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TWO COUNTEREXAMPLES IN ABSTRACT FACTORIZATION

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ABSTRACT. We give examples that provide negative answers to three questions about abstract factorization posed by Anderson and Frazier. We show that (1) an atomic domain need not be τ-atomic for τ divisive, (2) an atomic domain need not be a comaximal factorization domain (CFD) and (3) for τ divisive, a nonzero nonunit of a τ-UFD need not be a τ-product of τ-primes. Along the way, we generalize the theorem of Anderson and Frazier that a UFD is a τ-UFD for τ divisive (with a simplified proof), and we demonstrate a method for constructing domains with no pseudo-irreducible elements.

1. Introduction. Let D be an integral domain. We denote the units of D by $U(D)$ and then set $D^* = D \setminus (0)$ and $D^\# = D^* \setminus U(D)$. We call an element of $D^\#$ *irreducible* or an *atom* if it cannot be written as a product of two elements of $D^\#$. A *reducible* element of D is an element of $D^\#$ that is not irreducible. We use $\mathrm{Red}(D)$ and $\mathrm{Irr}(D)$ to denote the reducible and irreducible elements of D, respectively. We call D *atomic* if every element of $D^\#$ has a factorization into atoms; such a factorization is called an *atomic factorization*. It is possible to define weaker kinds of irreducible if we only admit certain factorizations of elements, and these give rise to different abstractions of atomicity. The τ-factorizations introduced in **[3]** give a good framework for studying generalized factorization and give us one way of defining weaker notions of irreducible. Here τ is a symmetric relation on $D^\#$ and a τ*-factorization* of $a \in D^\#$ is a factorization $a = \lambda a_1 \cdots a_n$, where $n \geq 1$, $\lambda \in U(D)$, each $a_i \in D^\#$, and $a_i \tau a_j$ for $i \neq j$. The second section of the paper reviews the definitions and some of the basic theory laid out in **[3]**, **[4]**, **[9]** and **[10]**. The relationship between the different kinds of atomic domains will be the topic of the third section

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of this paper. Here $a \in D^\#$ is a τ*-atom* if each of its τ-factorizations is trivial ($a = \lambda(\lambda^{-1}a)$) and D is τ*-atomic* if every element of $D^\#$ has a τ-factorization into τ-atoms. In the case where $a\tau b \Leftrightarrow (a,b) = D$, τ-atomic domains called *comaximal factorization domains* (CFD's) were studied in [**9**]. We show that an atomic domain need not be a CFD, thus answering in the negative a question raised in [**3**]. Along the way, we demonstrate a method for constructing the comaximal factorization analog of the antimatter domains studied in [**6**].

It is well known that an integral domain D is a unique factorization domain (UFD) if and only if D is atomic and each atom is prime, or equivalently, every element of $D^\#$ is a product of prime elements. The notion of a UFD is easily extended to τ-factorization. A domain D is defined to be a τ*-UFD* if (1) D is τ-atomic and (2) if $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-factorizations where each a_i, b_j is a τ-atom, then $n = m$ and after reordering, if necessary, a_i and b_i are associates, denoted $a_i \sim b_i$. In the fourth section we investigate the relationship between D being a τ-UFD and the elements of $D^\#$ having a τ-factorization into various forms of "τ-primes." In particular, we give an example of a τ-UFD where τ is both multiplicative and divisive (see Section 2) in which a τ-atom need not be τ-prime. This answers a question raised in [**3**]. Furthermore, this investigation will lead us to a simplified proof of (a generalization of) [**3,** Theorem 2.11], which shows that a UFD is a τ-UFD for τ divisive.

2. Review of τ-factorization. In this section we review some of the definitions and basic theory concerning τ-factorization introduced in [**3**]. Unless noted otherwise, all of the definitions from τ-factorization theory that we will use in this paper come from Section 2 of that paper.

Let D be an integral domain and τ be a symmetric relation on $D^\#$. A (*reduced*) *factorization* of $a \in D^\#$ is a product $a = \lambda a_1 \cdots a_n$, where $\lambda \in U(D)$ ($\lambda = 1$) and each $a_i \in D^\#$; we call n the *length* of the factorization. A factorization of length 1 is called *trivial*. A τ-*factorization* is a factorization $a = \lambda a_1 \cdots a_n$ where $a_i \tau a_j$ for $i \neq j$; we say that each a_i τ*-divides* a, written $a_i \mid_\tau a$. We sometimes call a τ-factorization $\lambda a_1 \cdots a_n$ a τ*-product* of the a_i's, each of which we call a τ*-factor*. An element of $D^\#$ is τ*-irreducible* or a τ*-atom* if it has no nontrivial τ-factorizations. A τ*-atomic factorization* is a τ-factorization into τ-atoms. We call D τ*-atomic* if every element of $D^\#$

has a τ-atomic factorization. We say D satisfies the *τ-ascending chain condition on principal ideals* (τ-ACCP) if for each sequence $\{a_n\}_{n=1}^{\infty}$ in $D^{\#}$ with each $a_{n+1} \mid_{\tau} a_n$, there is an $N \geq 1$ with $a_N \sim a_{N+1} \sim \cdots$. If D is τ-atomic, we say that D is a

(1) *τ-bounded factorization domain* (τ-BFD) if every element of $D^{\#}$ has an upper bound on the lengths of its τ-atomic factorizations;
(2) *τ-finite factorization domain* (τ-FFD) if every element of $D^{\#}$ has only finitely many τ-atomic factorizations up to associates and order;
(3) *τ-half factorial domain* (τ-HFD) if every element of $D^{\#}$ has all of its τ-atomic factorizations of the same length; and
(4) *τ-unique factorization domain* (τ-UFD) if the τ-atomic factorizations of an element of $D^{\#}$ are unique up to order and associates.

