ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 1, 2014

COMPARISON OF INVARIANT METRICS

HYUNSUK KANG, LINA LEE AND CRYSTAL ZEAGER

ABSTRACT. We estimate the boundary behavior of the Kobayashi metric on $\mathbf{C}\setminus\{0,1\}$. We also compare the Bergman metric on the ring domain in $\mathbf{C}^2$ to the Bergman metric on the ball and study the comparison between the Kobayashi, Carathéodory, Azukawa and Sibony metric.

1. Introduction. The Kobayashi and the Carathéodory metrics are the generalizations of the Poincaré metric in higher dimensions, and it is known that the Kobayashi metric is the largest and the Carathéodory metric is the smallest among such metrics, i.e., metrics that coincide with the Poincaré metric on the unit disc in $\mathbf{C}$ and satisfy the non-increasing property under holomorphic mappings. For example, the Sibony and the Azukawa metrics, whose definitions can be found in Section 2, are both examples of such metrics, and we always have the following inequality: Carathéodory metric $\leq$ Sibony metric $\leq$ Azukawa metric $\leq$ Kobayashi metric.

On the other hand, the Bergman metric is neither equal to the Poincaré metric on the unit disc in $\mathbf{C}$, nor does it satisfy the non-increasing property under holomorphic mappings. Hence, the inequality between the Bergman metric and the Carathéodory metric or the Kobayashi metric is not obvious. It was Hahn who proved that the Bergman metric is always greater than or equal to the Carathéodory metric in [**5**]. Diederich and Fornæss [**2**] and Diederich, Fornæss and Herbort [**3**] showed that there is no simple inequality that always holds between the Kobayashi metric and the Bergman metric.

1.1. Comparison of metrics in C. The first question we ask is regarding the Kobayashi, Carathéodory, Azukawa and Sibony metrics. As stated above, one can always find a simple inequality between these metrics. Then one may wonder how differently these metrics can behave. That leads to the following question: can one find a domain

Received by the editors on January 12, 2010, and in revised form on August 3, 2011.

DOI:10.1216/RMJ-2014-44-1-157 Copyright ©2014 Rocky Mountain Mathematics Consortium

where one metric does not vanish anywhere but another metric vanishes at some point? In other words, can one find a domain, where the hyperbolicity of these metrics are not equivalent?

The domain $\mathbf{C}\setminus\{0,1\}$ is an example of such a domain. The Kobayashi metric does not vanish everywhere since its covering space is the unit disc and the Kobayashi metric does not vanish on the unit disc. The Carathéodory metric, on the other hand, vanishes everywhere, since there does not exist a bounded holomorphic function on $\mathbf{C}\setminus\{0,1\}$. The Sibony and Azukawa metrics are also identically zero on $\mathbf{C}\setminus\{0,1\}$ because the bounded plurisubharmonic functions on $\mathbf{C}\setminus\{0,1\}$ are all constant.

A higher-dimensional example of the same phenomenon is given by $\Omega\times\Delta^n$ where $\Omega=\mathbf{C}\setminus\{0,1\}$. It is easy to check that this domain is Kobayashi hyperbolic using the known formula for the Kobayashi metric on product domains [**6**, page 106]. But this domain is not Carathéodory, Azukawa, or Sibony hyperbolic because it contains copies of Ω.

We then study how the Kobayashi metric behaves as the point approaches one of the punctures and obtained the following theorem:

Theorem 1. *Let $\Omega=\mathbf{C}\setminus\{0,1\}$, and let $\operatorname{dist}(p,0)=\delta$, $\xi=1$. Then, for $\delta>0$ sufficiently small, we have*

$$F_K^{\Omega}(p,\xi)\approx\frac{1}{\delta\log 1/\delta}, \tag{1.1}$$

where $F_K^{\Omega}(p,\xi)$ denotes the Kobayashi metric on Ω at the point p in the direction ξ.

We prove the above result in Section 3 by using the elliptic modular function and calculating its derivatives.

1.2. Comparison of the Bergman metric on a ring domain in $\mathbf{C}^n$. If a metric F satisfies the non-increasing property under holomorphic mappings, then it is immediate that $F^A\geq F^B$ if $A\subset B$, where F^A and F^B denote the metric F on A and B, respectively, since the metric should not increase under the inclusion mapping.

We consider a ring domain $\Omega = \{r < \|z\| < 1\} \subset \mathbf{C}^n$, $r \in (0,1)$. Since $\Omega \subset \mathbf{B}^n = \{\|z\| < 1\}$, we have the following inequality $F^\Omega \geq F^{\mathbf{B}^n}$, where F denotes any of the Kobayashi, Carathéodory, Azukawa and Sibony metrics. In fact, the Sibony, Azukawa and Kobayashi metrics blow up near the inner boundary of Ω in the normal direction, see [**4, 8**].

In Section 4, we study how the Bergman metric behaves on a ring domain in $\mathbf{C}^n$. First, we show that the Bergman metric does not blow up near the inner boundary. In Proposition 2, we show this in a more general setting: if $K \subset\subset \Omega \subset\subset \mathbf{C}^n$ and K, Ω are domains, then $F_B^{\Omega\setminus K} \approx F_B^\Omega$, where F_B denotes the Bergman metric. We then show the following strict inequality in the tangential direction on a ring domain in $\mathbf{C}^n$:

Theorem 2. *Let $\mathbf{B}^n$ be the unit ball in $\mathbf{C}^n$ and Ω the ring domain, $\Omega = \{z \in \mathbf{C}^n : r < |z| < 1\}$, $r \in (0,1)$. Let $p \in \Omega$ and $\xi \in T_p\Omega$ be such that $\xi \cdot \overline{p} = 0$, i.e., ξ is in the tangential direction to the inner boundary. Then*

$$F_B^\Omega(p,\xi) \lneq F_B^{\mathbf{B}^n}(p,\xi), \quad \text{for all } p \in \Omega. \tag{1.2}$$

Note that we have the reverse inequality, $F^\Omega(p,\xi) \geq F^{\mathbf{B}^n}(p,\xi)$ for the Kobayashi, Carathéodory, Azukawa and Sibony metrics for all $p \in \Omega$ and for all $\xi \in \mathbf{C}^n$.

