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NEGATIVE PISOT AND SALEM NUMBERS AS ROOTS OF NEWMAN POLYNOMIALS

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ABSTRACT. A Newman polynomial has all its coefficients in $\{0,1\}$ and constant term 1. It is known that every root of a Newman polynomial lies in the slit annulus $\{z \in \mathbf{C} : \tau^{-1} < |z| < \tau\} \setminus \mathbf{R}^{+}$, where τ denotes the golden ratio, but not every polynomial having all of its conjugates in this set divides a Newman polynomial. We show that every negative Pisot number in $(-\tau, -1)$ with no positive conjugates, and every negative Salem number in the same range obtained by using Salem's construction on small negative Pisot numbers, is satisfied by a Newman polynomial. We also construct a number of polynomials having all their conjugates in this slit annulus, but that do not divide any Newman polynomial. Finally, we determine all negative Salem numbers in $(-\tau, -1)$ with degree at most 20, and verify that every one of these is satisfied by a Newman polynomial.

1. Introduction. A *Newman polynomial* is a univariate polynomial with all its coefficients in $\{0,1\}$ and constant term 1. Let $\mathcal{N}$ denote the set of Newman polynomials. In 1993, Odlyzko and Poonen [**19**] established a number of facts about Newman polynomials. For a positive real number ρ, we let A_ρ denote the open annulus

$$A_\rho = \{z \in \mathbf{C} : \rho^{-1} < |z| < \rho\}.$$

Odlyzko and Poonen proved that if $\alpha \in \mathbf{C}$ is a root of a polynomial $f \in \mathcal{N}$, then $\alpha \in A_\tau$, where $\tau = (1+\sqrt{5})/2$ denotes the golden ratio. They also demonstrated that the set W of roots of $\mathcal{N}$ in the complex plane has a fractal appearance and showed that its closure $\overline{W}$ is path-connected. In addition, they remarked that the set $\overline{W}$ is unlikely to

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be the full annulus $\overline{A_\tau}$, as this annulus appears to contain a number of zero-free regions with positive area.

Certainly, not every algebraic integer $\alpha \in A_\tau$ is a root of a Newman polynomial, since α or one of its conjugates may be a positive real number. Dubickas [**11**] has shown that an algebraic number has no positive real conjugates if and only if it is satisfied by a polynomial having all nonnegative coefficients. However, it follows from recent work of Drungilas and Dubickas [**9**] that α may not be the root of a Newman polynomial, even if α and all of its conjugates lie in $A_\tau \setminus \mathbf{R}^+$. In fact, they proved that, for each real number $\rho \in (1, 2]$, the set of algebraic units whose conjugates all lie in the annulus A_ρ, but do not occur as the root of any polynomial with $\{-1, 0, 1\}$ coefficients, is dense in $\overline{A_\rho}$. It is straightforward to modify their construction so that the algebraic units in question have no positive real conjugates. It follows that the set of algebraic units whose conjugates all lie in $A_\tau \setminus \mathbf{R}^+$, but do not occur as a root of any Newman polynomial, is dense in $\overline{A_\tau}$.

Next, recall that the *Mahler measure* of a polynomial

$$f(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - \alpha_k)$$

is defined by

$$M(f) = |a_n| \prod_{k=1}^{n} \max\{1, |\alpha_k|\}.$$

If α is an algebraic number, then we denote by $M(\alpha)$ the Mahler measure of its minimal polynomial. It is well known (owing to a classical result of Kronecker) that $M(f) = 1$ for $f(z) \in \mathbf{Z}[z]$ if and only if $f(z)$ is a product of cyclotomic polynomials, and a power of z. In 1933, Lehmer [**13**] asked whether there exists a positive number ε such that if $f \in \mathbf{Z}[z]$ has $M(f) \neq 1$ then $M(f) \geq 1 + \varepsilon$. This is known as *Lehmer's problem*, and it remains open. The smallest known value of the measure greater than 1 is $1.17628\ldots$, attained by a polynomial noted by Lehmer:

$$\ell_1(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1. \tag{1}$$

Several partial results are known in Lehmer's problem. For example, if a polynomial $f \in \mathbf{Z}[z]$ exists with $1 < M(f) < M(\ell_1)$, then

$\deg(f) \geq 56$ [**17**]. For more information, we refer the reader to the survey article [**23**].

It is remarkable that a simple condition involving the Mahler measure suffices for deducing that an algebraic integer is satisfied by some polynomial with $\{-1, 0, 1\}$ coefficients. We say a polynomial with $\{-1, 0, 1\}$ coefficients has *height* 1. Pathiaux [**20**] and Mignotte [**14**] proved that if α is algebraic and $M(\alpha) < 2$, then there exists a polynomial $F(z)$ with height 1 such that $F(\alpha) = 0$. Furthermore, it follows from Siegel's lemma that if $f(z) \in \mathbf{Z}[z]$ has $M(f) < 2$, then there exists a polynomial $F(z)$ with height 1 such that $f(z) \mid F(z)$, even if f is not irreducible [**3**].

Since every root of a polynomial with height 1 and nonzero constant term lies in the annulus A_2, one might ask if a similar property may hold for the set of Newman polynomials. We investigate this question in this article.

Problem 1. *Does there exist a real number $\sigma > 1$ such that if $f(z) \in \mathbf{Z}[z]$ has no nonnegative real roots and $M(f) < \sigma$, then $f(z) \mid F(z)$ for some $F(z) \in \mathcal{N}$?*

By analogy with the known results for height 1 polynomials, one might surmise that the value of σ could be taken to be τ, the golden ratio. Certainly this is the largest possible value for σ. However, we show that this cannot be the case in subsection 2.1, where we construct a number of polynomials that have all their roots in the slit annulus $A_\tau \setminus \mathbf{R}^+$, exhibit Mahler measure less than τ, and yet do not divide any Newman polynomial. The smallest Mahler measure in this list is approximately 1.556 (see Table 1 in Section 2).

Problem 1 certainly requires that if $f(z)$ is a product of cyclotomic polynomials with $(z-1) \nmid f(z)$, then $f(z)$ must divide a Newman polynomial. In 2003, Dubickas [**10**] proved precisely this statement, providing an explicit construction, and showing that indeed one may require the Newman polynomial itself to be a product of cyclotomic polynomials.

It is natural then to consider the problem of representing other classes of polynomials by Newman polynomials. Recall that a *Pisot number* is a real algebraic integer $\beta > 1$ whose conjugates all lie inside the open unit disk $\{z : |z| < 1\}$. Much is known about this set: it is closed, its minimal element is the real root of $z^3 - z - 1$ $(1.32471\ldots)$, and its smallest limit point is the golden ratio τ. In 1966, Amara [**1**] determined all of the limit points of Pisot numbers in $(1, 2]$, along with all of the associated sequences of Pisot numbers approaching each limit point. Furthermore, Boyd [**6, 8**] developed algorithms that can identify all of the Pisot numbers in $[1, 2-\delta]$ for any $\delta > 0$, including the limit points. Using all of this information, it is well known (see for instance [**2**]) that every Pisot number less than the golden ratio is a root of one of the following polynomials, for some positive integer n:

$$\begin{aligned} p_{2n}(z) &= z^{2n+1} - z^{2n-1} - z^{2n-2} - \cdots - z - 1, \\ q_{2n+1}(z) &= z^{2n+1} - z^{2n} - z^{2n-2} - \cdots - z^2 - 1, \\ r_n(z) &= z^n(z^2 - z - 1) + z^2 - 1, \\ g(z) &= z^6 - 2z^5 + z^4 - z^2 + z - 1. \end{aligned} \tag{2}$$

We say β is a *negative Pisot number* if $-\beta$ is a Pisot number. It is convenient to define the following families of polynomials when analyzing the small negative Pisot numbers:

$$\begin{aligned} P_n(z) &= z^n(z^2 + z - 1) + 1, \\ Q_n(z) &= z^n(z^2 + z - 1) - 1, \\ R_n(z) &= z^n(z^2 + z - 1) + z^2 - 1, \\ S_n(z) &= z^n(z^2 + z - 1) - z^2 + 1, \\ G(z) &= z^6 + 2z^5 + z^4 - z^2 - z - 1. \end{aligned} \tag{3}$$

Since $(z+1)p_{2n}(-z) = -P_{2n}(z)$, $(z^2-1)q_{2n+1}(-z) = -Q_{2n+1}(z)$, $r_{2n}(-z) = R_{2n}(z)$, $r_{2n+1}(-z) = -S_{2n+1}(z)$ and $g(-z) = G(z)$, every negative Pisot number $\beta > -\tau$ occurs as a root of one of these polynomials. In addition, each polynomial in each of these families has a negative Pisot number as a root, although in some cases (such as $P_{2n+1}(z)$ and $Q_{2n}(z)$) this number is less than $-\tau$. We need these negative Pisot numbers as well for the second class of algebraic integers that we consider.