Several properties that the relation τ can possess have been defined and studied. The following will be useful for this paper. We call τ

(1) *associate-preserving* (respectively, *divisive*) if whenever $a, a', b, b' \in D^{\#}$, $a' \sim a$ (respectively, $a' \mid a$), and $b' \sim b$ (respectively, $b' \mid b$), we have $a\tau b \Rightarrow a'\tau b'$;
(2) *multiplicative* if ($a\tau b$ and $a\tau c$) $\Rightarrow a\tau bc$;
(3) *refinable* [**4**] if, whenever $\lambda a_1 \cdots a_n$ and $a_i = b_1 \cdots b_m$ are τ-factorizations, then so is $\lambda a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n$; and
(4) *combinable* [**4**] if, whenever $\lambda a_1 \cdots a_n$ is a τ-factorization, then so is each factorization $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1}) a_{i+2} \cdots a_n$.

Note that, if τ is combinable, then any nonzero nonunit with a nontrivial τ-factorization has a τ-factorization of length 2. In [**3,** Proposition 2.2] it is shown that a divisive relation is refinable and associate-preserving, and that a multiplicative relation is combinable. Another way to see the latter fact is to observe that τ is combinable if and only if ($a\tau b$, $a\tau c$, and $b\tau c$) $\Rightarrow a\tau bc$. However, a refinable and associate-preserving relation need not be divisive, and a combinable relation need not be multiplicative. For example, if $a, b \in D^{\#}$ are reducible non-associates and τ is the symmetric relation on $D^{\#}$ determined by $(\lambda a)\tau(\mu b)$ for $\lambda, \mu \in U(D)$, then τ is refinable, associate-preserving, and combinable, but neither divisive nor multiplicative. While one may be hard-pressed to come up with a natural combinable relation that is not multiplicative, there are some interesting relations that are refinable and associate-preserving but not divisive. For example, the symmetric relation $\tau_{(2)}$ on $\mathbf{Z}^\#$ given by $a\tau_{(2)}b \Leftrightarrow a \equiv b \mod 2$ is refinable, associate-preserving, and multiplicative, but is not divisive. (See [**3,** Section 5] for details.)

We pause to review several of the more interesting examples of τ-factorization that have been studied.

Example 2.1 ([**3,** Example 2.1(1)-(2)])**.** We use τ_D to denote the symmetric relation $D^\# \times D^\#$. The relation τ_D is clearly multiplicative and divisive, and the τ_D-factorizations are simply the factorizations. The notions of τ_D-atomic, τ_D-ACCP, τ_D-BFD, τ_D-FFD, τ_D-HFD, and τ_D-UFD reduce to the classical factorization concepts of atomic, ACCP, BFD, FFD, HFD, and UFD. A good survey of these topics for standard factorizations can be found in [**1**]. At the other extreme, if we take $\tau_\emptyset = \emptyset$, every $\tau_\emptyset$-factorization is trivial, every element of $D^\#$ is a $\tau_\emptyset$-atom, and $a \mid_{\tau_\emptyset} b \Leftrightarrow a \sim b$.

Example 2.2 ([**3,** Example 2.1(5)])**.** We use τ_d to denote the symmetric relation on $D^\#$ given by $a\tau_d b \Leftrightarrow (a,b) = D$. One familiar with star operations (see [**8,** Section 32]) can generalize this as follows. If $\star$ is a star operation on D, we define a symmetric relation $\tau_\star$ on $D^\#$ by $a\tau_\star b \Leftrightarrow (a,b)^\star = D$. It is easy to see that $\tau_\star$ is multiplicative and divisive. (In particular, the τ_d relation is multiplicative and divisive.) The *comaximal factorizations* studied in [**9**] are simply τ_d-factorizations. (Technically, they study the reduced τ_d-factorizations, but the fact that τ_d is associate-preserving means that it does not make any difference for our purposes whether we study τ_d-factorizations or reduced τ_d-factorizations, and we will follow the convention of [**3**] of simply considering a comaximal factorization to be a τ_d-factorization.) In the terminology of [**9**], a *pseudo-irreducible* is a τ_d-atom, a *complete comaximal factorization* is a τ_d-atomic factorization, a *comaximal factorization domain* (CFD) is a τ_d-atomic domain, a *bounded* CFD is a τ_d-BFD, and a *unique comaximal factorization domain* (UCFD) is a τ_d-UFD. In [**9,** Theorem 1.7], UCFD's were characterized as the CFD's in which every 2-generated invertible ideal is principal. The paper [**3**] generalized the comaximal factorizations to the $\star$*-factorizations*, which are simply $\tau_\star$-factorizations. (And every $\star$-factorization phrase is similarly abbreviated: "$\star$-atom" for "$\tau_\star$-atom," "$\star$-UFD" for "$\tau_\star$-UFD," etc.) Some results about $\star$-factorizations generalizing the earlier ones about comaximal factorizations were proved. For example, if every 2-generated $\star$-invertible $\star$-ideal is principal, then D is a $\star$-UFD. Whether the converse is true remains an open question.