We show strict inequality in the normal direction on a ring domain in $\mathbf{C}^2$ near the inner boundary when the inner radius is small enough, see Proposition 3. Hence, we obtain strict inequality in all directions in the ring domains in $\mathbf{C}^2$ in such cases:

Theorem 3. *Let $\mathbf{B}^2$ be the unit ball in $\mathbf{C}^2$ and Ω the ring domain, $\Omega = \{z \in \mathbf{C}^2 : r < |z| < 1\}$ for $r > 0$ small. Let $p = (r+\varepsilon, 0)$ for $\varepsilon > 0$ small, and let $\xi \in \mathbf{C}^2$. Then we have*

$$F_B^\Omega(p,\xi) \lneq F_B^{\mathbf{B}^2}(p,\xi). \tag{1.3}$$

This paper is organized as follows. In Section 2 we give definitions and the background for the metrics. We prove Theorem 1 in Section 3 and Theorems 2 and 3 in Section 4.

2. Definitions and background. In this section, we give definitions and properties of the metrics which are used in later sections. For a more detailed discussion of the metrics, see [**6**].

Definition 1. Let $\Omega \subset \mathbf{C}^n$ be a domain, $p \in \Omega$, and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n$. Let Δ be the unit disk in $\mathbf{C}$, and let $\mathbf{B}^n(p,r) \subset \mathcal{C}^n$ be the ball of radius r centered at p.

- Carathéodory pseudometric $F_C^\Omega(p,\xi)$ is defined as

$$F_C^\Omega(p,\xi) = \sup\{|f'(p)\cdot\xi| : f \in \mathcal{O}(\Omega,\Delta), f(p) = 0\},$$

where $\mathcal{O}(\Omega,\Delta)$ is the set of holomorphic functions from Ω to Δ.

- Kobayashi pseudometric $F_K^{\mathcal{W}}(p,\xi)$ is defined as

$$F_K^\Omega(p,\xi) = \inf\left\{|\alpha| : f \in \mathcal{O}(\Delta,\Omega), f(0) = p,\ \exists\alpha > 0, \alpha f'(0) = \xi\right\},$$

where $\mathcal{O}(\Delta,\Omega)$ is the set of holomorphic functions from Δ to Ω.

- Sibony pseudometric $F_S^\Omega(p,\xi)$ is defined as

$$F_S^\Omega(p,\xi) = \sup\left\{(\partial\overline{\partial}u(p)(\xi,\overline{\xi}))^{1/2} = \left(\sum_{i,j=1}^n \frac{\partial^2 u(p)}{\partial z_i \partial \overline{z_j}}\xi_i\overline{\xi_j}\right)^{1/2} : u \in \mathcal{S}_\Omega(p)\right\},$$

where $\mathcal{S}_\Omega(p)$ is the set of functions u such that $u : \Omega \to [0,1)$ vanishes at p, $\log u$ is plurisubharmonic and u is C^2 near p.

- Azukawa pseudometric $F_A^\Omega(p,\xi)$ is defined as

$$F_A^\Omega(p,\xi) = \sup\left\{\limsup_{\lambda\searrow 0}\frac{1}{|\lambda|}u(p+\lambda\xi) : u \in \mathcal{K}_\Omega(p)\right\},$$

where $\mathcal{K}_\Omega(p)$ is the set of functions u such that $u : \Omega \to [0,1)$, $\log u$ is plurisubharmonic and there exist $M > 0$, $r > 0$ such that $\mathbf{B}^n(p,r) \subset \Omega$ and $u(z) \le M\|z-p\|$ for all $z \in \mathbf{B}^n(p,r)$.

Let K_Ω be the Bergman kernel of Ω.

Definition 2. The Bergman metric $F_B^\Omega(p,\xi)$ is defined by

$$F_B^\Omega(p,\xi) = \left(\sum_{\nu,\mu=1}^{n} \frac{\partial^2}{\partial z_\nu \partial \overline{z_\mu}} \log K_\Omega(z,z)\xi_\nu \overline{\xi_\mu} \right)^{1/2},$$

provided that K_Ω is nonvanishing on Ω.

Except for the Bergman metric, the other four metrics are non-increasing with respect to holomorphic mappings, that is, if $\Phi : \Omega_1 \to \Omega_2$ is holomorphic, then $F^{\Omega_1}(p,\xi) \geq F^{\Omega_2}(\Phi(p), \Phi_*(\xi))$ where F^{Ω_i} is one of $F_C^{\Omega_i}$, $F_S^{\Omega_i}$, $F_A^{\Omega_i}$ and $F_K^{\Omega_i}$. Moreover, they satisfy the following relationship:

$$F_C^\Omega(p,\xi) \leq F_S^\Omega(p,\xi) \leq F_A^\Omega(p,\xi) \leq F_K^\Omega(p,\xi) \tag{2.1}$$

for all p and ξ. The Bergman metric behaves differently from the rest of the metrics in the sense that it does not have non-increasing property, nor does it fit in the comparison (2.1). Between the Carathéodory and the Bergman metric the following is known.

Theorem 4 [**5**]. *In any complex manifold Ω, the Bergman metric F_B^Ω is always greater than or equal to the Carathéodory differential metric F_C^Ω if M admits them*:

$$F_C^\Omega(p,\xi) \leq F_B^\Omega(p,\xi). \tag{2.2}$$

However, Kobayashi and Bergman metrics do not have any such relation, and they are in fact incomparable as discussed in the introduction.

The boundary behavior of the Sibony metric on the ring domain in $\mathbf{C}^2$ near the inner boundary is a special case of domains studied in [**4**].

Theorem 5 [**4**]. *Let $\Omega = \{(1/4) < |z_1|^2 + |z_2|^2 < 1\} \subset \mathbf{C}^2$ and $P_\delta = (1/2+\delta, 0)$, and $\xi = (1,0)$. Then we have*

$$F_S^\Omega(p,\xi) \approx \frac{1}{\delta^{1/2}} \tag{2.3}$$

for $\delta > 0$ small enough.

In particular, since the Sibony metric blows up towards the inner boundary in the normal direction, so does the Azukawa metric by (2.1). In Section 4, we show that Bergman is bounded in the inner boundary, hence not comparable with either the Sibony or the Azukawa metric.

For convenience to the readers, we provide the formulae for Bergman kernel and Bergman metric on the unit ball:

$$K_{\mathbf{B}^n}(z,\zeta) = \frac{n!}{\pi^n}\frac{1}{(1-\sum z_j\bar{\zeta}_j)^{n+1}}$$

and

$$F_B^{\mathbf{B}^n}(p;\xi) = \sqrt{n+1}\left(\frac{|\langle p,\xi\rangle|^2}{(1-|p|^2)^2} + \frac{|\xi|^2}{1-|p|^2}\right)^{1/2} \tag{2.4}$$

where $z = (z_1, \dots , z_n)$ and $\zeta = (\zeta_1, \dots , \zeta_n)$.