A *Salem number* is a real algebraic integer $\alpha > 1$ having $1/\alpha$ as one conjugate, and the rest of its conjugates lying on the unit circle. The set of Salem numbers is not as well understood as the set of

Pisot numbers. In particular, it is not known if a minimal Salem number exists. The smallest one known is $1.17628\ldots$, the real root greater than 1 of the polynomial $\ell_1(z)$ from (1). However, some results connecting Salem numbers and Pisot numbers are known. If $p(z)$ is the minimal polynomial of a Pisot number β, we let $p^*(z) = z^{\deg(p)}p(1/z)$ denote the *reciprocal polynomial* of $p(z)$. Salem [**21**] proved that, for sufficiently large integers m, the polynomials $z^m p(z) + p^*(z)$ and $z^m p(z) - p^*(z)$ have a Salem number as a root, and that these two associated sequences of Salem numbers approach β in the limit, one sequence from above and one sequence from below. Furthermore, Boyd [**5**] showed that every Salem number arises as a member of one of these sequences associated with some Pisot number–in fact, each Salem number is generated infinitely often through these sequences. More information on Salem numbers may be found in [**2, 4, 22**].

We say α is a *negative Salem number* if $-\alpha$ is a Salem number. It is natural then to consider all of the negative Salem numbers associated with each of the negative Pisot numbers from (3) in the context of Problem 1. We define the families

$$
\begin{aligned}
P^+_{m,n}(z) &= z^m P_n(z) + P^*_n(z), & P^-_{m,n}(z) &= z^m P_n(z) - P^*_n(z),\\
Q^+_{m,n}(z) &= z^m Q_n(z) + Q^*_n(z), & Q^-_{m,n}(z) &= z^m Q_n(z) - Q^*_n(z),\\
R^+_{m,n}(z) &= z^m R_n(z) + R^*_n(z), & R^-_{m,n}(z) &= z^m R_n(z) - R^*_n(z),\\
S^+_{m,n}(z) &= z^m S_n(z) + S^*_n(z), & S^-_{m,n}(z) &= z^m S_n(z) - S^*_n(z),\\
G^+_{m}(z) &= z^m G(z) + G^*(z), & G^-_{m}(z) &= z^m G(z) - G^*(z).
\end{aligned}
\tag{4}
$$

We investigate the negative Salem numbers in these sequences and determine whether every such number $\alpha > -\tau$ can be represented by a Newman polynomial. Note that some of the negative Salem numbers α from the polynomials in (4) satisfy $\alpha > -\tau$, even when the associated negative Pisot number β satisfies $\beta < -\tau$. This occurs, for example, with the polynomials $Q^+_{m,n}(z)$ when n is even, m is odd and $m \le n+1$.

In this article, we show that the best possible result holds for negative Pisot and Salem numbers by proving the following results.

Theorem 1. *If β is a negative Pisot number with $\beta > -\tau$ and β has no positive real conjugates, then there exists a Newman polynomial F with $F(\beta) = 0$.*

Theorem 2. *If α is a negative Salem number satisfied by one of the polynomials* (4) *and $\alpha > -\tau$, then there exists a Newman polynomial F with $F(\alpha) = 0$.*

It is also interesting that Theorem 2 extends a result of Mukunda [**18**] in a natural way. In his article, Mukunda first adapted Boyd's algorithm to show that every negative Pisot number whose minimal polynomial is a Newman polynomial is a root of $Q_n(z)$ with n odd. He then shows that each of these negative Pisot numbers is a limit point, from below, of negative Salem numbers, each of which is satisfied by some Newman polynomial. Our Theorem 2 then not only produces a sequence of negative Salem numbers converging to each of these negative Pisot numbers from above as well, but shows that this approximation property holds for *every* small negative Pisot number, not just those whose minimal polynomial is a Newman polynomial.

This paper is organized in the following way. Section 2 describes some algorithms for identifying Newman polynomials having a prescribed factor, and for determining when certain algebraic units are not satisfied by any Newman polynomial. Section 3 constructs the Newman polynomials for the Pisot case, proving Theorem 1, and Section 4 handles the Salem case and establishes Theorem 2. Section 5 then describes the determination of every negative Salem number $\alpha > -\tau$ with degree at most 20 and verifies that each of these 502 algebraic numbers occurs as the root of a Newman polynomial.

2. Algorithms. Let $f \in \mathbf{Z}[z]$ be a polynomial having all its roots in $A_\tau \setminus \mathbf{R}^+$. If a polynomial $F \in \mathcal{N}$ exists with $f \mid F$, we would like an effective method for constructing such a polynomial. Certainly, a trivial method would merely test every Newman polynomial of degree $n \geq \deg(f)$ until finding a qualifying polynomial. In this section, we describe two much more efficient methods for finding these polynomials, one of which can also determine when no such Newman polynomial exists, at least in certain cases. We then show how these methods may be employed to determine Newman polynomials representing families like (3) and (4).

2.1. Values attained in an interval. Given a negative real algebraic integer $\beta \in (-\tau, -1)$, our first method calculates all of the values in a certain interval which are attained by the set of Newman polynomials of bounded degree. It then determines if a Newman polynomial F with a zero at β exists within this degree range by testing whether 0 is ever realized. More precisely, let $I(\beta)$ denote the real interval

$$I(\beta) = \left[\frac{-1}{\beta^2 - 1}, \frac{-\beta}{\beta^2 - 1}\right].$$

Next, for a nonnegative integer d, let

$$\mathcal{N}(\beta, d) = \{F(\beta) : F \in \mathcal{N} \quad \text{and} \quad \deg(F) \leq d\} \cap I(\beta),$$

and let

$$\mathcal{N}(\beta) = \bigcup_{d \geq 0} \mathcal{N}(\beta, d).$$

It is straightforward to verify that, if $F \in \mathcal{N}$ and $F(\beta) \notin I(\beta)$, then $\beta F(\beta) \notin I(\beta)$ and $\beta F(\beta) + 1 \notin I(\beta)$. It follows that, for a positive integer n, the set $\mathcal{N}(\beta, n)$ may be computed by using a simple recursive strategy. Clearly, $\mathcal{N}(\beta, 0) = \{1\}$, and for $d > 0$, we have that

$$\begin{aligned} \mathcal{N}(\beta, d+1) = \mathcal{N}(\beta, d) &\cup \big(\{\beta\omega : \omega \in \mathcal{N}(\beta, d)\} \cap I(\beta)\big) \\ &\cup \big(\{\beta\omega + 1 : \omega \in \mathcal{N}(\beta, d)\} \cap I(\beta)\big). \end{aligned} \tag{5}$$

We may therefore determine whether β is satisfied by a Newman polynomial of degree at most n by checking whether any of the sets $\mathcal{N}(\beta, d)$ for $d \leq n$ contain 0.

We also note that, if $\mathcal{N}(\beta, d+1) = \mathcal{N}(\beta, d)$ for some d, then $\mathcal{N}(\beta) = \mathcal{N}(\beta, d)$, and in this case the method then determines whether β is satisfied by any Newman polynomial. While we can only guarantee that this iteration stabilizes when β is a negative Pisot number, in practice we find that it often terminates for other algebraic integers.

For β a negative Pisot number, a result of Garsia [**12**] shows that there exists a constant $c := c(\beta) > 0$, independent of d, such that, for $x, y \in \mathcal{N}(\beta)$, either $x = y$ or $|x - y| > c > 0$. As a result, we see that the number of distinct elements of $\mathcal{N}(\beta)$ is finite and bounded, and hence this algorithm must terminate. Upon termination, we can check whether $\mathcal{N}(\beta)$ contains 0.

There is a second way that this algorithm may terminate: if 0 is shown to be in $\mathcal{N}(\beta)$ at some point during the calculation, then we may terminate with an affirmative. In this case, it does not matter whether $\mathcal{N}(\beta)$ is finite or infinite. Most of the Salem numbers discussed in this paper possess this property.