Example 2.3 ([**3,** Example 2.1(6)])**.** Two elements a and b of an integral domain D are said to be *relatively prime* if they have no common nonunit factor, in which case we write $[a, b] = 1$. We use $\tau_{[\,]}$ to denote the symmetric relation on $D^{\#}$ given by $a\tau_{[\,]}b \Leftrightarrow [a, b] = 1$. It is easy to see that $\tau_{[\,]}$ is divisive. However, in general it is not multiplicative nor even combinable. A nice characterization of when $\tau_{[\,]}$ is multiplicative exists [**5,** Theorem 3.1]: the relation $\tau_{[\,]}$ is multiplicative if and only if D satisfies the conclusion of Gauss's lemma, i.e., when the product of any two primitive polynomials in $D[x]$ is primitive. For an example of when $\tau_{[\,]}$ is not combinable, we consider the following one from [**3**]. Let $K \subsetneq L$ be fields and $D = K + xL[[x]]$. The paper [**3**] shows that for each $u \in L \setminus K$ the element $u(u + 1)x^3$ has the nontrivial $\tau_{[\,]}$-factorization $x(ux)((u + 1)x)$, but it has no $\tau_{[\,]}$-factorization of length 2. Thus $\tau_{[\,]}$ is not combinable in this case.

Let D be an integral domain. The relations on $D^{\#}$ are partially ordered by inclusion, and we correspondingly write $\tau_1 \leq \tau_2$ in place of $\tau_1 \subseteq \tau_2$. For example, for τ any symmetric relation we have $\tau_{\emptyset} \leq \tau \leq \tau_D$, and for any star operation $\star$ we have $\tau_d \leq \tau_\star \leq \tau_v \leq \tau_{[\,]}$, where v is the v-operation. Let $\tau_1 \leq \tau_2$ be symmetric relations on $D^{\#}$. The following diagram of implications holds, where a dotted line indicates that the implication holds if the relations involved are refinable and associate-preserving, and two parallel dotted lines indicate that it holds if τ_1 is divisive and τ_2 is refinable and associate-preserving. In particular, all of the implications in the diagram below hold when the relations involved are divisive.

$$
\begin{tikzcd}
& \tau_2\text{-HFD} \arrow[dr] & & & \\
\tau_2\text{-UFD} \arrow[ur] \arrow[r] \arrow[d, Rightarrow, dotted] & \tau_2\text{-FFD} \arrow[r] \arrow[d, dotted] & \tau_2\text{-BFD} \arrow[r, dotted] \arrow[d, dotted] & \tau_2\text{-ACCP} \arrow[r, dotted] \arrow[d] & \tau_2\text{-atomic} \\
\tau_1\text{-UFD} \arrow[r] & \tau_1\text{-FFD} \arrow[r] & \tau_1\text{-BFD} \arrow[r, dotted] & \tau_1\text{-ACCP} \arrow[r, dotted] & \tau_1\text{-atomic}
\end{tikzcd}
$$

The fact that the above diagram holds in the special case with $\tau_2 = \tau_D$ and τ_1 divisive was demonstrated in [**3,** Section 2], and we observe that, with the notable exception of the UFD case, trivial changes to the proofs given there give the most general version stated above. The fact that a UFD is a τ-UFD for τ divisive is shown in [**3,** Theorem 2.11], and its proof was later adapted in [**10,** Theorem 4.12] to prove the case for any divisive relations $\tau_1 \leq \tau_2$. Both proofs are somewhat involved and make essential use of the fact that τ_2 is divisive and not merely refinable and associate-preserving. We will present a simplified proof that shows the strongest result, but it makes use of the concept of "τ-primes," so we will defer this until Section 4.

Examples given in [**3**] and [**1**] show that no further nontrivial implications can be added to the above diagram in the case where $\tau_2 = \tau_D$ and τ_1 is divisive. The one exception is the question posed by the authors of [**3**] of whether atomic implies τ-atomic for τ divisive. Two related additional questions that they posed are if an atomic domain is $\star$-atomic, and if an atomic domain is a CFD. In the next section we will answer all three questions in the negative by constructing an atomic domain that is not a CFD. So, in the language of τ-factorization, an atomic domain is not necessarily τ-atomic, even if τ is both multiplicative and divisive.

3. An atomic domain that is not a CFD. In this section we construct an atomic domain that is not a CFD. We can start our construction with any domain that is not a CFD. While any non-CFD will do, we might as well spend a moment constructing a family of non-CFD's with some nice properties. In particular, we will deviate from CFD's as far as possible, constructing domains that have no pseudo-irreducibles; we call such a domain an *anti*-CFD. (Clearly fields are the only domains that can simultaneously be a CFD and an anti-CFD.) The analogous standard factorization concept is an *antimatter domain*: a domain with no atoms. These were first studied in [**6**]. Among many other interesting results, that paper proved [**6,** Theorem 2.13] that every domain is a subring of an antimatter domain that is not a field. We briefly note another such embedding in which the antimatter domain is a one-dimensional valuation domain. Let D be a domain with quotient field K and M be the maximal ideal of the monoid domain $K[x; \mathbf{Q}^+]$ consisting of the elements with zero constant term. Because the elements of $K[x; \mathbf{Q}^+]_M$ are (up to associates) the monomials in $K[x; \mathbf{Q}^+]$, we see that $K[x;\mathbf{Q}^+]_M$ is a one-dimensional (see [**7,** Theorems 17.1, 21.4]) antimatter valuation domain. (See the proof of [**6,** Theorem 2.13] for a completely different embedding method.) Of course, we cannot hope to have an exactly analogous result for anti-CFD's since valuation domains are quasilocal and all of their nonzero nonunits are pseudo-irreducible, but we will prove the next best thing: every domain is a subring of a one-dimensional Bezout anti-CFD.

