \times S)$, $f(b, s) = \int s \circ f b \, d\mu$; this mapping from $L_1(E)$ to $C(B \times S)$ is one-to-one, continuous with $\sigma(L_1(E), L_\infty(E'))$ on $L_1(E)$ and pointwise topology on $C(B \times S)$. We will again denote by f the image of $f \in L_1(E)$ in $C(B \times S)$.

The paper is set up like this. Starting with a relatively countably compact subset Q of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$, we prove some lemmas about Q. These lemmas are used in the proof of our theorem that $\overline{Q}$ is Eberlein compact. In the next section, we start with a compact $Q \subset (L_1(E), \sigma(L_1(E), L_\infty(E')))$, denote its closed convex hull in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ by W and prove that W is compact and is also equal to the closed convex hull of Q in $(L_1(E), \|\cdot\|_1)$.

2. Eberlein compactness. We first prove some lemmas. In the first lemma we prove that the $\|\cdot\|_1$-norm of an element of $L_1(E)$ equals its sup over the elements of the closed unit ball of $L_\infty(E')$.

Lemma 1. *For a* $q \in L_1(E)$, $\|q\|_1 = \sup\{|\langle q, g\rangle| : g \in L_\infty(E'),\ \|g\|_\infty \le 1\}$. *Also we have*

$$\|q\|_1 = \sup\{\langle q, g\rangle : g \in L_\infty(E'),\ \|g\|_\infty \le 1\}$$
$$\|q\|_1 = \sup\{|\langle q, g\rangle| : g \in L_\infty(E'),\ g \text{ simple } \|g\|_\infty \le 1\}$$
$$\|q\|_1 = \sup\{\langle q, g\rangle : g \in L_\infty(E'),\ g \text{ simple } \|g\|_\infty \le 1\}.$$

Proof. When q is simple, it is trivially true. In the general case assume that $\|q\|_1 > \sup\{|\langle q, g\rangle| : g \in L_\infty(E'),\ \|g\|_\infty \le 1\} + 4c$ for some $c > 0$. Take a simple $q_0 \in L_1(E)$ such that $\|q - q_0\|_1 < c$. This means $\|q\|_1 < \|q_0\|_1 + c$. Select a simple $g \in L_\infty(E')$ with $\|g\|_\infty \le 1$ such that $\|q_0\|_1 < |\langle q_0, g\rangle| + c$. Since $\|q - q_0\|_1 < c$, we have $|\langle q_0, g\rangle| \le |\langle q, g\rangle| + c$. Thus, $\|q\|_1 \le \|q_0\|_1 + c < |\langle q_0, g\rangle| + 2c \le |\langle q, g\rangle| + 3c \le \|q\|_1 - 4c + 3c = \|q\|_1 - c$ which is a contradiction. The others are easily verified. □

Corollary 2. *If Q is a relatively countably compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$, then Q is bounded in $(L_1(E), \|\cdot\|_1)$.*

Proof. We first prove that Q is point-wise bounded on $L_\infty(E')$. Suppose this is not the case. This means there is a sequence $\{q_n\} \subset Q$ and a $g \in L_\infty(E')$ such that $\langle q_n, g\rangle \geq n$ for all n. Since this sequence has a cluster point in $L_1(E)$ with $\sigma(L_1(E), L_\infty(E'))$ topology, this is a contradiction. Since $L_\infty(E')$ is a Banach space and the elements of Q can be considered as linear continuous mappings $L_\infty(E') \to R$, by the uniform boundedness principle, Q are uniformly bounded on the unit ball of $L_\infty(E')$. The result now follows from Lemma 1. □

The next lemma establishes a result about the uniform convergence of some integrals relative to the elements of Q.