3. Boundary behavior of the Kobayashi metric on $\mathbf{C} \setminus \{0, 1\}$. One technique for studying the Kobayashi metric of a domain is to study instead the Kobayashi metric of its covering space. In general, if $\pi : \widetilde{\Omega} \to \Omega$ is the covering map, it is known that $F_K^{\tilde{\Omega}} = \pi^* F_K^{\Omega}$ [**7**, page 91]. In the case that the covering space is a half-plane $\mathbf{H}$, we have the following equation. We provide a proof for the convenience of the reader.

Lemma 1. *Let Ω be a connected domain in $\mathbf{C}$ whose universal covering is $\mathbf{H}$. Let $q \in \mathbf{H}$, and let $m : \mathbf{H} \to \Delta$ be the biholomorphism such that $m(q) = 0$. Let $p \in \Omega$, $\xi \in \mathbf{C}^n$ and $\pi : \mathbf{H} \to \Omega$, with $\pi(q) = p$. Then*

$$F_K^{\Omega}(p,\xi) = \frac{|m'(q)|}{|\pi'(q)|}\|\xi\|.$$

Proof. Let f be a candidate for the Kobayashi metric at the point p, that is, $f(0) = p$ and $f'(0)$ is the multiple of ξ. Since the unit disc is simply connected, there exists the unique lifting $\widetilde{f} : \Delta \to \mathbf{H}$ such that $\widetilde{f}(0) = q$ as in the following diagram.

$$
\begin{tikzcd}
 & \Delta \\
 & \mathbf{H} \arrow[u, "m(z)"'] \arrow[d, "\pi"] \\
\Delta \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & \Omega
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & 0 \\
 & q \arrow[u, mapsto, "m"'] \arrow[d, mapsto, "\pi"] \\
0 \arrow[r, mapsto, "f"'] \arrow[ur, mapsto, "\tilde{f}"] & p
\end{tikzcd}
$$

Let π^{-1} be the local inverse in a neighborhood of p. Since $m \circ \widetilde{f}(0) = 0$ and $\widetilde{f} = \pi^{-1} \circ f$, the Schwarz lemma gives

$$|m'(q) \cdot (\pi^{-1})'(p) \cdot f'(0)| \leq 1,$$

implying

$$\frac{1}{|f'(0)|} \geq \frac{|m'(q)|}{|\pi'(q)|}.$$

This estimate holds for any candidate function, and the map $\pi \circ m^{-1}$ is itself a candidate for the Kobayashi metric. The claim follows. □

We will use Lemma 1 to estimate boundary behavior of the Kobayashi metric on the domains $\Delta \setminus \{0\}$ and $\mathbf{C} \setminus \{0, 1\}$.

Proposition 1. *Let p be the point in $\Delta \setminus \{0\}$ such that* $\operatorname{dist}(p, 0) = \delta$, *and let $\xi = 1$. For every $\delta > 0$ we have*

$$F_K^{\Delta\setminus\{0\}}(p, \xi) = \frac{1}{2\delta \log 1/\delta}.$$

Proof. The map $z \mapsto ze^{i\theta}$ is a self map of $\Delta \setminus \{0\}$; hence, it is enough to show the proposition in the case that $p = \delta$. Let $\mathbf{H}_{\text{left}}$ denote the left half plane $\{\operatorname{Re}(z) < 0\} \subset \mathbf{C}$. The covering map is given by $\pi : \mathbf{H}_{\text{left}} \to \Delta \setminus \{0\}$, $z \mapsto e^z$. Let $q = \log \delta$. Then, $m : \mathbf{H}_{\text{left}} \to \Delta$ is

$$m(z) := \frac{z - \log \delta}{z + \log \delta} \quad \text{with} \quad m'(q) = \frac{1}{2 \log \delta}.$$

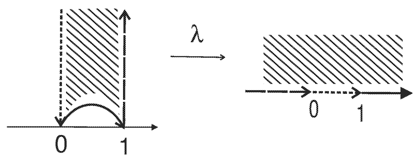

FIGURE 3.1.

By Lemma 1, we have

$$F_K^{\Delta\setminus\{0\}}(\delta, 1) = \frac{1}{2\delta|\log\delta|} = \frac{1}{2\delta\log(1/\delta)}. \qquad \square$$

In order to use Lemma 1 to obtain a boundary estimate for $\Omega = \mathbf{C}\setminus\{0,1\}$ we consider the elliptic modular function as the covering map from the half-plane to Ω and estimate its derivative. Let $\lambda(\tau)$ be the elliptic modular function. We include some properties and estimates related to λ to be used for our purpose. For a more complete reference on λ, see [**1**, Chapter 7, subsection 3.4].

It is known that the elliptic modular function can be written as follows:

$$\lambda(\tau) = \frac{\sum_{n=-\infty}^{\infty}\left[\frac{1}{\cos^2(\pi(n-\frac{1}{2})\tau)} - \frac{1}{\sin^2(\pi(n-\frac{1}{2})\tau)}\right]}{\sum_{n=-\infty}^{\infty}\left[\frac{1}{\cos^2(\pi n\tau)} - \frac{1}{\sin^2(\pi(n-\frac{1}{2})\tau)}\right]} =: \frac{N(\tau)}{D(\tau)}. \tag{3.1}$$

We first restrict λ on a strip shaded in the figure. The figure shows how λ maps the region and its boundary to the upper half plane. By the Schwarz reflections, we can see images of λ will then cover the complex plane except for $\{0,1\}$ since they are the images of boundary points $\{0,1\}$ of the upper half plane.

We also notice that λ maps the infinity to the origin. To estimate the Kobayashi metric on $\mathcal{C}\setminus\{0,1\}$ near the origin at the point p, we will consider the preimage $q = \lambda^{-1}(p)$ in the upper half-plane with large positive imaginary part.