TABLE 1. Some polynomials with all roots in $A_\tau \setminus \mathbf{R}^+$ that do not divide any Newman polynomial.

Polynomial	Mahler measure
$z^6 - z^5 - z^3 + z^2 + 1$	1.556014485
$z^7 - z^6 - z^5 + z^4 + z^3 - z^2 + 1$	1.558378942
$z^8 - z^7 + z^2 + 1$	1.604364647
$z^9 - z^8 - z^6 + z^2 + 1$	1.615829244
$z^8 - z^7 - z^5 + z^2 + 1$	1.617538308
$z^8 + z^7 + 2z^6 + z^5 + z^4 + z^3 + 2z^2 + z + 1$	1.618530599
$z^9 - z^8 + z^7 + z^5 + z^4 + z^2 + 1$	1.621082531
$z^8 - z^7 + z^5 + z^3 - z + 1$	1.624147966
$z^7 + z^5 - z^4 - z + 1$	1.646642716
$z^8 + z^7 + 2z^6 + 2z^5 + z^4 + z^3 + z^2 + z + 1$	1.652235034

In theory, if $0 \in \mathcal{N}(\beta)$, then this algorithm will terminate. In practice, it is impossible to distinguish between β where $0 \notin \mathcal{N}(\beta)$ and those where the calculation runs out of memory. An example of this is given by $s_1(z)$ in equation (13) at the end of this paper.

We employ a variation of this algorithm to find a number of polynomials that have all their roots in $A_\tau \setminus \mathbf{R}^+$ but do not divide any Newman polynomial. Table 1 exhibits ten such polynomials, four of which have Mahler measure less than τ. None of these polynomials has a negative root between -2 and -1, so we cannot use the algorithm described above directly. We describe here a modified version for polynomials having no real roots.

Let β be any algebraic integer with $|\beta| > 1$. Define

$$I'(\beta) = \left\{ z \in \mathbf{C} : \ |z| \le \frac{|\beta|}{|\beta| - 1} \right\},$$
$$\mathcal{N}'(\beta, d) = \{F(\beta) : F \in \mathcal{N} \quad \text{and} \quad \deg(F) \le d\} \cap I'(\beta),$$

and

$$\mathcal{N}'(\beta) = \bigcup_{d \geq 0} \mathcal{N}'(\beta, d).$$

It is straightforward to verify that, if $F \in \mathcal{N}$ and $F(\beta) \notin I'(\beta)$, then $\beta F(\beta) \notin I'(\beta)$ and $\beta F(\beta) + 1 \notin I'(\beta)$. We may thus calculate the sets $\mathcal{N}'(\beta, d)$ just as in (5), and if we find that $\mathcal{N}'(\beta, d+1) = \mathcal{N}'(\beta, d)$ for some d, then $\mathcal{N}'(\beta)$ is finite. In this case, if $0 \notin \mathcal{N}'(\beta)$, then we may conclude that β is not the root of a Newman polynomial. This was the case for each example in Table 1.

2.2. Sieving, and reciprocal polynomials. Our second method for constructing Newman polynomials with a prescribed factor $f \in \mathbf{Z}[z]$ was developed by the second author in 2003 [**15**]. We briefly describe the method here.

Clearly we may represent a Newman polynomial F by the integer $F(2)$. If $f \mid F$, then certainly $f(2) \mid F(2)$, so we need only consider odd multiples of $f(2)$ as initial candidates for $F(2)$. Next, for a positive integer $x = \sum_{i=0}^{n} x_i 2^i$, with each $x_i \in \{0, 1\}$, let $\gamma(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} x_{2i} 2^{2i} - \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} x_{2i+1} 2^{2i+1}$, so that $\gamma(F(2)) = F(-2)$. We may thus compute $F(-2)$ from $F(2)$ in just three elementary operations: first use a mask to obtain the even-indexed bits in $F(2)$, then another mask to obtain the odd-indexed bits and then subtract these two values. Our method then tests whether $f(-2) \mid \gamma(u)$ for each positive odd integer u that is a multiple of $f(2)$, and on each success we construct $F(z)$ from $u = F(2)$ and test whether $f \mid F$.

In most situations in this research the polynomial f is *reciprocal*, that is, $f = f^*$, so we might expect that many Newman polynomials having f as a factor will also be reciprocal. As in [**15**], we adapt this method to search for reciprocal Newman polynomials having a prescribed reciprocal factor f. If $F \in \mathcal{N}$ is reciprocal with degree n, select k and m so that $2k + m = n + 1$ and $f(2) < 2^m$. Then write

$$F(2) = 2^{k+m} a^* + 2^k b + a, \tag{6}$$

where a is odd, $0 < a < 2^k$, $0 \leq b < 2^m$, and a^* denotes the integer obtained by reversing the bits of a, interpreted as a bit string of length k. The condition that $f(2) \mid F(2)$ is then equivalent to the requirement that

$$b \equiv -2^m a^* - 2^{-k} a \bmod f(2), \tag{7}$$

and if we select m as small as possible so that $f(2) < 2^m$ and $m + 2k = n + 1$, then for each odd $a < 2^k$ there are at most three values of b that satisfy (7). The requirement that $f(-2) \mid F(-2)$ can also be adapted to the reciprocal case. Using (6), it follows that

$$F(-2) = (-2)^{k+m}\gamma(a^*) + (-2)^k\gamma(b) + \gamma(a),$$

so we require that

$$\gamma(b) \equiv -(-2)^m\gamma(a^*) - (-2)^{-k}\gamma(a) \bmod f(-2). \tag{8}$$

Thus, given a reciprocal polynomial f and a target degree n, our second method first selects m and k so that $m + 2k = n + 1$ and $2^{m-2} < f(2) < 2^m$, and then for each odd integer $a \in (0, 2^k)$, it computes all integers $b \in [0, 2^m)$ satisfying (7) and checks if any of these satisfies (8). If a and b pass both of these tests, then we construct F from a and b by using (6) and determine whether $f \mid F$. We remark that the various powers of ± 2 modulo $f(2)$ and $f(-2)$ required in (7) and (8) may be computed at initialization for efficiency. In addition, using a Gray code to step through the different possible values of a also greatly improves performance.

We note that this algorithm does not require that the given polynomial f be irreducible, and the adaptation to searching for reciprocal polynomials greatly extends its reach. We also note that this method may produce some non-reciprocal multiples of f as well–while it guarantees that the leading k coefficients match the last k coefficients, the middle m coefficients need not be symmetric.

2.3. Example. We show how these methods were employed to determine families of Newman polynomials that represent our negative Pisot and Salem numbers by presenting some details in one case.

We consider the class $Q^+_{m,n}(z)$ when m is odd, n is even and $n \geq m-1$. We first note that these polynomials always have $(z-1)^2(z+1)$ as a factor in this case, and we remove these factors before applying our second algorithm. Some initial experiments with our methods indicated that Newman representatives for these polynomials could only be found when m and n were both rather small, suggesting that factors of sizable degree may be required in general. When $(m, n) = (5, 4)$ or $(7, 6)$, we find that we can manufacture many examples, but no patterns were immediately apparent. We might hope, however, that if $(Q^+_{m,n}(z)/(z-1)^2(z+1)) \cdot g(z)$ is a Newman polynomial, then perhaps $g(z)h(z)$ has a particularly simple representation, for some auxiliary polynomial $h(z)$. Multiplying our polynomials $g(z)$ by combinations of factors of the form $z^k \pm 1$, $z^k + z^{k-1} + \cdots + 1$ and $z^j \pm z^k + 1$ for various values of j and k, we found that the reciprocal Newman polynomial of minimal degree that is divisible by $Q^+_{5,4}(z)/(z-1)^2(z+1)$ can be written as

$$\frac{Q^+_{5,4}(z)(z^{24}-1)(z^5+1)}{(z^8-1)(z^6-1)(z^2-1)},$$

and, for the case $(m, n) = (7, 6)$, a reciprocal Newman polynomial of degree 58 is

$$\frac{Q^+_{7,6}(z)(z^{60}-1)(z^7+1)}{(z^{12}-1)(z^{10}-1)(z^2-1)}.$$

(In this case, examples with degree as small as 50 exist.) We then turn to the case $(m, n) = (9, 8)$, which is more difficult, but still tractable, as our algorithm from subsection 2.2 finds that the reciprocal Newman polynomial with minimal degree that is divisible by $Q^+_{9,8}(z)/(z-1)^2(z+1)$ has degree 80. (The prescribed factor has degree 16 here.) We find that this Newman polynomial may be written as

$$\frac{Q^+_{9,8}(z)(z^{80}-1)(z^9+1)}{(z^{16}-1)(z^{10}-1)(z^2-1)}.$$

These examples suggested further experiments with polynomials of the form

$$\frac{Q^+_{m,m-1}(z)(z^{ab/2}-1)(z^m+1)}{(z^a-1)(z^b-1)(z^2-1)}, \tag{9}$$

with ab even, and we found that selecting $a = 3m-1$ and $b = m+1$ always appeared to produce a Newman polynomial. Additional work then demonstrated that, in the general case with m odd, n even, and

$m \geq n-1$, we should use (9) with $a = m + 2n + 1$ and $b = n + 2$, and that we should write the factor $z^m + 1$ in the numerator of (9) as $z^{n+1} + 1$. We verify that the resulting polynomial (11) is indeed always a Newman polynomial in subsection 4.3.