Lemma 3. *Let Q be a relatively countably compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. Then, for a disjoint sequence $\{A_n\} \subset \mathcal{A}$, $\int_{A_n} \|f(x)\|\, d\mu \to 0$ uniformly for $f \in Q$.*

Proof. Suppose this is not true. Then there is a sequence $\{q_n\} \subset Q$ such that $\int_{A_n} \|q_n(x)\|\, d\mu > 4c$ for all n for some $c > 0$. Using Lemma 1, for every n, take a finite disjoint sequence $\{A_i^n : 1 \leq i \leq p(n)\} \subset \mathcal{A}$, $A_i^n \subset A_n$, and a finite sequence $\{g_i^n : 1 \leq i \leq p(n)\} \subset S$ such that $\sum_{i=1}^{p(n)} \langle q_n,\, \chi_{A_i^n} g_i^n\rangle > 4c$. The elements of the countable set $\{\chi_{A_i^n} g_i^n : 1 \leq i \leq p(n), n \in N\}$ are denoted by $\{h_n\}$. For any subset $M \subset N$, $\sum_{n \in M} h_n$ is in the closed unit ball of $L_\infty(E')$. Put $Y = \{\sum_{n\in M} h_n : M \subset N\}$ and $Y_0 = \{\sum_{n \in M} h_n : M$ a finite subset of $N\}$. Y_0 is countable. We now prove that Y_0 is dense in $(Y, \sigma(L_\infty(E'), L_1(E))$; to prove this, we have to simply prove that, for a disjoint sequence $\{B_n\}$, in $\mathcal{A}$, a sequence $\{g_n\} \subset S$, and $f \in L_1(E)$, we have $\sum_{1\leq i \leq n} \langle f, \chi_{B_i} g_i\rangle \to \sum_{1 \leq i < \infty} \langle f, \chi_{B_i} g_i\rangle$ in L_1. Since $|(\sum_{1 \leq i \leq n} \chi_{B_i} g_i \circ f)(x)| \leq \|f(x)\|$ on X, it follows from the dominated convergence theorem. Now $Q \subset C(Y)$, Y is separable and Q is relatively countably compact in the topology of pointwise convergence on Y and so, by [**6**, Theorem 2.1, page 538], there is a subsequence of $\{q_n\}$, which again we denote by $\{q_n\}$ and $q \in L_1(E)$ such that $q_n \to q$, pointwise on Y. Now define $\lambda_n : 2^N \to R$, $\lambda_n(M) = \sum_{m \in M} \langle q_n, h_m\rangle$; they are easily verified to be countably additive and $\lim \lambda_n(M)$ exists for every $M \subset N$. By [**4**, Lemma 2.2], the convergence is uniform on 2^N. Let $\lambda_n \to \lambda$; then λ is also countably additive. So there is an $m_0 \in N$ such that $|\lambda_n - \lambda| < c$ on 2^N, for all $n \geq m_0$. Now choose an $m \in N$, $m \geq m_0$ such that $|\lambda(U)| \leq c$ for any $U \subset P_m$ where $P_m = \{n \in N : n \geq m\}$. Combining these results, we get $\lambda_n(U) \leq 2c$ for any $U \subset P_m$. This is a contradiction. □

The next lemma uses Lemma 3 to prove some additional results about the uniform convergence of some new integrals relative to the elements of Q.

Lemma 4. *Let Q be a relatively countably compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. Then*

(i) *For any decreasing sequence $\{A_n\} \subset \mathcal{A}$ with $A_n \downarrow A$, A being μ-null, $\int_{A_n} \|f(x)\|\, d\mu \to 0$ uniformly for $f \in Q$;*