Our construction will make use of some of the monoid domain techniques employed in [**2**]. In analogy with the pure monoids introduced in [**2**], for each $k \geq 0$ we call an additive monoid S *k-pure* if (1) it is order isomorphic to a submonoid of the additive monoid $\mathbf{Q}^+ = \{r \in \mathbf{Q} \mid r \geq 0\}$, (2) it is *locally cyclic*, meaning every finitely generated submonoid is cyclic, and (3) for each $s \in S$ there is a natural number $n \geq 2$ (depending on s) with $s/n \in S$ and n not a power of k. A *pure monoid* is a 0-pure monoid. In [**2**] it is shown that if S is a pure monoid, then $s_2 - s_1 \in S$ for every $s_1, s_2 \in S$ with $s_1 < s_2$; in fact, the proof applies to any locally cyclic submonoid of $\mathbf{Q}^+$. Two examples of pure monoids given in [**2**] are $(\mathbf{Q}^+, +)$ and $(\mathbf{Z}_T^+, +)$, where $T \neq \{1\}$ is a multiplicative subset of $\mathbf{Z}^+ = \{n \in \mathbf{Z} \mid n \geq 0\}$. One last fact that we will use several times in this section is that for a domain D, the units of the Laurent polynomial ring $D[x, x^{-1}]$ are the elements of the form ux^m, where $u \in U(D)$ and $m \in \mathbf{Z}$.

Theorem 3.1. *Let K be a field of characteristic $p \geq 0$, and let S be a locally cyclic submonoid of $\mathbf{Q}^+$. Then $K[x;S]$ is Bezout and every non-constant monomial is pseudo-irreducible. If K is algebraically closed, then the following are equivalent.*

(1) *S is p-pure.*
(2) *No non-monomial is pseudo-irreducible.*

Proof. The fact that $K[X;S]$ is Bezout follows from [**7,** Theorem 13.6].

Because no two non-constant monomials are relatively prime, in order to show that every non-constant monomial is pseudo-irreducible it will suffice to show that the non-constant monomials form a saturated subset of $K[x;S]^\#$. Take any $f, g \in K[x;S]^\#$ with fg a non-constant monomial. By the locally cyclic property we get that fg is a non-constant monomial in $K[x^s]$ for some $s \in S$ with $f, g \in K[x^s]$. Therefore $fg \in U(K[x^s, 1/x^s])$, and hence $f, g \in U(K[x^s, 1/x^s])$. It is well-known that this implies that f and g have the form $a(x^s)^m = ax^{ms}$ for some $m \in \mathbf{Z}$ and $a \in K$, and this leads us to the conclusion that f and g are non-constant monomials.

Assume K is algebraically closed.

(1) $\Rightarrow$ (2). Assume that S is p-pure and let $f \in K[x; S]^{\#}$ be any non-monomial. Write $f = a_1x^{s_1} + \cdots + a_nx^{s_n}$, where $n \geq 2$, $a_1, \ldots, a_n \in K^*$, and $0 \leq s_1 < \cdots < s_n$. Then $f = x^{s_1}(a_1 + a_2x^{s_2-s_1} + \cdots + a_nx^{s_n-s_1})$ is a factorization in $K[x; S]$. If $s_1 > 0$, then x^{s_1} and $a_1 + \cdots + a_nx^{s_n-s_1}$ are relatively prime (hence comaximal since $K[x; S]$ is Bezout) by the fact that every non-unit divisor of the former is a non-constant monomial and the latter has no non-constant monomial divisors. So let us assume $s_1 = 0$. It will suffice to consider the case $a_n = 1$. By the fact that S is locally cyclic, $f = a_1 + \cdots + a_{n-1}x^{s_{n-1}} + x^{s_n}$ is a polynomial in x^q for some $q \in S$. Because K is algebraically closed, this polynomial splits into a product of monic linear polynomials in x^q, say $f = (x^q + b_1)^{k_1} \cdots (x^q + b_m)^{k_m}$, where $m, k_1, \ldots, k_m \geq 1$ and $b_1, \ldots, b_m \in K^*$ are distinct. If $m \geq 2$, then this is a nontrivial comaximal factorization, so we may assume $m = 1$. By the fact that S is p-pure there are $j \geq 0$ and $N \geq 2$ with $p \nmid N$ and $q/(p^jN) \in S$. (Here we regard 0^0 as 1). Then $x^{q/p^j} + b_1^{1/p^j} = (x^{q/(p^jN)})^N + b_1^{1/p^j}$ splits into N distinct monic linear polynomials in $x^{q/(p^jN)}$ by the fact that K is algebraically closed and $p \nmid N$. The fact that $(x^q + b_1)^k = (x^{q/p^j} + b_1^{1/p^j})^{p^jk}$ then reduces this case to the previously covered $m \geq 2$ case.

(2) $\Rightarrow$ (1). Assume that every non-monomial has a nontrivial comaximal factorization and take any nonzero $s \in S$. By the locally cyclic property, the non-monomial $x^s + 1$ has a nontrivial comaximal factorization in $K[x^t]$ for some $t \in S$. By the fact that s is the sum of the leading exponents of the factors in this factorization, we get that $t = s/N$ for some $N \geq 2$. If $N = p^k$ for some $k \geq 1$, then $p > 0$ and the unique atomic factorization of $x^s + 1$ in the UFD $K[x^t]$ is $x^s + 1 = (x^t + 1)^{p^k}$, so the factors in the aforementioned nontrivial comaximal factorization have a common divisor of $x^t + 1$, contradicting comaximality. Therefore, N is not a power of p, as desired. $\square$

The last part of the above theorem is false if we drop the algebraically closed hypothesis. For example, the paper **[2]** notes that $\mathbf{Q}[x; \mathbf{Q}^+]$ has an irreducible non-monomial $x - 2$.