3. Negative Pisot numbers. Using the methods from Section 2, we may identify families of Newman polynomials having the qualifying negative Pisot numbers from (3) as roots. We consider each of these families in turn in the following proof.

Proof of Theorem 1. From (2), we need only consider the polynomials $P_n(z)$ and $R_n(z)$ when n is even, $Q_n(z)$ and $S_n(z)$ when n is odd, and $G(z)$. Since $R_n(0) = -1$ and $R_n(1) = 1$, we see that $R_n(z)$ has a positive real root for every n, and a direct calculation reveals this to be the case for $G(z)$ as well. Next, when n is odd, we find that

$$\frac{Q_n(z)}{z^2-1} = z^n + \sum_{k=0}^{(n-1)/2} z^{2k}$$

and

$$\frac{S_n(z)(z^n+1)(z^{n+1}-1)}{z^2-1} = z^{3n+1} + z^{2n+1} \sum_{k=0}^{(n-1)/2} z^{2k} + z^{n+3} \sum_{k=0}^{(n-5)/2} z^{2k} + 1,$$

which are both Newman polynomials. Finally, if n is even, then

$$\frac{P_n(z)(z^{n+1}-1)}{z^2-1} = z^{2n+1} + (z^{n+2}+1) \sum_{k=0}^{(n/2)-1} z^{2k},$$

which is again a Newman polynomial. $\square$

4. Negative Salem numbers. In order to prove Theorem 2, we consider each of the ten families of polynomials from (4) in turn. We let $\Phi_k(z)$ denote the cyclotomic polynomial with index k throughout this section.

4.1. The polynomials $P^+_{m,n}(z)$. If m and n are both odd, then it is straightforward to verify that $P^+_{m,n}(-\tau) < 0$ and $P^+_{m,n}(-3) > 0$, so the Salem number in question is always less than $-\tau$ in this case. If m is odd and n is even, we find that $P^+_{m,n}(z)$ again has a root in $(-3,-\tau)$ when either $m \geq 5$ and $n \geq m-1$, or $m \leq 3$ and $n \geq m+1$. After verifying that $P^+_{3,2}(z)\Phi_3(z)\Phi_6(z)\Phi_{18}(z)$ is a Newman polynomial, we need only consider odd $m \geq 5$ and even $n \leq m-3$ to complete this case. For this, we note the identity

$$\begin{aligned} \frac{P^+_{m,n}(z)(z^{m-1}-1)}{z^2-1} &= z^{2m+n-1} + z^{n+2}\sum_{k=0}^{m-2} z^{2k} - z^{m-1} \\ &\quad + z\sum_{k=0}^{m-2} z^{2k} - z^{m+n} + 1, \end{aligned} \tag{10}$$

which is valid for all positive integers m and n, and we see that if m is odd, n is even, $m \geq 5$, and $n \leq m-3$, then the two negative terms in (10) cancel with positive ones in the sums, producing a Newman polynomial. If m is even and n is odd, the fact that $P^+_{m,n}(z) = P^+_{n+2,m-2}(z)$ reduces this problem to the prior case except when $m=2$, and $P^+_{2,n}(z) = (z^{n+3}+1)(z+1)$ is cyclotomic (and a Newman polynomial). Last, if m and n are both even, then by using the symmetry we need only consider the case $m \geq n+2$, and in this case the two negative terms in (10) cancel with positive ones in the sums, producing a Newman polynomial.

We remark that Lehmer's polynomial $\ell_1(z)$ from (1) is a factor of $P^+_{4,6}(-z)$, and thus the identity (10) produces the Newman polynomial $z^{13}+z^{12}+z^8+z^5+z+1$ that represents the smallest (in absolute value) known negative Salem number.

4.2. The polynomials $P^-_{m,n}(z)$. We find that $P^-_{m,n}(z)/(z^2-1)$ is a Newman polynomial precisely when $P^-_{m,n}(z)$ has no real roots less than $-\tau$. More precisely, it is straightforward to verify that $P^-_{m,n}(z)$ has a real root in $(-3,-\tau)$ in the following cases: m and n both odd and $n \leq m-4$; m is even and n is odd; and m and n are both even and $n \geq m$. For the remaining cases, if m is odd, and either n is even, or n is odd and $n \geq m-2$, then

$$\frac{P^-_{m,n}(z)}{z^2-1} = z^{m+n} + z(z^{n+1}+1)\sum_{k=0}^{(m-3)/2} z^{2k} + 1$$

is a Newman polynomial. If m and n are both even and $n \le m-2$, then

$$\frac{P^-_{m,n}(z)}{z^2-1} = z^{m+n} + z \sum_{k=0}^{[(m+n)/2]-1} z^{2k} + z^{n+2} \sum_{k=0}^{[(m-n)/2]-2} z^{2k} + 1$$

is again in $\mathcal{N}$.

4.3. The polynomials $Q^+_{m,n}(z)$. Using the intermediate value theorem, one may verify that $Q^+_{m,n}(z)$ has a real root in $(-3,-\tau)$ when m and n are both even, and also when m is odd, n is even, and $n \le m-3$. Further, the symmetry $Q^+_{m,n}(z) = Q^+_{n+2,m-2}(z)$ allows us to ignore the case when m is even, n is odd, and $m \ge 4$, and it is easy to check that $Q^+_{2,n}(z)$ has a real root in $(-2,-\tau)$ for odd $n \ge 3$. The polynomial $Q^+_{2,1}(z) = (z-1)^2(z+1)^3$ is cyclotomic. This leaves two recalcitrant cases. If m and n are both odd, some considerable experimentation and lengthy algebra reveals that

$$\frac{Q^+_{m,n}(z)\left(z^{(m+n)(m-1)/2}-1\right)}{(z^{m+n}-1)(z^{m-1}-1)(z^2-1)} = \sum_{k=0}^{[(m+n)/2]-1} z^{k(m-1)} + z\left(\sum_{k=0}^{(n-3)/2} z^{2k}\right)\left(\sum_{k=0}^{(m-3)/2} z^{k(m+n)}\right)$$

is a Newman polynomial, and if m is odd, n is even, and $n \ge m-1$, then with more experimentation (summarized in subsection 2.3) we find that

$$\frac{Q^+_{m,n}(z)\left(z^{(m+2n+1)(n+2)/2}-1\right)\left(z^{n+1}+1\right)}{(z^{m+2n+1}-1)(z^{n+2}-1)(z^2-1)} \tag{11}$$
$$= z(z^n+1)\left(\sum_{k=0}^{(m-3)/2} z^{2k}\right)\left(\sum_{k=0}^{n/2} z^{k(m+2n+1)}\right) + \sum_{k=0}^{n+[(m-1)/2]} z^{k(n+2)}$$

is again in $\mathcal{N}$.

4.4. The polynomials $Q^-_{m,n}(z)$. Since $Q^-_{m,n}(z) = P^-_{n+2,m-2}(z)$, we need only check the cases with $m \le 2$. Using the intermediate value theorem, we find that $Q^-_{1,n}(z)$ has a real root in $(-3,-\tau)$ for $n \ge 2$, and $Q^-_{1,1}(z) = (z-1)(z+1)^3$ and $Q^-_{2,n}(z) = (z^{n+3}-1)(z+1)$ are both cyclotomic.