(ii) $\lim_{\mu(A)\to 0}(\int_A \|f(x)\| d\mu) = 0$ *uniformly for $f \in Q$.*

Proof. (i) Suppose this is not true. Then there is a sequence $\{q_n\} \subset Q$ such that $\int_{A_n} \|q_n(x)\|\, d\mu > 2c$ for all n for some $c > 0$. Put $n_0 = 1$ and select an $n_1 \in N$, $n_1 > n_0$, such that $\int_{A_{n_1}} \|q_{n_0}(x)\|\, d\mu < c$ (note that $\lim_{n\to\infty} \int_{A_n} \|q_{n_0}(x)\|\, d\mu = 0$). This implies that $\int_{A_{n_0} \setminus A_{n_1}} \|q_{n_0}(x)\|\, d\mu > c$. Now select an $n_2 \in N$, $n_2 > n_1$ such that $\int_{A_{n_2}} \|q_{n_1}(x)\|\, d\mu < c$. This implies that $\int_{A_{n_1} \setminus A_{n_2}} \|q_{n_1}(x)\|\, d\mu > c$. Continuing this process, we get an increasing sequence $\{n_k\} \subset N$ such that $\int_{A_{n_{k-1}} \setminus A_{n_k}} \|q_{n_{k-1}}(x)\|\, d\mu > c$ for all k. Since the elements of the sequence $\{A_{n_{k-1}} \setminus A_{n_k}\}$ are mutually disjoint, this contradicts Lemma 3.

(ii) Suppose this is not true. That means there is a sequence $\{B_n\}$, in $\mathcal{A}$, a sequence $\{q_n\} \subset Q$ such that $\mu(B_n) \leq 1/2^{n+1}$ and $\int_{B_n} \|q_n(x)\|\, d\mu > c$ for all n for some $c > 0$. Put $A_n = \cup_{n \leq i < \infty} B_i$. Now $\mu(A_n) \leq 1/2^n$ and $A_n \downarrow$, and so $A_n \downarrow A$ where A is μ-null. By (i), $\int_{A_n} \|q_n(x)\|\, d\mu \to 0$. Since $B_n \subset A_n$ for all n, we get $\int_{B_n} \|q_n(x)\|\, d\mu \to 0$. This is a contradiction. □

Now we come to the main theorem of this section.

Theorem 5. *$(X, \mathcal{A}, \mu)$ is finite measure space, E a Banach space and Q is a relatively countably compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. Then $\overline{Q}$ is Eberlein compact.*

Proof. We consider $Q \subset (C(B \times S), \|\cdot\|)$. In pointwise topology, $Q \subset (C(B\times S), \|\cdot\|)$ is relatively countably compact and, by Corollary 2, is norm bounded and so its closure in $C(B \times S)$ is pointwise compact. This implies that it is relatively weakly compact in the Banach space $(C(B \times S), \|\cdot\|)$.

Now we want to prove that Q is relatively compact in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. Put $p = \sup\{\|q\|_1 : q \in Q\}$ (by Corollary 2, Q is bounded in $(L_1(E), \|\cdot\|_1)$). Take a net $\{f_\alpha\} \subset Q$; there is a subnet, which again we denote by $\{f_\alpha\}$, such that $f_\alpha \to f \in C(B \times S)$ (pointwise topology). So there is a sequence $\{f_n\} \subset \{f_\alpha\}$ such that $f_n \to f \in C(B \times S)$ (pointwise topology). Since Q is a relatively countably compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$, this sequence $\{f_n\}$ has a cluster point, say $f_0 \in L_1(E)$. It is easily seen that $f = f_0$ on $B \times S$. Now we will show that $f_\alpha \to f_0$ pointwise on $L_\infty(E')$. First we prove that $\|f_0\|_1 \le p$. This follows from the fact that f_0 is a cluster point of $\{f_n\} \subset Q$ in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ and from Lemma 1.

