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A NOTE ON FREE PRODUCTS

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ABSTRACT. We prove that a theorem by Stallings on finitely generated subgroups of free groups is also valid for free products of groups.

1. Introduction. Let F be any free group. Let U and V be finitely generated subgroups of F. Stallings [**6**] proved that if $U \cap V$ is of finite index in both U and V, then it is of finite index in the subgroup $\langle U \cup V \rangle$ of F generated by U and V. In this note we prove that the same result holds for free products.

We will say a group G has the Stallings property if the following holds for any finitely generated subgroups U, V of G with $U \cap V$ nontrivial: if $U \cap V$ is of finite index in U and in V, then it is of finite index in $\langle U \cup V \rangle$.

Theorem 1.1. *Let $G = \prod_{\alpha \in J} * G_\alpha$ be a free product of groups. Then G has the Stallings property if and only if G_α has the Stallings property for every $\alpha \in J$.*

Note that, in Theorem 1.1, we require that $U \cap V$ be nontrivial. The theorem is not true without this condition. A simple example showing this is that $G = G_1 * G_2$, $U = G_1$ and $V = G_2$ where both G_1 and G_2 are finite groups. Also, the condition on the factors are necessary only for the "degenerate" case, the case when $U \cap V$ is contained in some free factor of G.

Theorem 1.2. *Let G be as in Theorem* 1.1. *Let U and V be finitely generated subgroups of G such that $U \cap V$ is nontrivial and of finite index in both U and V. If $U \cap V$ is not contained in any conjugate of any factor G_α, then it is of finite index in $\langle U \cup V \rangle$.*

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After we proved the results, we came to know recently that Sykiotis [**7**] proved that the same property holds for graphs of groups with suitable conditions. Theorems 1.1 and 1.2 are therefore essentially special cases of Sykiotis's theorem. However, Sykiotis worked with Bass-Serre machinery which is different from our direct algebraic/combinatorial arguments.

2. Preliminaries. Let J be a set and G_α a nontrivial group for each $\alpha \in J$. Let $G = \prod_{\alpha \in J} * G_\alpha$ be the free product of G_α, $\alpha \in J$. Every element $g \in G$ is uniquely represented as a reduced word of the form

$$g = g_1 g_2 \cdots g_k, \tag{2.1}$$

where $k \geq 0, 1 \neq g_i \in G_{\alpha_i}$, $i = 1, \ldots, k$ and $\alpha_i \neq \alpha_{i+1}$, $i = 1, \ldots, k-1$.

If g is represented as in (2.1), we will denote the first letter g_1 by $\lambda(g)$, the last letter g_k by $\varepsilon(g)$, and the length of g by $l(g)$. We also denote the index α_k of the last letter $\varepsilon(g)$ by $\omega(g)$. If $\lambda(g) \neq \varepsilon(g)^{-1}$, we say that g is cyclically reduced. If X is a subset of G, we define X_+ to be the set of all initial segments of elements in X. In particular, X_+ contains $1 \in G$. A set B of elements of G is a Schreier system if any initial segment of any element of B is again an element of B, that is, if $B = B_+$.

Lemma 2.1. *Let $G = \prod_{\alpha \in J} * G_\alpha$ be a free product, and let $w \in G$ be a cyclically reduced element of G. If w is a torsion element, then w is a torsion element of some free factor G_α of G.*

Proof. Since w is cyclically reduced, it is easy to see that if $l(w) \geq 2$, then $w^n \neq 1$ for any $n > 0$. Thus, $l(w) = 1$ which means w is in some factor G_α of G. □

Lemma 2.2. *Let G be as above and $h \in G$ be a nontrivial element. The following statements are equivalent*:

(a) *There exists $N > 0$ such that $\lambda(h) \neq \lambda(h^N)$ or $\varepsilon(h) \neq \varepsilon(h^N)$*;

(b) *h is a torsion element*;

(c) *h is a conjugate of a torsion element of a factor G_α of G.*

Proof. We show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

Suppose (a) is true. Write $h = uwu^{-1}$ where u is reduced and w is cyclically reduced and uwu^{-1} is reduced as written. $u = 1$ if and only if h is cyclically reduced. For any $n > 0$,

$$h^n = uw^n u^{-1}$$

which is reduced as written if w^n is reduced first and $w^n \neq 1$. So $\lambda(h) = \lambda(h^n)$ and $\varepsilon(h) = \varepsilon(h^n)$ unless $w^n = 1$. It follows that $w^N = 1$. By Lemma 2.1, w is a torsion element of some free factor G_α of G. Hence, h is also a torsion element and is a conjugate of a torsion element of a factor G_α of G. Thus (b) is true.