4.5. The polynomials $R^+_{m,n}(z)$ and $S^+_{m,n}(z)$. We consider both of these families simultaneously since $R^+_{m,n}(z) = S^+_{n,m}(z)$. We first verify that $R^+_{m,n}(z)$ has a real root in $(-3, -\tau)$ when m is even and n is odd, when both are odd and $n \le m$, and when both are even and $n \ge m$. If m is odd and n is even, then

$$\begin{aligned}\frac{R^+_{m,n}(z)(z^{m+n+1}-1)}{z^2-1} = {}& z^{2m+2n+1}\\ &+ \left(z^{2m+2n} + z^{2m+2n-2} + \cdots + z^{m+n+3}\right)\\ &- z^{m+2n+1} + z^{2m+n+1}\\ &+ \left(z^{m+n-2} + z^{m+n-4} + \cdots + z\right) - z^m + z^n + 1,\end{aligned}$$

and the two terms with coefficient -1 each cancel with a term in the prior sum. If m and n are both odd and $n > m \ge 3$, then

$$\begin{aligned}&\frac{R^+_{m,n}(z)(z^{m+1}-1)(z^m+1)}{z^2-1} = z^{3m+n+1}\\ &\quad + \left(z^{3m+n} + z^{3m+n-2} + \cdots + z^{2m+n+1}\right) + z^{3m+1} + z^n\\ &\quad + \left(z^{2m+n-2} + z^{2m+n-4} + \cdots + z^{m+n+3}\right)\\ &\quad + \left(z^{2m-2} + z^{2m-4} + \cdots + z^{m+3}\right) + \left(z^m + z^{m-2} + \cdots + z\right) + 1,\end{aligned}$$

a Newman polynomial. To complete this case, we note that

$$R^+_{1,n}(z)\Phi_3(z) = z^{n+5} + z^{n+4} + z^n + z^5 + z + 1.$$

Finally, suppose m and n are both even and $m > n$. For this case, we first find that

$$\begin{aligned}&\frac{R^+_{m,n}(z)(z^n-1)(z^{n+1}+1)}{z^2-1} = z^{m+3n+1} - z^{3n+1} - z^m\\ &\quad + \left(z^{m+3n} + z^{m+3n-2} + \cdots + z^{m+2n}\right)\\ &\quad + \left(z^{m+2n-1} + z^{m+2n-3} + \cdots + z^{m+n+3}\right)\\ &\quad + \left(z^{2n-2} + z^{2n-4} + \cdots + z^{n+2}\right) + \left(z^{n+1} + z^{n-1} + \cdots + z\right) + 1,\end{aligned}$$

and the two negative terms cancel with positive ones if $n < m < 2n$. Suppose then that $m \geq 2n$. We compute that

$$\begin{aligned}
\frac{R^+_{m,n}(z)(z^n-1)(z^{2n-1}+1)(z^m+1)}{z^2-1} &= z^{2m+4n-1} \\
&+ \left(z^{2m+4n-2} + z^{2m+4n-4} + \cdots + z^{2m+3n}\right) \\
&+ z^{2m+2n} \\
&+ \left(z^{2m+2n-3} + z^{2m+2n-5} + \cdots + z^{2m+n+1}\right) \\
&+ \left(z^{m+4n-2} + z^{m+4n-4} + \cdots + z^{m+2n}\right) - z^{2m} \\
&+ \left(z^{m+2n-1} + z^{m+2n-3} + \cdots + z^{m+1}\right) - z^{4n-1} \\
&+ \left(z^{3n-2} + z^{3n-4} + \cdots + z^{2n+2}\right) + z^{2n-1} \\
&+ \left(z^{n-1} + z^{n-3} + \cdots + z\right) + 1,
\end{aligned} \tag{12}$$

and this is a Newman polynomial provided that $2n \leq m < 4n$. For $m \geq 4n$, note that if we multiply (12) by $z^{2n}+1$, then the resulting product still has height 1, the z^{4n-1} and z^{2m+2n} terms both cancel, and the result is a Newman polynomial if $4n \leq m < 6n$. For $6n \leq m < 8n$, a similar argument shows that multiplying (12) by the factor $z^{4n}+z^{2n}+1$ suffices, and in general, if $m > 2n$ then

$$\frac{R^+_{m,n}(z)(z^n-1)(z^{2n-1}+1)(z^m+1)(z^{2n\lfloor m/2n\rfloor}-1)}{(z^2-1)(z^{2n}-1)}$$

is a Newman polynomial.

4.6. The polynomials $R^-_{m,n}(z)$. Since $R^-_{m,n}(z) = R^-_{n,m}(z)$, we need only consider the case $n \geq m$. We first note that $R^-_{m,n}(z)$ has a real root in $(-3, -\tau)$ when m is odd and $n \geq m$. If m and n are both even, then

$$\frac{R^-_{m,n}(z)}{z^2-1} = z^{m+n} + z^m + z^n + 1 + z \sum_{k=0}^{[(m+n)/2]-1} z^{2k}$$

is a Newman polynomial unless $m = n$. For this case, since $R^-_{m,m}(z) = (z^m+1)(z^{m+2}+z^{m+1}-z^m+z^2-z-1)$, we find that

$$\frac{R^-_{m,m}(z)}{(z^m+1)(z^2-1)} = z^m + 1 + z \sum_{k=0}^{m/2-1} z^{2k}.$$

Suppose then that m is even, n is odd and $n > m$. Let $k < m/2$ be a positive integer. We compute

$$\begin{aligned}
&\frac{R^-_{m,n}(z)\left(z^{3m+n+2} - z^{2m+n+2} + z^{2m+n+1-2k} + z^{m+2k+1} - z^m + 1\right)}{z^2-1} \\
&= z^{4m+2n+2} + \left(z^{4m+2n+1} + z^{4m+2n-1} + \cdots + z^{3m+2n+3}\right) \\
&\quad + z^{3m+2n-2k+1} + \left(z^{3m+2n-2k+3} + z^{3m+2n-2k+1} + \cdots + z^{3m+n+3}\right) \\
&\quad - z^{2m+2n+2} + z^{2m+2n-2k+1} + z^{4m+n+2} + z^{3m+n-2k+1} \\
&\quad + z^{2m+n+2k+1} + z^{2m+n+1} - z^{2m+n+2} - z^{2m+n} \\
&\quad + \left(z^{2m+n+2k} + z^{2m+n+2k-2} + \cdots + z^{2m+n-2k+2}\right) + z^{2m+n-2k+1} \\
&\quad + z^{m+n+2k+1} + \left(z^{m+n-1} + z^{m+n-3} + \cdots + z^{m+2k+2}\right) - z^{2m} + z^n \\
&\quad + z^{2m+2k+1} + z^{m+2k+1} + \left(z^{m-1} + z^{m-3} + \cdots + z\right) + 1.
\end{aligned}$$

There are four terms with coefficient -1, but each one cancels with another term since $2k+2 \le m < n$. In addition, we determine that all of these monomials are distinct if $m \ne 4k$, $n \ne m+2k+1$, $n \ne 2m+2k+1$ and $n > 4k-1$. If $k = 1$, we find that this is therefore a Newman polynomial provided that $m \ge 6$, $n \ne m+3$ and $n \ne 2m+3$, and if $k = 2$, it is a Newman polynomial provided that $m = 6$ or $m \ge 10$, $n > 7$, and $n \notin \{m+5, 2m+5\}$. These two cases therefore cover all pairs (m, n) with m even, n odd and $n > m \ge 6$, except for $(8, 11)$ and $(8, 19)$. For these, we find that

$$R^-_{8,11}(z)\Phi_6(z)\Phi_{10}(z)\Phi_{30}(z)\Phi_{48}(z)/\Phi_1(z),$$
$$R^-_{8,19}(z)\Phi_4(z)\Phi_8(z)\Phi_{75}(z)/\Phi_1(z)\Phi_5(z)$$

are both Newman polynomials.