If K is an algebraically closed field and S is a pure monoid, then [**2,** Theorem 1] shows that $K[x;S]$ is an antimatter domain. However, Theorem 3.1 shows that if K has positive characteristic p, then we need S to be p-pure and not merely pure if we want to have this monoid domain as close to an anti-CFD as possible. It is interesting to note that $K[x;S]$ can even end up being a UCFD if we only assume S is pure.

Example 3.2. Let K be an algebraically closed field of positive characteristic p and let S be the pure monoid $\mathbf{Z}_T^+$, where $T = \{p^m \mid m \geq 0\}$. We claim that $K[x;S]$ is a UCFD. Because [**9,** Corollary 1.9] asserts that a Bezout CFD is a UCFD, it will suffice to show that $K[x;S]$ is a CFD. The proof of "(1) $\Rightarrow$ (2)" in Theorem 3.1 shows that, up to associates, we can write every nonzero nonunit of $K[x;S]$ as a comaximal product whose factors are each either a non-constant monomial or of the form $(x^{1/p^n} + b)^k$ for some $n \geq 0$, $k \geq 1$, and $b \in K^*$. So it will suffice to show that elements of these two forms are pseudo-irreducible. Both of these cases are covered by Theorem 3.1, the former explicitly, and the latter by adapting the proof of "(2) $\Rightarrow$ (1)."

From Theorem 3.1 we obtain the following embedding result in the spirit of the ones about antimatter domains mentioned above.

Corollary 3.3. *Every domain is a subring of a one-dimensional Bezout anti-*CFD.

Proof. Let D be a domain, K be the algebraic closure of its quotient field, $R = K[x;\mathbf{Q}^+]$, and T be the saturated multiplicatively closed subset of $K[x;\mathbf{Q}^+]$ consisting of the nonzero monomials. By Theorem 3.1 and [**7,** Theorems 17.1, 21.4], we see that R is a one-dimensional Bezout domain and that its only pseudo-irreducibles are the non-constant monomials. It follows that R_T is a one-dimensional Bezout domain, and it only remains to show that it is an anti-CFD. Note that any non-monomial in R with nonzero constant term has a nontrivial comaximal factorization into elements of that same form. By the fact that the nonzero nonunits of R_T are (up to associates) the non-monomials in R with nonzero constant term, we see that R_T has no pseudo-irreducible elements. $\square$

We next review a construction used in [**11**]. Let D be an integral domain with quotient field K. For $S \subseteq D^*$, define $L(D;S) =$

$D[\{x_s, sx_s^{-1} \mid s \in S\}]$, where $\{x_s\}_{s\in S}$ is a family of algebraically independent indeterminates. We will abbreviate $L(D;\{s_1,\dots,s_n\})$ by $L(D;s_1,\dots,s_n)$. The domain $L(K;S)$ is simply the Laurent polynomial ring over K in the variables $\{x_s\}_{s\in S}$. As a consequence of algebraic independence, each element of $L(K;S)$ has a unique Laurent polynomial representation. Writing $x = x_s$, the domain $L(D;s)$ is the set of Laurent polynomials $\sum_{k=m}^{n} c_k x^k$, where $m \leq n$ are integers, $c_k \in D$ for $k \geq 0$, and $c_k \in (s^{-k})$ for $k < 0$. The following facts are shown to hold for $\emptyset \subsetneq S \subseteq D^{\#}$ in [**11**, Lemma 3.2], where $L = L(D;S)$.

(1) $L \cap K = D$.
(2) $U(L) = U(D)$, so $D^{\#} \subseteq L^{\#}$.
(3) $\mathrm{Red}(D) \subseteq \mathrm{Red}(L)$.
(4) If $S \subseteq \mathrm{Red}(D)$, then $\mathrm{Irr}(D) \subseteq \mathrm{Irr}(L)$.
(5) Each $s \in S$ can be written as a product $s = x_s(s/x_s)$ of two nonassociate atoms in L.

We add the following additional fact related to comaximal factorizations.

Lemma 3.4. *Let D be an integral domain and $\emptyset \subsetneq S \subseteq D^{\#}$. For each $a \in D^{\#}$, the comaximal factorizations of a in D coincide with the comaximal factorizations of a in $L(D;S)$. So D is a* CFD (*respectively, bounded* CFD, d-FFD, d-HFD, UCFD, *anti*-CFD) *if $L(D;S)$ is.*

Proof. Take any $a \in D^{\#}$. It follows from fact (2) above that any comaximal factorization of a in D is a comaximal factorization of a in $L(D;S)$.