Now take a $g \in L_\infty(E')$, $\|g\|_\infty \le 1$ and fix a $c > 0$. Using Lemma 4 (ii), select an $\eta > 0$ such that $\int_A \|f(x)\|\, d\mu < c/4$ for all $f \in Q$, as well as for f_0, and for all $A \in \mathcal{A}$, when $\mu(A) < \eta$. Now get an $A \in \mathcal{A}$, with $\mu(A) < \eta$, and a simple function $g_0 \in L_\infty(E')$ such that $\|g - g_0\|_\infty < c/[8(1+p)]$ on $X \setminus A$. Now, on X, $|\langle (f_\alpha - f_0)(x), g\rangle| \le \|f_\alpha(x)\| + \|f_0(x)\|$ and, on $X \setminus A$, $|\langle (f_\alpha - f_0)(x), g - g_0\rangle| \le (\|f_\alpha(x)\| + \|f_0(x)\|)c/[8(1+p)]$. Thus,

$$\begin{aligned}\left|\int (f_\alpha - f_0)(g)\, d\mu\right| &\le \left|\int_A (f_\alpha - f_0)(g)\, d\mu\right| \\ &\quad + \left|\int_{X\setminus A} (f_\alpha - f_0)(g - g_0)\, d\mu\right| \\ &\quad + \left|\int_{X\setminus A} (f_\alpha - f_0)(g_0)\, d\mu\right| \\ &\le 2\frac{c}{2} + \frac{2pc}{8(1+p)}\end{aligned}$$

$$+\left|\int_{X\setminus A}(f_\alpha - f_0)(g_0)\,d\mu\right|.$$

Since

$$\left|\int_{X\setminus A}(f_\alpha - f_0)(g_0)\,d\mu\right| \longrightarrow 0,$$

as we take the limit over α, the result follows. Thus closure $\overline{Q}$ of Q in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ is compact. Since $B \times S$ separates the points of $(L_1(E))$, $\overline{Q}$ is also a weakly compact subset of the Banach space $C(B \times S)$ and as such is Eberlein compact.

3. The closed convex hull. Now we come to the study of the closed convex hull of a compact subset Q of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. First we prove the result under the assumption that $(L_1(E), \|\cdot\|_1)$ is separable; for this, we use the technique of barycenters of probability measures studied in [**5**].

Lemma 6. *Suppose* $(L_1(E), \|\cdot\|_1)$ *is separable and* Q *is a compact subset of* $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. *Then*

(i) *the closed covex hull,* W, *of* Q *in* $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ *is compact*;

(ii) W *is also the closed covex hull of* Q *in* $(L_1(E), \|\cdot\|_1)$.

Proof. Let P be the set of all regular Borel probability measures on $(Q, \sigma(L_1(E_0), L_\infty(E_0')))$ and B_0 the closed unit ball of $(L_1(E), \|\cdot\|_1)'$. Fix a $\lambda \in P$ and put $L = \{h \in (L_1(E), \|\cdot\|_1)' : h\ \lambda\text{-measurable}\}$. We claim that $B_0 = \overline{L_\infty(E') \cap B_0}$ (closure in $\sigma((L_1(E, \|\cdot\|_1)', L_1(E)))$. Suppose this is not true. Then, by the separation theorem ([**7**, 9.2, page 65]), there is a $\phi \in B_0$ and a $g \in L_1(E)$ such that $\langle \phi, g\rangle > \sup\langle B_0 \cap L_\infty(E'), g\rangle$. This implies that $\|g\|_1 \geq \langle \phi, g\rangle > \|g\|_1$ (by Lemma 1), which is a contradiction.

Since B_0 is metrizable in $\sigma((L_1(E), \|\cdot\|_1)', L_1(E)))$, for any $\phi \in B_0$ there is a sequence $\{\phi_n\} \subset L_\infty(E') \cap B_0$ such that $\phi_n \to \phi$ pointwise on $L_1(E)$; this means that $\phi_{|Q}$ is λ-integrable. Consider the mapping $\psi : (L_1(E), \|\cdot\|_1)' \to R$, $\psi(\phi) = \int \phi_{|Q}\, d\lambda$. If a sequence, in B_0, $\phi_n \to 0$ pointwise on $L_1(\mu_0, E)$, then, by the dominated convergence theorem, $\psi(\phi_n) \to 0$, and so, by the Grothendieck completeness theorem ([**5**, page 149]), there is an $f_\lambda \in L_1(E)$ such that $\psi(\phi) = \phi(f_\lambda)$ for all $\phi \in (L_1(E), \|\cdot\|_1)'$ (in the terminology of [**5**], f_λ is the barycenter of λ).