We treat the case $m = 4$ by considering the residue of n modulo 10. If $n \equiv 1 \bmod 10$, then

$$\begin{aligned}
&\frac{R^-_{4,n}(z)\left(z^{2n-4} + z^{2n-6} + z^{n+4} - z^{n-1} + z^{n-2} - z^{n-3} + z^{n-8} + z^2 + 1\right)}{z^5-1} \\
&= \left(z^{3n-3} + z^{3n-8} + \cdots + 1\right) - z^{2n+3} + z^{3n-6} \\
&\quad + \left(z^{3n-4} + z^{3n-9} + \cdots + z^{2n+2}\right) + z^{2n+5} + z^{2n+4} + z^{2n-1} \\
&\quad + z^{2n-4} + z^{2n-5} + z^{2n-8} + z^{n+5} + z^{n+2} + z^{n-4} + z^{n-7} + z^{n-8} \\
&\quad + \left(z^{n-5} + z^{n-10} + \cdots + z\right) + z^3,
\end{aligned}$$

and here the lone negative term cancels with a term in the prior sum. Also, all the terms are distinct provided $n > 11$, so this is a Newman polynomial except when $n = 11$. For this case, we note that $R^-_{4,11}(z)\Phi_{14}(z)\Phi_{24}(z)\Phi_{48}(z)/\Phi_1(z) \in \mathcal{N}$. If $n \equiv 3 \bmod 10$, then

$$\begin{aligned}\frac{R^-_{4,n}(z)\left(z^{2n-2}+z^{2n-4}+z^{n+3}-z^{n-1}+z^{n-5}+z^2+1\right)}{z^5-1}\\ = \left(z^{3n-1}+z^{3n-6}+\cdots+z^{2n+7}\right)\\ +\left(z^{3n-2}+z^{3n-7}+\cdots+z^{2n-4}\right)\\ +z^{3n-4}+z^{2n+4}\\ +\left(z^{2n+3}+z^{2n-2}+\cdots+z^{n-4}\right)+z^{2n-1}+z^{2n-5}\\ +\left(z^{2n+3}+z^{2n-2}+\cdots+z^{n-4}\right)+z^{n+4}+z^{n}+z^{n-5}\\ +\left(z^{n+3}+z^{n-2}+\cdots+z\right)\\ +\left(z^{n-8}+z^{n-13}+\cdots+1\right)+z^3,\end{aligned}$$

and this is a Newman polynomial for all $n > m$ in this class. If $n \equiv 5 \bmod 10$, then

$$\begin{aligned}\frac{R^-_{4,n}(z)\left(z^{2n-4}+z^{2n-6}+z^{n-2}+z^2+1\right)}{z^5-1}\\ = \left(z^{3n-4}+z^{3n-9}+\cdots+z\right)+z^{3n-6}\\ +\left(z^{3n-3}+z^{3n-8}+\cdots+z^{2n-3}\right)+z^{2n-1}\\ +z^{2n-2}+z^{2n-5}+z^{2n-6}+z^{n+3}+z^{n+2}+z^{n-1}\\ +z^{n-2}+\left(z^{n}+z^{n-5}+\cdots+1\right)+z^3\end{aligned}$$

suffices except when $n = 5$, and we find that $R^-_{4,5}(z)\Phi_{14}\Phi_{24}(z)/\Phi_1(z) \in \mathcal{N}$. If $n \equiv 7 \bmod 10$, then

$$\begin{aligned}\frac{R^-_{4,n}(z)\left(z^{2n+14}+z^{2n+12}+z^{n+7}+z^2+1\right)}{z^5-1}\\ = z^{3n+12}+\left(z^{3n+15}+z^{3n+10}+\cdots+z^{2n+7}\right)\\ +\left(z^{3n+14}+z^{3n+9}+\cdots+z^{2n+11}\right)+z^{2n+15}\\ +\left(z^{2n+4}+z^{2n-1}+\cdots+z^{n+11}\right)+\left(z^{n+8}+z^{n+3}+\cdots+1\right)\\ +\left(z^{n+4}+z^{n-1}+\cdots+z\right)+z^{n}+z^3\end{aligned}$$

is a Newman polynomial. Last, if $n \equiv 9 \bmod 10$, then

$$\begin{aligned}\frac{R^-_{4,n}(z)\left(z^{2n+8}+z^{2n+6}+z^{n+4}+z^2+1\right)}{z^5-1}\\ &= z^{3n+6}+z^{2n+9}+z^{2n+4}+(z^{3n+9}+z^{3n+4}+\cdots+z^{2n+8})\\ &\quad+(z^{3n+8}+z^{3n+3}+\cdots+z^{2n+7})\\ &\quad+(z^{2n+5}+z^{2n}+\cdots+z^{n+4})+(z^{n+2}+z^{n-3}+\cdots+z)\\ &\quad+z^{n+5}+(z^{n+1}+z^{n-4}+\cdots+1)+z^n+z^3\end{aligned}$$

is a Newman polynomial.

Finally, we establish the case $m = 2$ by considering the residue of n modulo 6. If $n \equiv 1 \bmod 6$, then

$$\begin{aligned}\frac{R^-_{2,n}(z)\left(z^{2n+4}+z^{n+3}-z^{n+2}+z^{n+1}+1\right)}{z^3-1}\\ &= (z^{3n+5}+z^{3n+2}+\cdots+z^{2n+3})+z^{3n+4}+z^{2n+5}\\ &\quad+z^{2n+4}+z^{2n+1}+(z^{2n-1}+z^{2n-4}+\cdots+z^{n+6})\\ &\quad+z^{n+4}+(z^{n+2}+z^{n-1}+\cdots+1)+z^{n+1}+z^n+z\end{aligned}$$

is a Newman polynomial when $n > 2$. If $n \equiv 3 \bmod 6$, then

$$\begin{aligned}\frac{R^-_{2,n}(z)\left(z^{2n}+z^{n+2}-z^{n}+z^{n-2}+1\right)}{z^3-1}\\ &= (z^{3n+1}+z^{3n-2}+\cdots+z^{2n-2})\\ &\quad+z^{3n}+z^{2n+3}\\ &\quad+(z^{2n+2}+z^{2n-1}+\cdots+z^{n-1})+z^{n-2}\\ &\quad+(z^{n+3}+z^{n}+\cdots+1)+z\end{aligned}$$

is a Newman polynomial except when $n = 3$, and $R^-_{2,3}(z)$ has a real root at $-1.63557\cdots < -\tau$. Last, if $n \equiv 5 \bmod 6$, then

$$\begin{aligned}\frac{R^-_{2,n}(z)\left(z^{2n-2}+z^{n+2}-z^{n-1}+z^{n+4}+1\right)}{z^3-1}\\ &= (z^{3n-1}+z^{3n-4}+\cdots+z^{2n-2})\\ &\quad+z^{3n-2}+z^{2n+3}+z^{2n+2}\\ &\quad+(z^{2n-3}+z^{2n-6}+\cdots+z^{n+2})+z^{2n-4}+z^{n+3}\\ &\quad+(z^{n+1}+z^{n-2}+\cdots+1)+z^{n-3}+z^{n-4}+z\end{aligned}$$

is Newman for $n > 5$, and $R^-_{2,5}(z)\Phi_6(z)\Phi_{12}(z)\Phi_{24}(z)/\Phi_1(z) \in \mathcal{N}$.

4.7. The polynomials $S^-_{m,n}(z)$. Since $S^-_{m,n}(z) = S^-_{n,m}(z)$, we need only consider the case $n \geq m$. When m is even, a short computation reveals that $S^-_{m,n}(z)$ has a real root in $(-3, -\tau)$. When m is odd, we compute that

$$\frac{S^-_{m,n}(z)(z^m+1)}{z^2-1} = z^{2m+n} - z^{2m} - z^n + 1 + \frac{z(z^m+1)(z^{m+n}-1)}{z^2-1}.$$

If n is odd, this reduces to

$$z^{2m+n} - z^{2m} - z^n + 1 + z(z^m+1)\sum_{k=0}^{[(m+n)/2]-1} z^{2k},$$

and it is straightforward to check that this is a Newman polynomial for $n \geq m$. If n is even, the same expression becomes

$$z^{2m+n} - z^{2m} - z^n + 1 + z^{m+1}\sum_{k=0}^{n/2-1} z^{2k} + z\sum_{k=0}^{m+n/2-1} z^{2k},$$

which is a Newman polynomial for all $n \geq m$ except when $n = 2m$. In this case, we see that $S^-_{m,2m}(z) = (z^m-1)(z^{2m}(z^2+z-1) + z^{m+1} - z^2 + z + 1)$, and determine that

$$\begin{aligned}\frac{S^-_{m,2m}(z)(z^{m+1}-1)}{(z^m-1)(z^2-1)} &= z^{3m+1} + z(z^{2m}+1)\sum_{k=0}^{(m-1)/2} z^{2k} \\ &\quad + z^{m+3}\sum_{k=0}^{(m-5)/2} z^{2k} + 1\end{aligned}$$

for $m \geq 3$, a Newman polynomial. The lone remaining case is $S^-_{1,2}(z)$, and this polynomial has a real root at $-1.72208\cdots < -\tau$.