The following convergence results are known (see [**1**, page 280]). Uniformly with respect to $\mathrm{Re}\,(\tau)$, as $\mathrm{Im}\,(\tau) \to \infty$,

$$D(\tau) \longrightarrow \pi^2, \tag{3.2}$$

$$\lambda(\tau)e^{-i\pi\tau} \longrightarrow 16. \tag{3.3}$$

Lemma 2. *As* $\mathrm{Im}\,(\tau) \to \infty$, *the derivatives of* $N(\tau)$ *are uniformly bounded by a constant*:

$$\left|\frac{d}{d\tau}D(\tau)\right| < C \tag{3.4}$$

for some $C > 0$ *and the derivatives of* $D(\tau)$ *satisfy the following estimate*:

$$\left|\frac{d}{d\tau}N(\tau)\right| \lesssim |e^{i\pi\tau}| = e^{-\pi\mathrm{Im}\,(\tau)}. \tag{3.5}$$

Proof of Lemma 2. For $z = x + iy$, we have

$$\begin{aligned}\sin(z) &= \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)})\\ &= \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^{y}).\end{aligned}$$

Hence, for $|y| > 1/2\ln 2$, we have

$$\frac{1}{4}e^{|y|} < |\sin z| < e^{|y|}. \tag{3.6}$$

Similarly, we have

$$\frac{1}{4}e^{|y|} < |\cos z| < e^{|y|}. \tag{3.7}$$

Using (3.6) and (3.7) the derivatives of cosine terms of $D(\tau)$ except for the term with $n = 0$ can be estimated as

$$\left|\frac{d}{d\tau}\frac{1}{\cos^2(\pi n\tau)}\right| = \left|\frac{2\pi n\sin(\pi n\tau)}{\cos^3(\pi n\tau)}\right| \lesssim \frac{|n|}{e^{2\pi|n|\mathrm{Im}\,(\tau)}} \tag{3.8}$$

as $\mathrm{Im}\,(\tau) \to \infty$.

Similarly, we have

$$\begin{aligned} \left| \frac{d}{d\tau} \frac{1}{\sin^2(\pi(n-\frac{1}{2})\tau)} \right| &= \left| \frac{2\pi(n-\frac{1}{2})\cos(\pi(n-\frac{1}{2})\tau)}{\sin^3(\pi(n-\frac{1}{2})\tau)} \right| \\ &\lesssim \frac{|n-1/2|}{e^{2\pi|n-1/2|\mathrm{Im}\,(\tau)}}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \left| \frac{d}{d\tau} \frac{1}{\cos^2(\pi(n-\frac{1}{2})\tau)} \right| &= \left| \frac{2\pi(n-\frac{1}{2})\sin(\pi(n-\frac{1}{2})\tau)}{\cos^3(\pi(n-\frac{1}{2})\tau)} \right| \\ &\lesssim \frac{|n-1/2|}{e^{2\pi|n-1/2|\mathrm{Im}\,(\tau)}} \end{aligned} \tag{3.9}$$

for all $n \in \mathbf{Z}$ as $\mathrm{Im}\,(\tau) \to \infty$.

Hence, using the derivative of $D(\tau)$, we have

$$\left| \frac{d}{d\tau} D(\tau) \right| \leq \sum \left| \frac{d}{d\tau} \frac{1}{\cos^2(\pi n \tau)} \right| + \sum \left| \frac{d}{d\tau} \frac{1}{\sin^2(\pi(n-\frac{1}{2})\tau)} \right|.$$

By (3.8) and (3.9), it is uniformly bounded above. Similarly the derivative of $N(\tau)$ is

$$\begin{aligned} \left| \frac{d}{d\tau} N(\tau) \right| &\lesssim \sum \left| \frac{d}{d\tau} \frac{1}{\sin^2(\pi(n-\frac{1}{2})\tau)} \right| + \sum \left| \frac{d}{d\tau} \frac{1}{\cos^2(\pi(n-\frac{1}{2})\tau)} \right| \\ &\lesssim \frac{1}{e^{\pi \mathrm{Im}\,(\tau)}}. \qquad \square \end{aligned}$$

Proof of Theorem 1. Let $\Omega = \mathbf{C} \setminus \{0, 1\}$, and let $\mathbf{H}_{\text{upper}}$ denote the upper half-plane $\{\mathrm{Im}\,(z) > 0\} \subset \mathbf{C}$. Let p be a point close to the origin, and let $\lambda : \mathbf{C} \to \mathbf{C} \setminus \{0, 1\}$ be the elliptic modular function as the covering map of Ω. From Figure 3.1, we see that an inverse image of p is a point q of the form $r + iM$ where $M > 0$, $M \to \infty$ as $p \to 0$. From the Schwarz reflection argument, if we allow r to be in $[0, 2]$, then $\lambda(q)$ would be approaching the origin in every direction. We estimate the Kobayashi metric at point p by applying Lemma 1.

The Möbius transformation $m : \mathbf{H}_{\text{upper}} \to \Delta$ sending $q = r + iM$ to the origin is given by

$$m(z) = \frac{z-q}{z-\bar{q}} = \frac{z-(r+iM)}{z-(r-iM)}.$$

Lemma 1 gives that

$$F_K^{\mathcal{W}}(p,\xi) = \frac{|m'(q)|}{|\lambda'(q)|} = \frac{1}{2M|\lambda'(r+iM)|}. \tag{3.11}$$

Using the notation we used in (3.1) the derivative of λ is

$$\lambda'(\tau) = \frac{N'(\tau)}{D(\tau)} - \lambda(\tau)\frac{D'(\tau)}{D(\tau)}.$$

From (3.2), (3.3) and Lemma 2, we have

$$|\lambda'(\tau)| \approx |N'(\tau) - \lambda(\tau)D'(\tau)| \lesssim |e^{i\pi\tau}|$$

as $\operatorname{Im}(\tau) \to \infty$. Hence, combining with (3.11) we have

$$F_K^{\Omega}(p,\xi) \gtrsim \frac{1}{2Me^{-\pi M}}. \tag{3.12}$$

The equation (3.3) gives $\delta = \operatorname{dist}(p,0) \approx e^{-\pi M}$, that is, $M \approx \log(1/\delta)$. It follows from (3.12) we have lower bound estimate:

$$F_K^{\Omega}(p,\xi) \gtrsim \frac{1}{\delta\log(1/\delta)}.$$

On the other hand, $\Delta \setminus \{0\}$ is a subset of $\mathbf{C} \setminus \{0,1\}$. By non-increasing property of Kobayashi metric, we have

$$F_K^{\Omega}(p,\xi) \leq F_K^{\Delta\setminus\{0\}}(p,\xi) = \frac{1}{2\delta\log 1/\delta}.$$

This completes the proof. □

Remark 1. The Kobayashi metric has the same boundary behavior near 1 as it nears 0 on the domain $\mathbf{C}\setminus\{0,1\}$. One can see this by noting that the map $z \mapsto 1-z$ is a biholomorphism that exchanges points 0 and 1 and leaves the length of the tangent vector unchanged.