Let $a = fg$ be any length 2 reduced comaximal factorization of a in $L(D;S)$. Then there is a nonempty finite subset $\{s_1,\dots,s_n\}$ of S with $a = fg$ a comaximal factorization in $L(D;s_1,\dots,s_n)$. Let us first consider the case $n = 1$. Write $s = s_1$ and $x = x_s$. The equation $a = fg$ implies that $f, g \in U(K[x,x^{-1}])$, so without loss of generality we have $f = cx^m$ and $g = dx^{-m}$ for some $m \geq 0$ and $c \in D$ and $d \in (s^m)$ with $cd = a$. The fact that f and g are comaximal in $L(D;s)$ implies that there are $h_1, h_2 \in L(D;s)$ with $fh_1 + gh_2 = 1$. Write $h_1 = a_k x^k + \cdots + a_{-k}x^{-k}$ and $h_2 = b_k x^k + \cdots + b_{-k}x^{-k}$ for some $k \geq m$, where $a_j, b_j \in D$ for $j \geq 0$ and $a_j, b_j \in (s^{-j})$ for $j < 0$. The coefficient of x^j in $fh_1 + gh_2$ is $ca_{j-m} + db_{j+m}$, where we set $a_i = b_i = 0$ for $i \notin \{-k,\dots,k\}$. We have $a_{-m}, d \in (s^m)$, so $1 = ca_{-m} + db_m \in (s^m)$, simultaneously showing that c and d are comaximal in D and that $m = 0$. Thus, $f = c$ and $g = d$, so $a = fg$ is a comaximal factorization in D. Now, if $n \geq 2$, we have $a = fg$ a comaximal factorization in $L(D; s_1, \ldots, s_{n-1})$ by the case $n = 1$, and hence $a = fg$ is a comaximal factorization in D by induction.

Now let $a = \lambda f_1 \cdots f_n$ be any comaximal factorization of a in $L(D; S)$. By fact (2) we get $\lambda \in U(D)$, so $f_1 \cdots f_n$ is a comaximal factorization of $\lambda^{-1}a \in D^{\#}$ in $L(D; S)$. If $n = 1$, then $f_1 = \lambda^{-1}a \in D^{\#}$, so λf_1 is a comaximal factorization of a in D. So, let us assume $n \geq 2$. Then, $f_1(f_2 \cdots f_n)$ is a length 2 reduced comaximal factorization of $\lambda^{-1}a$ in $L(D; S)$, so by the previous paragraph we have $f_1(f_2 \cdots f_n)$ a comaximal factorization in D, and by induction $f_2 \cdots f_n$ is a comaximal factorization in D, so $\lambda f_1 \cdots f_n$ is a comaximal factorization in D by the refinability of τ_d.

It follows from the above work that the complete comaximal factorizations of a given $a \in D^{\#}$ in D coincide with its complete comaximal factorizations in $L(D; S)$. The last statement of the lemma follows immediately from this observation. $\square$

In the above lemma, we can also note that D satisfies d-ACCP if $L(D; S)$ does. This same statement would be true with any domain T with $D^{\#} \subseteq T^{\#}$ in place of $L(D; S)$.

We continue with the construction from [**11**]. Let D be an integral domain. Inductively define $A^0(D) = D$ and $A^n(D) = L(A^{n-1}(D); \mathrm{Red}(A^{n-1}(D)))$ for $n \geq 1$. Define $A^\infty(D) = \bigcup_{n=0}^\infty A^n(D)$. The following facts were proved in [**11**, Theorem 3.3], where $A^\infty = A^\infty(D)$.

(1) $A^\infty \cap K = D$.
(2) $U(A^\infty) = U(D)$, so $D^{\#} \subseteq (A^\infty)^{\#}$.
(3) $\mathrm{Red}(D) \subseteq \mathrm{Red}(A^\infty)$.
(4) $\mathrm{Irr}(D) \subseteq \mathrm{Irr}(A^\infty)$.
(5) Any reducible element in A^∞ is a product of two non-associate atoms in A^∞.

(We have added "non-associate" to (5); this additional fact can be readily deduced from fact (2) and a careful reading of the proof of [**11**, Theorem 3.3 (5)].)

We augment these with the following.

Theorem 3.5. *Let D be an integral domain. For each $a \in D^\#$, the comaximal factorizations of a in D coincide with the comaximal factorizations of a in $A^\infty(D)$. So D is a* CFD (*respectively, bounded* CFD, d-FFD, d-HFD, UCFD, *anti*-CFD) *if $A^\infty(D)$ is.*

Proof. Take any $a \in D^\#$. It follows from fact (2) that any comaximal factorization of a in D is a comaximal factorization in A^∞. Now let $a = \lambda f_1 \cdots f_n$ be any comaximal factorization of a in A^∞. Then there is a $k \geq 1$ with $\lambda f_1 \cdots f_n$ a comaximal factorization in $A^k(D) = L(A^{k-1}(D); \mathrm{Red}(A^{k-1}(D)))$, and hence a comaximal factorization in $A^{k-1}(D)$ by Lemma 3.4, and hence a comaximal factorization in D by induction.

The proof of the last statement is similar to the proof of the last part of Lemma 3.4. $\square$

We are now ready to give the promised example. Coincidentally, this also will provide an example of a $\tau_{[\,]}$-atomic domain that is not a CFD.

Example 3.6. (Every domain is a subring of a domain that is atomic and $\tau_{[\,]}$-atomic but not a CFD.) Given a domain, Corollary 3.3 extends it to a domain R that is not a CFD, and we further extend it to $A^\infty(R)$, which is both atomic and $\tau_{[\,]}$-atomic by fact (5) from the above list but is not a CFD by Theorem 3.5.