Now, considering P with the topology of pointwise convergence on $C(Q)$ and $L_1(E)$ with the topology of pointwise convergence on $L_\infty(E)$, P is a compact convex set and the affine mapping $T : P \to L_1(E)$, $\lambda \to f_\lambda$, is continuous. Since P is compact, it follows that $T(P) = \overline{\text{convex}\,(Q)} = W$. Thus, W is compact.

Now put $W_0 =$, the closed convex hull of Q, in $((L_1(E), \|\cdot\|_1)$. Evidently, $W_0 \subset W$. To prove $W_0 \supset W$, it is enough to prove that each $f_\lambda \in W_0$. If $f_\lambda \notin W_0$, by the separation theorem ([**7**, 9.2, page 65]), there is a $\phi \in B_0$ such that $\int \phi\, d\lambda = \phi(f_\lambda) > \sup q(Q)$ (note that ϕ is λ-integrable). Since λ is a probability measure on Q, this is a contradiction. This proves the result. $\square$

By the next lemma, we reduce the general case to the case when $(L_1(E), \|\cdot\|_1)$ is separable.

Lemma 7. *Suppose that Q is separable compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. Then there is a separable closed subspace $E_0 \subset E$ and countably generated σ-algebra $\mathcal{B} \subset \mathcal{A}$ such that, for each $f \in Q$, $f(X) \subset E_0$ a.e$[\mu_0]$ and $f : X \to E_0$ is $\mathcal{B}$-measurable where $\mu_0 = \mu_{|\mathcal{B}}$. Also, $(L_1(E_0), \|\cdot\|_1)$ is separable and Q is a compact subset of $(L_1(E_0), \sigma(L_1(E_0), L_\infty(E_0')))$.*

Proof. Take a dense sequence $\{q_n\} \subset Q$. Evidently, there is a separable subspace $E_0 \subset E$ such that $q_n(X) \subset E_0$ almost everywhere $[\mu]$. We claim for any $q \in \overline{\{q_n\}}$, one gets $q(X) \subset E_0$ almost everywhere $[\mu]$. To prove this, we take a $q \in \overline{\{q_n\}}$. So there is a subsequence of $\{q_n\}$, which again we denote by $\{q_n\}$, such that $q_n \to q$ in $\sigma(L_1(E), L_\infty(E'))$ (Theorem 5). We claim $q(X) \subset E_0$ almost everywhere $[\mu]$. Suppose this is not true; this means there is a $\zeta > 0$ such that $q(X) \nsubseteq E^0 = \overline{E_0 + \zeta L}$ almost everywhere $[\mu]$, L being the closed unit ball of E (closure in E; note $E_0 = \cap_{n=1}^{\infty} \overline{(E_0 + (1/n)L)}$).

Take a separable closed subspace $E_1 \subset E$, $E_1 \supset E_0$, such that $q(X) \subset E_1$ almost everywhere $[\mu]$ and $\mu(q^{-1}(E_1 \setminus E^0)) > 0$. This implies that there is a $x_0 \in E_1$ and a $c > 0$ such that $\overline{B(x_0, c)} \cap E^0 = \varnothing$ and $\mu(A) > 0$ where $A = q^{-1}(B(x_0, c))$ (here $B(x_0, c)$ is the open ball, in E_1, with center at x_0 and radius c, and $\overline{B(x_0,c)}$ its closure in E_1) (here we are using that E_1 is separable). By the separation theorem ([**7**, 9.1, page 64]), there is an $f \in E'$, with $\|f\| = 1$, such that $\inf f(B(x_0,c)) \geq \sup f(E^0)$. This implies $f(E_0) = 0$ and $\sup f(E^0) = \zeta$. Thus, $\inf f(B(x_0,c))) \geq \zeta > 0$ Now $0 = \int_A f \circ q_n \, d\mu \to \int_A f \circ q \, d\mu \geq \zeta\mu(A) > 0$, a contradiction.