4.8. The polynomials $G_m^+(z)$. If m is odd and $m \geq 3$, we find that

$$\begin{aligned}\frac{G_m^+(z)\Phi_{15}(z)(z^{2m+10}-1)}{z^2-1} &= z^{3m+22} \\ &\quad + z^{3m+16} + z^{3m+12} + z^{3m+10} \\ &\quad + \left(z^{3m+21} + z^{3m+19} + \cdots + z^{2m+24}\right) + z^{2m+18} \\ &\quad + z^{2m+14} + z^{2m+12} \\ &\quad + \left(z^{2m+9} + z^{2m+7} + \cdots + z^{m+14}\right) \\ &\quad + z^{m+10} + z^{m+8} + z^{m+4} \\ &\quad + \left(z^{2m+8} + z^{2m+6} + \cdots + z^{m+13}\right) \\ &\quad + \left(z^{m-2} + z^{m-4} + \cdots + z\right) \\ &\quad + z^{12} + z^{10} + z^{6} + 1,\end{aligned}$$

which is a Newman polynomial. For the case $m = 1$, the same multiplier suffices, but produces the Newman polynomial $z^{25} + z^{20} + z^{19} + z^{16} + z^{15} + z^{14} + z^{11} + z^{10} + z^{9} + z^{6} + z^{5} + 1$. If m is even, then

$$\begin{aligned}&\frac{G_m^+(z)\left(z^{m+1}+1\right)\left(z^{m+14}+z^{m+9}+z^{m+1}-z^{13}-z^{5}-1\right)}{(z+1)(z^3-1)} \\ &= z^{3m+17} + z^{3m+16} + z^{3m+14} + z^{3m+12} \\ &\quad + z^{3m+11} + z^{3m+10} + z^{3m+8} + z^{3m+4} + z^{3m+2} \\ &\quad + \left(z^{3m+5} + z^{3m+3} + \cdots + z^{2m+19}\right) + z^{2m+13} + z^{2m+11} \\ &\quad + z^{2m+10} + z^{2m+9} + z^{2m+7} + z^{2m+4} + z^{m+13} + z^{m+10} + z^{m+8} \\ &\quad + \left(z^{2m+1} + z^{2m-1} + \cdots + z^{m+17}\right) \\ &\quad + \left(z^{2m} + z^{2m-2} + \cdots + z^{m+16}\right) \\ &\quad + z^{m+7} + z^{m+6} + z^{m+4} \\ &\quad + \left(z^{m-2} + z^{m-4} + \cdots + z^{12}\right) + z^{15} + z^{13} \\ &\quad + z^{9} + z^{7} + z^{6} + z^{5} + z^{3} + z + 1\end{aligned}$$

for $m \geq 14$. A direct computation reveals that the cases $m = 2$, $m = 4$ and $m = 6$ all exhibit a real root less than $-\tau$, and for the remaining cases, we find that the following are all Newman polynomials:

$$G_8^+(z)\Phi_6(z)\Phi_{16}(z)\Phi_{25}(z),$$
$$G_{10}^+(z)\Phi_2(z)\Phi_6(z)\Phi_{22}(z)\Phi_{25}(z),$$
$$G_{12}^+(z)\Phi_2(z)\Phi_5(z)\Phi_{10}(z)\Phi_{15}(z)\Phi_{22}(z).$$

4.9. The polynomials $G_m^-(z)$. If m is odd and $m \geq 19$, then

$$\begin{aligned}
&\frac{G_m^-(z)\left(z^{m+13}(z^{15}+z^{10}+z^5-z^3+1)+z^{15}-z^{12}+z^{10}+z^5+1\right)}{(z+1)(z^3-1)} \\
&= z^{2m+30}+\left(z^{2m+29}+z^{2m+27}+\cdots+z^{2m+19}\right) \\
&\quad + z^{2m+24}+z^{2m+20} \\
&\quad + z^{2m+15}+z^{2m+13}+\left(z^{2m+14}+z^{2m+12}+\cdots+z^{16}\right)+z^{m+30} \\
&\quad + z^{m+28}+z^{m+24}+z^{m+6}+z^{m+2}+z^m+z^{17}+z^{15}+z^{11}+z^{10} \\
&\quad + z^9+z^7+z^6+z^5+z^3+z+1
\end{aligned}$$

is a Newman polynomial. When $m = 1$, $m = 3$ or $m = 5$, the polynomial $G_m^-(z)$ has a real root less than $-\tau$. For the remaining cases, we verify that

$$\begin{gathered}
G_7^-(z)\Phi_6(z)\Phi_{15}(z)/\Phi_1(z), \\
G_9^-(z)\Phi_{15}(z)\Phi_{26}(z)(z^{48}-z^{38}+z^{28}-z^{22}+z^{16}-z^6+1)/\Phi_1(z), \\
G_{11}^-(z)\Phi_{10}(z)\Phi_{57}(z)/\Phi_1(z), \qquad G_{13}^-(z)\Phi_{15}(z)\Phi_{34}(z)\Phi_{72}(z)/\Phi_1(z), \\
G_{15}^-(z)\Phi_{10}(z)\Phi_{39}(z)\Phi_{72}(z)/\Phi_1(z), \qquad G_{17}^-(z)\Phi_6(z)\Phi_{55}(z)/\Phi_1(z)
\end{gathered}$$

are all Newman polynomials. If m is even, then

$$\begin{aligned}
\frac{G_m^-(z)\Phi_{15}(z)}{z^2-1} &= z^{m+12}+\left(z^{m+11}+z^{m+9}+\cdots+z\right) \\
&\quad + z^{m+6}+z^{m+2}+z^m+z^{12}+z^{10}+z^6+1,
\end{aligned}$$

which is a Newman polynomial when $m = 2$ or $m \geq 14$, and for the remaining cases we find that

$$\begin{gathered}
G_4^-(z)\Phi_6(z)\Phi_{18}(z)/\Phi_1(z)\Phi_2(z), \\
G_6^-(z)\Phi_6(z)\Phi_{20}(z)/\Phi_1(z)\Phi_2(z), \\
G_8^-(z)\Phi_{33}(z)/\Phi_1(z)\Phi_2(z), \\
G_{12}^-(z)\Phi_6(z)\Phi_{10}(z)\Phi_{20}(z)\Phi_{40}(z)/\Phi_1(z), \\
G_{10}^-(z)\Phi_6(z)\Phi_{10}(z)\Phi_{12}(z)\Phi_{24}(z)\Phi_{54}(z)/\Phi_1(z)
\end{gathered}$$

are all Newman polynomials, completing the proof of Theorem 2. □

5. Negative Salem numbers with small degree. Theorem 2 demonstrates that every negative Salem number $\alpha > -\tau$ arising from one of the families (4) is satisfied by a Newman polynomial. However, some negative Salem numbers in $(-\tau, -1)$ may not be represented by one of these families. For example, it appears that neither $-\alpha_2$ nor $-\alpha_3$, the second- and third-smallest known negative Salem numbers, is satisfied by any of the families (4). Here, $\alpha_2 = 1.18836\ldots$ and $\alpha_3 = 1.20002\ldots$, and these numbers have respective minimal polynomials

$$\begin{aligned}\ell_2(z) = z^{18} - z^{17} + z^{16} - z^{15} - z^{12} + z^{11} - z^{10}\\ + z^9 - z^8 + z^7 - z^6 - z^3 + z^2 - z + 1,\end{aligned}$$

and

$$\ell_3(z) = z^{14} - z^{11} - z^{10} + z^7 - z^4 - z^3 + 1.$$

It is natural then to test whether small negative Salem numbers like these occur as the root of a Newman polynomial. To check this, we determined the complete list of negative Salem numbers $\alpha > -\tau$ whose minimal polynomial has degree at most 20. This list was constructed by using Boyd's algorithm [**7**] to determine all reciprocal polynomials $f \in \mathbf{Z}[z]$ with $M(f) \leq \tau$, and then selecting the minimal polynomials for Salem numbers from this list. This last filter was implemented by using a change of variable. If $f(z)$ is a reciprocal polynomial of degree $2n$, let $g(y)$ be the polynomial obtained by substituting y for $z + z^{-1}$ in $z^{-n}f(z)$. (Note that $z^k + z^{-k} = 2T_k(y/2)$, where T_k denotes the kth Chebyshev polynomial, defined by $T_k(\cos\theta) = \cos(k\theta)$.) Since each pair of conjugate roots of f on the unit circle corresponds to a real root of g in $(-2, 2)$, we can determine whether f is the minimal polynomial for a Salem number by testing whether it is irreducible and checking that the corresponding polynomial g has exactly $n - 1$ real roots in $(-2, 2)$.