Corollary 1. *Let $\Omega \subset \mathbf{C}$ be a domain with discrete punctures, i.e., $\Omega = U\setminus J$, where U is a domain in $\mathbf{C}$ and J is a union of discrete points in U with at least two points and no limit point. As $p \in \Omega$ approaches a point $p_j \in J$ the Kobayashi metric on Ω satisfies the estimate*

$$F_K^{\Omega}(p,\xi) \approx \frac{1}{\delta \log(1/\delta)}$$

for $\|\xi\| = 1$.

Proof. Let $p_{j'} \in J$ with $j' \neq j$. Then, since the points in J are discrete, there exists a disk $\Delta(p_j, r)$ of radius r centered at p_j such that $\Delta(p_j,r)\setminus\{p_j\} \subset \Omega$. We have the following inclusions

$$\Delta(p_j,r)\setminus\{p_j\} \subset \Omega \subset \mathbf{C}\setminus\{p_j,p_{j'}\},$$

giving us corresponding inequalities of the metrics:

$$F_K^{\Delta(j,r)\setminus\{j\}} \geq F_K^{\Omega} \geq F_K^{\mathbf{C}\setminus\{j,j'\}}.$$

Proposition 1 and Theorem 1 imply the corollary. □

4. Estimation of the Bergman metric near the inner boundary of the ring domain. We show first that if Ω is a bounded domain in $\mathbf{C}^n$ and K is a relatively compact subset of Ω then the Bergman metric on $\Omega \setminus K$ is comparable to the Bergman metric on Ω. Hence, the metric does not blow up on $\Omega\setminus K$ near ∂K. We denote the space of square integrable functions holomorphic in Ω by $L_h^2(\Omega)$.

Proposition 2. *Let $\Omega \subset\subset \mathbf{C}^n$, $n \geq 2$ be a bounded domain and $K \subset\subset \Omega$ be a domain in $\mathbf{C}^n$. Then there exists a constant d, $0 < d < 1$, depending on K such that*

$$\sqrt{1-d^2}F_B^{\Omega}(z,\xi) \leq F_B^{\Omega\setminus K}(z,\xi) \leq \frac{1}{\sqrt{1-d^2}}F_B^{\Omega}(z,\xi).$$

Proof. We use the following property for the Bergman metric:

$$\big(F_B^{\mathcal{W}}(z,\xi)\big)^2 = \frac{b_\Omega^2(z,\xi)}{K_\Omega(z,z)} \tag{4.1}$$

where

$$b_\Omega^2(z,\xi) = \sup\{|\langle \partial f, \xi\rangle|^2 : f \in L_h^2(\Omega),\ f(z)=0,\ \|f\|_{L^2(\Omega)} = 1\}$$

and

$$K_\Omega(z,z) = \sup\{|g(z)|^2 : \|g\|_{L^2(\Omega)} = 1,\ g \in L_h^2(\Omega)\}.$$

Every holomorphic function f on $\Omega \setminus K$ holomorphically extends to Ω. We use the same f for extension.

We shall show that there exists a d such that

$$\|f\|_{L^2(K)} < d < 1 \quad \text{for all } f \in L_h^2(\Omega) \text{ with } \|f\|_{L^2(\Omega)} = 1.$$

If not, there exists a sequence f_n with $\|f\|_{L^2(K)} \geq 1 - 1/n$ for all $n \in \mathbf{N}$. Since L^2-norm of f_n is uniformly bounded, $|f(z)|$ is also uniformly bounded for all $z \in K$. Hence, there exists a convergent subsequence, call it again as f_n, whose limit function F is also in $L_h^2(\Omega)$. We have

$$\|F\|_{L^2(K)} = \lim \|f_n\|_{L^2(K)} = 1 \quad \text{whereas } \|F\|_{L^2(\Omega)} \leq 1,$$

implying $F = 0$ almost everywhere on $\Omega \setminus K$, hence $F \equiv 0$. This contradicts $\|F\|_{L^2(K)} = 1$.

We have, for $f \in L_h^2(\Omega) \setminus \{0\}$

$$(1-d^2)\|f\|^2_{L^2(\Omega)} < \|f\|^2_{L^2(\Omega\setminus K)} < \|f\|^2_{L^2(\Omega)}.$$

Equivalently,

$$\frac{1}{\|f\|^2_{L^2(\Omega)}} < \frac{1}{\|f\|^2_{L^2(\Omega\setminus K)}} < \frac{1}{(1-d^2)\|f\|^2_{L^2(\Omega)}}. \tag{4.2}$$

From (4.1) and (4.2) we have Proposition 4.2. $\square$

We now consider the case that Ω is the unit ball in $\mathbf{C}^2$ and K is a ball of smaller radius centered at the origin. In this special case we can obtain a specific constant d using the orthonormal basis. We let $\mathbf{B} = \{z \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ and $\Omega_r = \{z \in \mathbf{C}^2 : r^2 < |z_1|^2 + |z_2|^2 < 1\}$. Let $\{a_{jk} z_1^j z_2^k\}_{j \geq 0, k \geq 0}$ be an orthonormal basis of $L_h^2(\mathbf{B})$. Then we can easily check that $\{(1/r^{j+k+2}) a_{jk} z_1^j z_2^k\}_{j \geq 0, k \geq 0}$ is an orthonormal basis of $L_h^2(r\mathbf{B})$. If $f(z) = \sum c_{jk} z_1^j z_2^k$, then we have

$$\|f\|^2_{L^2(r\mathbf{B})} = \sum \frac{|c_{jk}|^2}{|a_{jk}|^2} r^{2j+2k+4} < r^4 \sum \frac{|c_{jk}|^2}{|a_{jk}|^2} = r^4 \|f\|^2_{L^2(\mathbf{B})}.$$

Hence, we can take $d = r^2$ and apply Proposition 2 to have

$$\sqrt{1 - r^4} F_B^{\mathbf{B}}(z, \xi) \leq F_B^{\Omega_r}(z, \xi) \leq \frac{1}{\sqrt{1 - r^4}} F_B^{\mathbf{B}}(z, \xi). \tag{4.3}$$