4. Generalizations of prime elements. Let D be an integral domain and τ a symmetric relation on $D^\#$. In [**3,** Section 2] the following two generalizations of primeness were defined: a nonzero nonunit is *τ-prime* (respectively, *$|_\tau$-prime*) if whenever it divides (respectively, τ-divides) a τ-factorization, it divides (respectively, τ-divides) one of the τ-factors. We add a third generalization, defining a nonzero nonunit to be *half $|_\tau$-prime* if whenever it τ-divides a τ-factorization, it divides one of the τ-factors. (These three are special cases of the τ_1-τ_2-τ_3-primes defined in [**3,** Section 2]. For three symmetric relations τ_1, τ_2, τ_3 on $D^\#$, we say that $a \in D^\#$ is *τ_1-τ_2-τ_3-prime* if, whenever $a \,|_{\tau_2}\, \lambda a_1 \cdots a_n$, a τ_1-factorization, $a \,|_{\tau_3}\, a_i$ for some i.) For example, the *pseudo-primes* studied in [**9**] are τ_d-primes. In [**3,** Proposition 2.4] it was shown that, if τ is multiplicative and divisive, then a τ-prime is $|_\tau$-prime. We note that their proof actually demonstrates the following stronger result. If

τ is combinable and divisive, then the half $|_\tau$-primes and $|_\tau$-primes coincide. In summary, the following diagram of implications holds, where a dotted line indicates that the implication holds if τ is both combinable and divisive.

$$\begin{tikzcd}
\text{prime} \arrow[r] & \tau\text{-prime} \arrow[r] & \text{half } |_\tau\text{-prime} \arrow[r] \arrow[d, bend left, dotted] & \tau\text{-irreducible} \\
 & & |_\tau\text{-prime} \arrow[u] &
\end{tikzcd}$$

There are no further nontrivial implications, even in the case where τ is both multiplicative and divisive. This we briefly demonstrate with the following examples:

(1) it is well known that an irreducible need not be prime;
(2) the previously defined relation $\tau_{(2)}$ on $\mathbf{Z}^\#$ is refinable, associate-preserving, and multiplicative, but the $|_{\tau_{(2)}}$-primes are precisely the odd primes (see [**3,** Section 5]);
(3) the relation $\tau_\emptyset$ is multiplicative and divisive and every nonzero nonunit is $\tau_\emptyset$ prime;
(4) Example 4.3 below exhibits a multiplicative and divisive relation τ and a $|_\tau$-prime that is not τ-prime.

In the case of standard factorization, there is a strong link between unique factorization and prime elements. Analogously, in τ-factorization, there is a strong link between unique τ-factorization and the various kinds of "τ-primes." For example, it follows immediately from [**10,** Lemma 4.9] that if $\tau_1 \leq \tau_2$ are divisive relations and every nonzero nonunit is a τ_2-product of τ_2-primes, then D is a τ_1-UFD. In particular, this provides a proof of the fact that a UFD is a τ-UFD for τ-divisive that considerably simplifies the one given in [**3,** Theorem 2.11]. However, in order to know whether the proof applies to the more general fact that a τ_2-UFD is a τ_1-UFD for divisive relations $\tau_1 \leq \tau_2$, we would need to know whether in this setup every nonzero nonunit is necessarily a τ_2-product of τ_2-primes. This motivates us to consider the following statements and the implications between them.

(1) D is a τ-UFD.
(2) D is τ-atomic and every τ-atom is $|_\tau$-prime.
(3) D is τ-atomic and every τ-atom is half $|_\tau$-prime.
(4) D is τ-atomic and every τ-atom is τ-prime.

One could study the implications between these statements with various degrees of hypotheses on τ, but by far the most natural and interesting hypothesis to impose is that τ is refinable and associate-preserving. This case is fully solved by the following theorem.

Theorem 4.1. *If τ is refinable and associate-preserving, then* (4) $\Rightarrow$ (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Proof. The fact that (2) and (4) each imply (3) is obvious, and the proof of (1) $\Leftrightarrow$ (2) can be obtained by small changes to the proof of [**3,** Theorem 2.7], so all that remains is (3) $\Rightarrow$ (1). For this, it will suffice to show that any two τ-factorizations of the same nonzero nonunit into half $|_\tau$-primes are equal up to order and associates. Suppose, to the contrary, that there are two such τ-factorizations $\lambda a_1 \cdots a_m = \mu b_1 \cdots b_n$ that are not equal up to order and associates. We can choose such an example with $m + n$ minimal, and we observe that this forces $a_i \not\sim b_j$ for each i, j. Therefore, using this fact along with the fact that the a_i's and b_j's are half $|_\tau$-prime, it follows easily by induction that we may reorder so that $a_1 \mid b_1 \mid a_2 \mid \cdots \mid a_n \mid b_n \mid a_{n+1}$. However, the element a_{n+1} is half $|_\tau$-prime, so it must divide (and hence be an associate of) some b_i, a contradiction. $\square$

We remark that for some of the implications one can drop one or both of the hypotheses on τ in the above theorem. Perhaps the most interesting observation along these lines is that the proof of (3) $\Rightarrow$ (1) above did not require any assumptions on τ. However, the full strength of the theorem's hypotheses was needed for (1) $\Rightarrow$ (3), as the following example shows.

Example 4.2. (In a τ-UFD, an atom (hence τ-atom) need not be half $|_\tau$-prime, even for τ associate-preserving or refinable.)