Now take a σ-algebra $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B}$ is countably generated and each q_n is measurable with respect to this σ-algebra. Putting $\mu_0 = \mu_{|\mathcal{B}}$, $L_1(\mu_0, E_0)$ is separable in norm topology. We claim $Q = \overline{\{q_n\}} \subset L_1(\mu_0, E_0)$. Suppose $q_n \to q$. Fix an $f \in E'$ and, for each $A \in \mathcal{B}$, define the measure $\nu_f(A) = \lim \int_A q_n \, d\mu_0$; it is easily verified that $\nu_f \ll \mu_0$, and so there is a $\phi_f \in L_1(\mu_0)$ such that $\nu_f = \phi_f \mu_0$. So we have $\phi_f = f \circ q$ almost everywhere $[\mu]$. Thus, $f \circ q$ is $\mathcal{B}$-measurable for every $f \in E'$. Since E_0 is separable, this means q is $\mathcal{B}$-measurable.

It is obvious that Q is a compact subset of $(L_1(E_0), \sigma(L_1(E_0), L_\infty(E_0'))))$. □

Now we prove the main theorem of this section in the general case.

Theorem 8. *Let Q be a compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E'))))$. Then the closed convex hull W of Q in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ is also compact, and W is also the closed convex hull of Q in $(L_1(E), \|\cdot\|_1)$.*

Proof. Let W_{00} be the closed convex hull of Q in $(L_1(E), \|\cdot\|_1)$ and W_0 the closed convex hull of Q in $C(B \times S)$ with weak topology of $C(B \times S)$. Fix a $q \in W_0$. Since the closed convex hull of Q, in $C(B \times S)$ with weak topology, is weakly compact, there is a sequence $\{q_n\} \subset Q$ such that q is in the closed convex hull, in $C(B \times S)$ with weak topology, of $\{q_n\}$. Let $Q_1 = \overline{\{q_n\}}$ (closure in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$). This means that Q_1 is a separable compact subset of $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. By Lemma 7, Q_1 is contained in a separable $(L_1(E_0), \|\cdot\|_1)$ and is compact in $(L_1(E_0), \sigma(L_1(E_0), L_\infty(E_0')))$. By Lemma 6, the closed convex hull W_1 of Q_1 in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$ is also compact and W_1 is also the closed convex hull of Q_1 in $(L_1(E), \|.\|_1)$. Since W_1 is also weakly compact in $C(B \times S)$, we get $q \in W_1$. This means $W_0 \subset W_{00}$. Thus, $W_0 = W_{00}$.

Now we want to prove that W_{00} is compact in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. We take a net $\{f_\alpha\} \subset W_0$ such that $f_\alpha \to f_0 \in W_0$, pointwise on $C(B \times S)$ (note W_0 is weakly compact in $C(B \times S)$). Since Q is bounded in $(L_1(E), \|\cdot\|_1)$ (Corollary 2), W_{00} is also bounded in $(L_1(E), \|\cdot\|_1)$ and so W_0 is also bounded in $(L_1(E), \|\cdot\|_1)$. By Lemma 4, we have $\lim_{\mu(A)\to 0}(\int_A \|f(x)\|\, d\mu) = 0$ uniformly for $f \in Q$ and from this it immediately follows that $\lim_{\mu(A)\to 0}(\int_A \|f(x)\|\, d\mu) = 0$ uniformly for $f \in W_{00}$. This is the main result used in Theorem 5 when it is proved that if $f_\alpha \to f_0$ pointwise on $C(B \times S)$, then $f_\alpha \to f_0$ pointwise on $L_\infty(E')$. So we get that W_0 is compact in $(L_1(E), \sigma(L_1(E), L_\infty(E')))$. This proves the result.

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