One can see that if Ω is the unit ball in $\mathbf{C}^n$, then we can take $d = r^n$ and have similar inequalities to (4.3). In the rest of this section we focus on improving the upper bound. We prove that the Bergman metric on Ω_r is *strictly smaller* than the Bergman metric on $\mathbf{B}$ near the inner boundary when the inner radius r is small enough. To do the comparison, we express the Bergman kernel $K_{\Omega_r}(z, z)$ in terms of the renormalized basis of $L_h^2(\mathbf{B})$ and compute the Levi form of $\log K_{\Omega_r}$ in normal and tangential directions. Let $\{a'_{jk} z_1^j z_2^k\}_{j \geq 0, k \geq 0}$ be an orthonormal basis of $L_h^2(\Omega_r)$. The coefficients a'_{jk} is a multiple of a_{jk}:

$$a'_{jk} = a_{jk} \frac{1}{\sqrt{1 - r^{2(j+k)+4}}}. \tag{4.4}$$

This is because

$$\begin{aligned}
\int_{\Omega_r} |z_1|^{2j} |z_2|^{2k} \, d\,\mathrm{Vol} &= \int_{\mathbf{B}} |z_1|^{2j} |z_2|^{2k} \, d\,\mathrm{Vol} - \int_{\|z\| < r} |z_1|^{2j} |z_2|^{2k} \, d\,\mathrm{Vol} \\
&= \int_{\mathbf{B}} |z_1|^{2j} |z_2|^{2k} \, d\,\mathrm{Vol} \\
&\quad - r^{2(j+k)+4} \int_{\mathbf{B}} |z_1|^{2j} |z_2|^{2k} \, d\,\mathrm{Vol}.
\end{aligned}$$

The Bergman kernel $K_{\Omega_r}(z,z)$ is $\sum |a'_{jk}|^2|z_1|^{2j}|z_2|^{2k}$. For later convenience, we compute the derivatives of K_{Ω_r}:

$$\frac{\partial^2 \log K_\Omega(z,z)}{\partial z_i \partial \overline{z}_j} = \frac{1}{K_\Omega^2}\left(K_\Omega(z,z)\frac{\partial^2 K_\Omega(z,z)}{\partial z_i \partial \overline{z}_j} - \frac{\partial K_\Omega(z,z)}{\partial z_i}\frac{\partial K_\Omega(z,z)}{\partial \overline{z}_j}\right).$$

We now prove Theorem 2 that the Bergman metric on the ring domain is *strictly smaller* than the Bergman metric on the unit ball in the tangential direction.

Proof of Theorem 2. Since the Bergman metric is invariant under biholomorphic mappings, we may assume $z_0 = (x,0)$, $x = r+\varepsilon$, $\varepsilon > 0$ and $T = (0,1)$. Let $c'_j = |a'_{j1}|^2$ and $c_j = |a_{j1}|^2$. Then, using equation (4.5), we have

$$(F_B^\Omega(z_0,T))^2 = \left.\frac{\partial^2 \log K_\Omega(z,z)}{\partial z_2 \partial \overline{z}_2}\right|_{z=z_0} = \frac{\sum_{j=0}^\infty c'_j x^{2j}}{\sum_{j=0}^\infty b'_j x^{2j}}.$$

Similarly, we have

$$(F_B^{\mathcal{B}}(z_0,T))^2 = \frac{\sum_{j=0}^\infty c_j x^{2j}}{\sum_{j=0}^\infty b_j x^{2j}}.$$

Again, from equation (4.4), $a'_{jk} = a_{jk}/\sqrt{1-r^{2(j+k)+4}}$ with $k=1$, we have

$$c'_j = \beta_j c_j \quad \text{where } \beta_j = \frac{1}{1-r^{2j+6}}.$$

The c_j's can be written explicitly:

$$c_j = \frac{(j+1)(j+2)(j+3)}{\pi^2}.$$

See [**6**, page 172]. Recall that the formulas for b_j, γ_j and β_j are given by equations (4.6) and (4.9) and that $b'_j = b_j\gamma_j$. We want to show the following:

$$F_\Omega(z_0,T) = \frac{\sum_{j=0}^\infty \beta_j c_j x^{2j}}{\sum_{j=0}^\infty \gamma_j b_j x^{2j}} < \frac{\sum_{j=0}^\infty c_j x^{2j}}{\sum_{j=0}^\infty b_j x^{2j}} = F_{\mathbf{B}}(z_0,T).$$

We cross multiply to have

$$\begin{aligned}\sum \beta_j c_j x^{2j} \sum b_k x^{2k} - \sum c_j x^{2j} \sum \gamma_k b_k x^{2k} \\ &= \sum_j (\beta_j - \gamma_j) b_j c_j x^{4j} \\ &\quad + \sum_{j>k\geq 0} (\beta_j c_j b_k + \beta_k c_k b_j - c_j \gamma_k b_k - c_k \gamma_j b_j) x^{2(j+k)} \\ &= \sum_j (\beta_j - \gamma_j) b_j c_j x^{4j} \\ &\quad + \sum_{j>k\geq 0} \{c_j b_k(\beta_j - \gamma_k) + c_k b_j (\beta_k - \gamma_j)\} x^{2(j+k)}.\end{aligned}$$

Note that the terms in the first summation are all negative since $\beta_j < \gamma_j$. Let

$$A_{jk} = c_j b_k (\beta_j - \gamma_k) + c_k b_j (\beta_k - \gamma_j).$$

Then, for every j, k,

$$\begin{aligned} A_{jk} &= \frac{(j+2)!(k+2)!}{\pi^4 j! k!}((j+3)(\gamma_j - \beta_k) + (k+3)(\gamma_k - \beta_j)) \\ &= \frac{(j+2)!(k+2)!}{\pi^4 j! k!}\Bigg((j+3)\bigg(\frac{r^{2j+6} - r^{2k+4}}{(1-r^{2j+6})(1-r^{2k+4})}\bigg) \\ &\qquad + (k+3)\bigg[\frac{r^{2k+6} - r^{2j+4}}{(1-r^{2k+6})(1-r^{2j+4})}\bigg]\Bigg) \\ &= \frac{(j+2)!(k+3)! r^{2k+4}}{\pi^4 j! k!}\Bigg[-\frac{j+3}{k+3}\bigg(\frac{1-r^{2(j-k)+2}}{(1-r^{2j+6})(1-r^{2k+4})}\bigg) \\ &\qquad + r^2 \bigg(\frac{1-r^{2(j-k-1)}}{(1-r^{2k+6})(1-r^{2j+4})}\bigg)\Bigg] \\ &= \frac{(j+2)!(k+3)! r^{2k+4}}{\pi^4 j! k!}[B_{jk}]. \end{aligned}$$

The term $B_{jk} < 0$ if

$$B := r^2 \frac{1-r^{2(j-k-1)}}{1-r^{2(j-k)+2}} \frac{(1-r^{2j+6})(1-r^{2k+4})}{(1-r^{2k+6})(1-r^{2j+4})} < \frac{j+3}{k+3}, \tag{4.7}$$

for all $j > k \geq 0$.

Since $j > k$, it is sufficient to show that $B < 1$.