(Associate-preserving case.) Let R be an integral domain, $D = R[x, y^2, xy]$, and τ be the symmetric relation on $D^\#$ determined by $(\lambda x^2)\tau(\mu y^2)$, $(\lambda xy)\tau(\mu xy)$, and $(\lambda x)\tau(\mu x)$ for $\lambda, \mu \in U(D)$. Note that τ is associate-preserving and that D is a τ-UFD. (The key fact to note for the latter is that τ-factorizations of the form $(\lambda x^2)(\mu y^2)$ are not τ-atomic, so the associates of $(xy)^2$ do indeed have unique τ-atomic factorizations.) However, because $(xy)^2 = (x^2)(y^2)$ are τ-factorizations and xy does not divide x^2 or y^2, the atom xy is not half $|_\tau$-prime.

(Refinable case.) Let R be an integral domain whose characteristic is not 2, $D = R[x^2, x^3]$, and τ be the symmetric relation on $D^\#$ determined by $(x^2)\tau(x^4)$, $(x^2)\tau(-x^2)$, and $(x^3)\tau(x^3)$. Observe that the only nontrivial τ-factorizations are those of the forms $\lambda(x^2)(x^4)$, $\lambda(x^2)(-x^2)$, and $\lambda(x^3)^n$ $(n \geq 2)$. From this we check that τ is refinable and D is a τ-UFD. (For the former, the key issue to note is that the τ-factorizations of the form $\lambda(x^2)(x^4)$ do not violate refinability because x^4 has no nontrivial reduced τ-factorization. The key issue for the latter is that of the τ-factorizations $\lambda(x^2)(x^4) = \lambda(x^3)^2$, only the second is τ-atomic.) However, because $(x^2)(x^4) = (x^3)^2$ are τ-factorizations and $x^2 \nmid x^3$, the atom x^2 is not half $|_\tau$-prime.

Of course all four statements are equivalent in the case $\tau = \tau_D$, and [**9,** Theorem 1.7] shows that they are also equivalent in the case $\tau = \tau_d$. These two relations are both divisive (and in fact also multiplicative), so the question that [**3**] poses is a natural one: does (1) $\Rightarrow$ (4) for τ divisive? If this were the case, then all four statements would be equivalent for τ divisive, and, as mentioned above, one could use [**10,** Lemma 4.9] to considerably simplify the previous proofs of the fact that τ_2-UFD implies τ_1-UFD for divisive $\tau_1 \leq \tau_2$. Unfortunately, however, the following example shows that (1) $\not\Rightarrow$ (4), even for τ both multiplicative and divisive.

Example 4.3. (In a τ-UFD, an atom (hence a τ-atom) need not be τ-prime, even if τ is both multiplicative and divisive.) Let R be an integral domain and $D = R[x^2, y^2, xy]$, where x and y are algebraically independent indeterminates over D. Define τ to be the symmetric relation on $D^\#$ determined by $(ux^{2m})\tau(vy^{2n})$ for $m, n \geq 1$ and $u, v \in U(R)$. Note that τ is divisive and multiplicative. The only elements in $D^\#$ that are not τ-atoms are those of the form $\lambda(x^{2m})(y^{2n})$, which also happens to be their unique τ-atomic factorization (up to associates and order). So D is a τ-UFD. Now, $xy \mid (x^2)(y^2)$, where the latter is a τ-factorization, but $xy \nmid x^2, y^2$, so the atom xy is not τ-prime.

Although that last example may seem discouraging, we end this paper on a positive note, showing how one can use the half $|_\tau$-primes to give the desired simplified proof. (This also allows one to weaken the requirement that the larger relation is divisive, which was an essential component of previous proofs.) This, along with the above example, leads us to propose that perhaps the half $|_\tau$-primes are more fundamental than the τ-primes.

Theorem 4.4. *Let $\tau_1 \leq \tau_2$ be symmetric relations on $D^\#$ with τ_1 divisive and τ_2 refinable and associate-preserving. If D is a τ_2-*UFD*, then it is a τ_1-*UFD*.*

Proof. As discussed in Section 2, a small modification of the corresponding proof in [**3,** Section 2] shows that a τ_2-BFD is a τ_1-BFD, so it will suffice to show that every τ_1-atom is half $|_{\tau_1}$-prime if D is a τ_2-UFD. So we assume that D is a τ_2-UFD and that $\lambda c a_1 \cdots a_m = \mu b_1 \cdots b_n$ are τ_1-factorizations with $m \geq 0$, $n \geq 1$, and c τ_1-irreducible, and we need to show that c divides some b_i. Because τ_2 is associate-preserving and D is τ_2-atomic, we can find reduced τ_2-factorizations of c, each a_i, and each b_j into τ_2-atoms. We replace each factor in the τ_2-factorizations $\lambda c a_1 \cdots a_m = \mu b_1 \cdots b_n$ with these reduced τ_2-factorization into τ_2-atoms, and by the refinable property the result is two τ_2-atomic factorizations of the same element. Using uniqueness and collecting factors as appropriate, we obtain $c = b_1' \cdots b_n'$, where each $b_i' \mid b_i$. Using divisiveness, we see that this is a τ_1-factorization (ignoring any unit factors). So, because c is τ_1-irreducible, exactly one b_i' is a nonunit, and we obtain $c \sim b_i' \mid b_i$, as desired. $\square$

We make one final remark that this is one of the few results where we need a relation to be divisive rather than merely associate-preserving and refinable. For example, recall the symmetric relation $\tau_{(2)}$ on $\mathbf{Z}^\#$ given by $a\tau_{(2)}b \Leftrightarrow a \equiv b \mod 2$. This relation is refinable, associate-preserving, and multiplicative, yet the UFD $\mathbf{Z}$ is a $\tau_{(2)}$-HFD but not a $\tau_{(2)}$-UFD. (See [**3,** Section 5] for further details.)

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