We find that there are exactly 502 negative Salem numbers $\alpha > -\tau$ having degree at most 20. This list is available at the website `http://www.cecm.sfu.ca/~mjm/Lehmer` [**16**]. By checking the families (4) with $\max\{m, n\} \leq 50$, we observe that Theorem 2 covers at least 272 of them. The minimal polynomials of nine of the remaining numbers are already Newman polynomials. This leaves 221 negative Salem numbers to check, including $-\alpha_2$ and $-\alpha_3$. We use the sieving algorithm for reciprocal polynomials of subsection 2.2 to check these numbers. This algorithm discovers a Newman polynomial with degree at most 97 (and $k \leq 37$ in the algorithm) for each of these negative Salem numbers, except for two of degree 18. For example, we find that

$$\ell_2(-z)\Phi_2(z)\Phi_4(z)\Phi_8(z)\Phi_{14}(z)\Phi_{20}(z)\Phi_{32}(z)$$

and

$$\ell_3(-z)\Phi_3(z)\Phi_8(z)\Phi_{15}(z)$$

are the reciprocal Newman polynomials of minimal degree (respectively, 55 and 28) that satisfy $-\alpha_2$ and $-\alpha_3$.

The remaining two negative Salem numbers are $\gamma_1 = -1.49604\ldots$ and $\gamma_2 = -1.57574\ldots$, with respective minimal polynomials

$$\begin{aligned} s_1(z) = z^{18} &+ 3z^{17} + 4z^{16} + 4z^{15} + 4z^{14} + 5z^{13} + 6z^{12} \\ &+ 7z^{11} + 7z^{10} + 7z^9 + 7z^8 + 7z^7 + 6z^6 + 5z^5 \\ &+ 4z^4 + 4z^3 + 4z^2 + 3z + 1 \end{aligned} \tag{13}$$

and

$$\begin{aligned} s_2(z) = z^{18} &+ 2z^{17} + 2z^{16} + 3z^{15} + 2z^{14} + z^{13} - z^{12} \\ &- 3z^{11} - 4z^{10} - 5z^9 - 4z^8 - 3z^7 - z^6 \\ &+ z^5 + 2z^4 + 3z^3 + 2z^2 + 2z + 1. \end{aligned}$$

For γ_1, since $s_1(1) = 89$, certainly any Newman polynomial having $s_1(z)$ as a factor must have $89k$ monomials, for some integer k. By using this observation in concert with the algorithm of subsection 2.2 for reciprocal polynomials, we find that

$$\begin{aligned} s_1(z)\Phi_{24}(z)(&z^{90} - 2z^{89} + 2z^{88} - z^{87} + z^{83} - 2z^{82} + 2z^{81} - z^{80} \\ &+ z^{79} - 2z^{78} + 2z^{77} - z^{76} + z^{72} - 2z^{71} + 3z^{70} - 3z^{69} + 2z^{68} - z^{67} \\ &+ z^{65} - 2z^{64} + 3z^{63} - 3z^{62} + 2z^{61} - z^{60} + z^{59} - z^{58} \\ &+ z^{56} - z^{55} + z^{54} - z^{53} + z^{52} - z^{51} + z^{50} - z^{49} \\ &+ z^{47} - z^{46} + z^{45} - z^{44} + z^{43} - z^{41} + z^{40} - z^{39} \\ &+ z^{38} - z^{37} + z^{36} - z^{35} + z^{34} - z^{32} + z^{31} - z^{30} \\ &+ 2z^{29} - 3z^{28} + 3z^{27} - 2z^{26} + z^{25} - z^{23} + 2z^{22} \\ &- 3z^{21} + 3z^{20} - 2z^{19} + z^{18} - z^{14} + 2z^{13} \\ &- 2z^{12} + z^{11} - z^{10} + 2z^9 - 2z^8 + z^7 \\ &- z^3 + 2z^2 - 2z + 1) \end{aligned}$$

is a Newman polynomial of degree 116. In fact, this is the minimal degree of a reciprocal Newman polynomial having $s_1(z)$ as a factor.

Finally, we determine a Newman polynomial for γ_2 by employing an auxiliary polynomial. We first construct 30 multiples of $s_2(z)$ having height 1, degree at most 60, and no positive real roots, and then we use our sieving method to search for a Newman multiple of each of these polynomials. With the auxiliary factor $\Phi_{14}(z)\Phi_{21}(z)\Phi_{42}(z)$, our method discovers the Newman polynomial

$$s_2(z)\Phi_2(z)\Phi_3^2(z)\Phi_6^2(z)\Phi_9(z)\Phi_{12}(z)\Phi_{14}(z)\Phi_{18}(z)\Phi_{21}(z)\Phi_{42}(z)\Phi_{55}(z)$$

with degree 113.

REFERENCES

1. M. Amara, *Ensembles fermés de nombres algébriques*, Ann. Sci. École Norm. **83** (1966), 215–270.

2. M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse and J.-P. Schreiber, *Pisot and Salem numbers*, Birkhäuser, Basel, 1992.

3. E. Bombieri and J.D. Vaaler, *Polynomials with low height and prescribed vanishing*, in *Analytic number theory and Diophantine problems*, A.C. Adolphson, J.B. Conrey, et al., eds., Prog. Math. **70**, Birkhäuser, Boston, 1987.

4. P. Borwein, *Computational excursions in analysis and number theory*, CMS Books in Mathematics/Ouvrages de Math. SMC **10**, Springer, New York, 2002.

5. D.W. Boyd, *Small Salem numbers*, Duke Math. J. **44** (1977), 315–328.

6. ———, *Pisot and Salem numbers in intervals of the real line*, Math. Comp. **32** (1978), 1244–1260.

7. ———, *Reciprocal polynomials having small measure*, Math. Comp. **35** (1980), 1361–1377.

8. ———, *Pisot numbers in the neighbourhood of a limit point*, I, J. Numb. Theor. **21** (1985), 17–43.

9. P. Drungilas and A. Dubickas, *Roots of polynomials of bounded height*, Rocky Mountain J. Math. **39** (2009), 527–543.

10. A. Dubickas, *The divisors of Newman polynomials*, Fiz. Mat. Fak. Moksl. Semin. Darb. **6** (2003), 25–28.

11. ———, *On roots of polynomials with positive coefficients*, Manuscr. Math. **123** (2007), 353–356.

12. A.M. Garsia, *Arithmetic properties of Bernoulli convolutions*, Trans. Amer. Math. Soc. **102** (1962), 409–432.

13. D.H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. Math. **34** (1933), 461–479.

14. M. Mignotte, *Sur les multiples des polynômes irréductibles*, Bull. Soc. Math. Belg. **27** (1975), 225–229.

15. M.J. Mossinghoff, *Polynomials with restricted coefficients and prescribed noncyclotomic factors*, LMS J. Comp. Math. **6** (2003), 314–325.

16. ———, *Lehmer's problem*, 2011. `http://www.cecm.sfu.ca/~mjm/Lehmer`.

17. M.J. Mossinghoff, G. Rhin and Q. Wu, *Minimal Mahler measures*, Exper. Math. **17** (2008), 451–458.

18. K. Mukunda, *Pisot numbers from* $\{0,1\}$*-polynomials*, Canad. Math. Bull. **53** (2010), 140–152.

19. A.M. Odlyzko and B. Poonen, *Zeros of polynomials with* $0,1$ *coefficients*, Enseign. Math. **39** (1993), 317–348.

20. M. Pathiaux, *Sur les multiples de polynômes irréductibles associés à certains nombres algébriques*, Sem. Delange-Pisot-Poitou **14** (1972/73), Secrétariat Mathématique, Paris, 1972/73.

21. R. Salem, *Power series with integral coefficients*, Duke Math. J. **12** (1945), 153–172.

22. ———, *Algebraic numbers and Fourier analysis*, D.C. Heath, Boston, 1963.

23. C. Smyth, *The Mahler measure of algebraic numbers*: *A survey*, in *Number theory and polynomials*, J. McKee and C. Smyth, eds., Lond. Math. Soc. Lect. Note **352**, Cambridge University Press, Cambridge, 2008.

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