If $j - k = 1$, then $B = 0$. Hence (4.7) is satisfied. If $j - k > 1$, then we have the following:

$$B = r^2 \frac{h_{2(j-k)-3}}{h_{2(j-k)+1}} \frac{h_{2j+5}}{h_{2k+5}} \frac{h_{2k+3}}{h_{2j+3}},$$

where $h_n = 1 + r + \cdots + r^n$. Since $h_n/h_m < 1$ if $n < m$, it is enough to show that

$$\frac{h_{2j+5}}{h_{2k+5}} \frac{h_{2k+3}}{h_{2j+3}} < 1.$$

Since $a/b < (a+\varepsilon)/(b+\varepsilon)$, if $0 < a < b$ and $\varepsilon > 0$, we get the following inequality:

$$\frac{h_{2k+3}}{h_{2j+3}} < \frac{h_{2k+3} + r^{2j+4} + r^{2j+5}}{h_{2j+5}} < \frac{h_{2k+5}}{h_{2j+5}},$$

where the second inequality follows from $j > k$ and $r \in (0, 1)$. Hence, (4.8) is proved.

Therefore, we have $B_{jk} < 0$ for all $j > k \geq 0$ and $F_K^{\Omega_r}(z_0, T) \lneq F_B^{\mathbf{B}}(z_0, T)$. □

Theorem 2 can be easily generalized to higher dimensions and also to a polydisc minus polydisc. We do not know at this point whether the same estimate holds on other rings of Reinhardt domains.

In the normal direction, we have the estimate for the ring domain in $\mathbf{C}^2$ when the inner boundary is close to zero. Our computation is restricted to the case of the ring domain in $\mathbf{C}^2$.

Proposition 3. *Let $\mathbf{B}^2$ be the unit ball in $\mathbf{C}^2$ and Ω the ring domain, $\Omega = \{z \in \mathbf{C}^2 : r < |z| < 1\}$, $r \in (0, 1)$. Let $z_0 \in \Omega$. For sufficiently small $r > 0$, we have*

$$F_B^{\Omega_r}(z_0, N) \lneq F_B^{\mathbf{B}^2}(z_0, N), \quad N = z_0/|z_0|,$$

for z_0 close to the inner boundary of Ω_r.

Proof. Because the Bergman metric is invariant under biholomorphic mappings we may assume $z_0 = (x, 0)$ with $x = r + \varepsilon$, $\varepsilon > 0$. Let

$b_j = |a_{j0}|^2$ and $b'_j = |a'_{j0}|^2$. The coefficients a_{jk}s are known explicitly, see [**6**, page 172]. From (4.4) with $k = 0$, we have

$$b_j = \frac{1}{\pi^2}(j+1)(j+2) \quad \text{and} \quad b'_j = \gamma_j b_j \text{ where } \gamma_j = \frac{1}{1 - r^{4+2j}}. \tag{4.9}$$

Let $z_0 = (x, 0)$ and $x \in (r, 1)$. Then we have

$$K_{\Omega_r}(z_0, z_0) = \sum_{j=0}^{\infty} b'_j |z_1^j|^2 = \sum_{j=0}^{\infty} b'_j x^{2j}. \tag{4.10}$$

For notational convenience, we use $\Omega = \Omega_r$. Using (4.5) and $N = (1, 0)$, one can calculate the Bergman metric on Ω as follows:

$$\begin{aligned}
\left(F_B^{\mathcal{W}}(z_0, N)\right)^2 &= \partial\overline{\partial} \log K_\Omega(z, z)_{z_0}(N, \overline{N}) \\
&= \frac{\partial^2 \log K_\Omega(z, z)}{\partial z_1 \partial \overline{z}_1}\bigg|_{z=z_0} \\
&= \frac{1}{K_\Omega(z_0)^2}\left(\sum_{j=0}^{\infty} b'_j x^{2j} \sum_{k=1}^{\infty} b'_k k^2 x^{2k-2} - \left(\sum_{k=1}^{\infty} b'_k k x^{2k-1}\right)^2\right).
\end{aligned}$$

We can simplify this as

$$\begin{aligned}
&F_B^{\Omega}(z_0, N)^2 K_\Omega(z_0)^2 \\
&\quad = \sum_{j=0}^{\infty} b'_j x^{2j} \sum_{k=1}^{\infty} b'_k k^2 x^{2k-2} - \left(\sum_{k=1}^{\infty} b'_k k x^{2k-1}\right)^2 \\
&\quad = (b'_0 + \sum_{j=1}^{\infty} b'_j x^{2j})\left(\sum_{k=1}^{\infty} b'_k k^2 x^{2k-2}\right) - \sum_{j,k \geq 1}^{\infty} b'_j b'_k jk x^{2(k+j)-2} \\
&\quad = b'_0 \sum_{k=1}^{\infty} b'_k k^2 x^{2k-2} + \sum_{j,k \geq 1}^{\infty} b'_j b'_k k^2 x^{2(k+j)-2} \\
&\qquad - \sum_{j,k \geq 1}^{\infty} b'_j b'_k jk x^{2(k+j)-2}.
\end{aligned}$$

Rewriting the second and third terms we have

$$\sum_{j,k \geq 1}^{\infty} b'_j b'_k k^2 x^{2(k+j)-2} = \sum_{j > k \geq 1}^{\infty} b'_j b'_k (k^2 + j^2) x^{2(k+j)-2}$$

$$+\sum_{k=1}^{\infty}(b'_k)^2k^2x^{4k-2}$$

$$\sum_{j,k\geq 1}^{\infty} b'_jb'_kjkx^{2(k+j)-2} = 2\sum_{j>k\geq 1}^{\infty} b'_jb'_kjkx^{2(k+j)-2} + \sum_{k=1}^{\infty}(b'_k)^2k^2x^{4k-2}.$$

Putting this all together, we have

$$\begin{aligned} F_B^{\Omega}(z_0,N)^2K_{\Omega}(z_0)^2 &= b'_0\sum_{k=1}^{\infty} b'_kk^2x^{2k-2} + \sum_{j>k\geq 1}^{\infty} b'_jb'_k(k-j)^2x^{2(k+j)-2} \\ &= \frac{1}{2}\sum_{j,k\geq 0}^{\infty} b'_jb'_k(k-j)^2x^{2(k+j)-2}. \end{aligned}$$

We use the formula (2.3) for Bergman metric on the unit ball provided in Section 2 to have $(F_B^{\mathbf{B}}(z_0,N))^2 = 3/(1-x^2)^2$. Since $K_{\Omega}(z_0,z_0) = \sum b'_jx^{2j}$ and $b'_j = \gamma_jb_j$, our claim of the proposition is

(4.11)
$$(F_B^{\Omega}(z_0,N))^2 = \frac{\sum_{j,k\geq 0}\gamma_k\gamma_jb_kb_jx^{2(j+k-1)}(k-j)^2}{2(\sum_{j=0}^{\infty}\gamma_jb_jx^{2j})^2} < \frac{3}{(1-x^2)^2}.$$

We rewrite (4.11) as

(4.12)
$$(1-2x^2+x^4)\sum_{j,k\geq 0}\gamma_k\gamma_jb_kb_jx^{2(j+k-1)}(k-j)^2 - 6\left(\sum_{j=0}^{\infty}\gamma_jb_jx^{2j}\right)^2 = \sum_{l\geq 0}C_lx^{2l} < 0.$$

We shall show that, for x close to r and sufficiently small $r>0$,

$$\begin{aligned} &C_0 < 0 \quad \text{and} \quad C_0 = O(r^4), \\ &C_1 < 0 \quad \text{and} \quad C_1x^2 = O(r^8), \\ &C_2x^4 = O(r^8). \end{aligned}$$

For $l \geq 3$, $C_l x^{2l} = O(r^{2l}) = O(r^6)$ for $l \geq 3$. Hence, the constant and x^2 terms dominate the higher order terms, and they are negative. Hence, the claim follows. The rest of the proof is explicit computation of coefficients. We recall the formulas for b_j and γ_j given in (4.9): $b_j = (j+1)(j+2)/\pi^2, b'_j = \gamma_j b_j$ and $\gamma_j = 1/(1-r^{4+2j})$. The constant term is:

$$\begin{aligned}
C_0 &= 2\gamma_0 b_0 \gamma_1 b_1 - 6\gamma_0^2 b_0^2 \\
&= 2\gamma_0 b_0 \frac{1}{\pi^2}\left(\frac{6}{1-r^6} - \frac{6}{1-r^4}\right) \\
&= -\frac{24}{\pi^4}\frac{r^4-r^6}{(1-r^6)(1-r^4)^2}.
\end{aligned}$$

The coefficient of the x^2 term is:

$$\begin{aligned}
C_1 &= 2\gamma_2\gamma_0 b_2 b_0 2^2 - 2(2\gamma_1\gamma_0 b_1 b_0) - 6\cdot 2\gamma_1\gamma_0 b_1 b_0 \\
&= 8\gamma_2 b_2 \gamma_0 b_0 - 16\gamma_1\gamma_0 b_1 b_0 \\
&= 8\gamma_0 b_0(\gamma_2 b_2 - 2\gamma_1 b_1) \\
&= -\frac{192}{\pi^4}\frac{r^6(1-r^2)}{(1-r^4)(1-r^8)(1-r^6)}.
\end{aligned}$$

The coefficient of the x^4 term is:

$$\begin{aligned}
C_2 = &\sum_{j,k\geq 0}^{j+k-1=2} \gamma_k\gamma_j b_k b_j x^{2(j+k-1)}(k-j)^2 - 2\sum_{j,k\geq 0}^{j+k=2}\gamma_k\gamma_j b_k b_j x^{2(j+k)}(k-j)^2 \\
&+ \sum_{j,k\geq 0}^{j+k+1=2} \gamma_k\gamma_j b_k b_j x^{2(j+k+1)}(k-j)^2 - 6\sum_{j,k\geq 0}^{j+k=2}\gamma_k\gamma_j b_k b_j x^{2(j+k)} \\
= &\frac{24r^4(3! - 2r^2 - 37r^4 - 66r^6 - 96r^8 - 69r^{10} - 34r^{12} + r^{14})}{(1+r^2)(1+r^2+r^4+r^6+r^8)(-1+r^6)(1+r^2+r^4)(-1+r^8)}. \qquad \square
\end{aligned}$$

Proof of Theorem 3. We may assume $z_0 = (x, 0)$, $x \in \mathbf{R}$, since the Bergman metric is invariant under biholomorphic mappings. Let $\xi = (\xi_1, \xi_2) = \xi_1 N + \xi_2 T$. Then

$$(F_B^\Omega(z_0,\xi))^2 = \sum_{j,k=1}^{2} \frac{\partial^2 \log K_\Omega(z,z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k.$$

Using equation (4.5) and the fact that $K_\Omega(z,z) = \sum a_{jk}|z_1|^{2j}|z_2|^{2k}$, we have

$$\frac{\partial^2 \log K_\Omega(z,z)}{\partial z_1 \partial \overline{z}_2}\bigg|_{(x,0)} = 0.$$

Hence,

$$(F_B^\Omega(z_0,\xi))^2 = |\xi_1|^2 (F_B^\Omega(z_0,N))^2 + |\xi_2|^2 (F_B^\Omega(z_0,T))^2.$$

This equality also holds for the unit ball. Therefore, by Theorem 2 and Proposition 3, we have that $F_B^\Omega(z_0,\xi) < F_B^{\mathbf{B}}(z_0,\xi)$, assuming r is small enough and z_0 is close to the inner boundary. □

REFERENCES

1. L. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, Third edition, McGraw-Hill Book Company, New York, 1979.

2. K. Diederich and J.E. Fornaess, *Comparison of the Bergman and the Kobayashi metric*, Math. Ann. **254** (1980), 257–262.

3. K. Diederich, J.E. Fornaess and G. Herbort, *Boundary behavior of the Bergman metric*, Proc. Symp. Pure Math. **41** (1984), 59–67.

4. J.E. Fornaess and L. Lee, *Kobayashi, Carathéodory, and Sibony metrics*, Complex Variable Elliptic Eq. **54** (2009), 293–301.

5. K.T. Hahn, *Inequality between the Bergman metric and Carathéodory differential metric*, Proc. Amer. Math. Soc. **68** (1978), 193–194.

6. M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*, Walter de Gruyter, Boston, 1993.

7. S. Kobayashi, *Hyperbolic complex spaces*, Springer, New York, 1998.

8. S.G. Krantz, *The boundary behavior of the Kobayashi metric*, Rocky Mountain J. Math. **22** (1992), 227–233.

Mathematics Department, The University of Michigan, East Hall, Ann Arbor, MI 48109
Email address: hskang@umich.edu

Mathematics Department, University of California Riverside, Riverside, CA 92521
Email address: linalee@math.ucr.edu

Mathematics Department, The University of Michigan, East Hall, Ann Arbor, MI 48109
Email address: zeagerc@umich.edu