Suppose now that (b) is true. So $h^{N_1} = 1$ for some $N_1 > 0$. To prove (c), write $h = uwu^{-1}$ as above. Then $h^{N_1} = uw^{N_1}u^{-1}$. It follows that $w^{N_1} = 1$. Again, by Lemma 2.1, w is a torsion element of some free factor G_α of G. Then (c) follows.

Finally, (c) clearly implies (a). This completes the proof of Lemma 2.2. □

Remark. It follows from Lemma 2.2 that if h is not a torsion element, then $\lambda(h) = \lambda(h^n)$ and $\varepsilon(h) = \varepsilon(h^n)$ for any $n > 0$.

Let H be a subgroup of G. By a (right) coset of H in G, we mean a subset of G of the form Hc, $c \in G$. The set of all (right) cosets of H will be denoted by $\mathcal{C}_H$ or simply $\mathcal{C}$. For any subgroup K of G, the set of all (H, K) double cosets will be denoted by $\mathcal{D}_{(H,K)}$. A set of coset representatives for H is a subset of G which contains exactly one element of each coset of H. Equivalently, it is a map $R : \mathcal{C}_H \to G$ such that $R(C) \in C$, $C \in \mathcal{C}_H$. Also recall that a *uniform Schreier system of coset representatives* [**5**], or simply a uniform Schreier system, for H a subgroup of $G = \prod_{\alpha \in J} *G_\alpha$ is a collection of sets R_α, one for each $\alpha \in J$, of coset representatives for H such that:

(1) $R_\alpha(H) = 1$ for all $\alpha \in J$;

(2) $R_\alpha(Ca) \in R_\alpha(C)G_\alpha$, for any $C \in \mathcal{C}_H$, $\alpha \in J$, $a \in G_\alpha$;

(3) For any $R_\alpha(C) = sa$ such that $s \neq 1$, $\omega(s) \neq \alpha$, and $a \in G_\alpha$, we have $R_\alpha(Hs) = s = R_{\omega(s)}(Hs)$.

Let $\{R_\alpha \mid \alpha \in J\}$ be a uniform Schreier system for H. Let $T = \bigcup_{\alpha \in J} R_\alpha$. T is a Schreier system. An element of T is usually called a transversal element. For each $\alpha \in J$, let $D_\alpha = \{s \mid s \in R_\alpha, \omega(s) \neq \alpha\}$. D_α is then a set of (H, G_α) double coset representatives.

By MacLane's proof of the Kurosh subgroup theorem [**5**], we can always choose a uniform Schreier system $\{R_\alpha \mid \alpha \in J\}$ for H. The subgroup H is then a free product itself,

$$H = F * \prod_{\alpha \in J} * \Bigg(\prod_{s \in D_\alpha} *(H \cap sG_\alpha s^{-1}) \Bigg). \tag{2.2}$$

Here F is a free group generated (not freely in general) by all the nontrivial elements $\{R_\alpha(C)R_\beta(C)^{-1} \mid R_\alpha(C)R_\beta(C)^{-1} \neq 1, C \in \mathcal{C}, \alpha, \beta \in J\}$. Letting $T_f = \{R_\alpha(C) \mid \alpha \in J, C \in \mathcal{C}$, there exists $\beta \in J$ such that $R_\beta(C) \neq R_\alpha(C)\}$ and $T_p = \{s \in D_\alpha \mid \alpha \in J, H \cap sG_\alpha s^{-1} \neq \{1\}\}$. Let $T_s = T_f \cup T_p$. T_f, T_p and T_s are all subsets of T. Finally, let $T_{s+} = (T_s)_+$ be the set of all initial segments of elements in T_s.

Lemma 2.3. *Let $G = \prod_{\alpha \in J} *G_\alpha$ be a free product and H a nontrivial subgroup of G. If H is torsion, then H is contained in a conjugate of a free factor of G.*

Proof. If H is not contained in any conjugate of a free factor, the Kurosh subgroup theorem implies that H is either a free group or a nontrivial free product. Then H is not torsion. ◻

Lemma 2.4 (Baumslag [**1**]). *An element $h \in H$ either ends with some*

$$xs^{-1}$$

such that $x \in G_\alpha$, $s \in T_p \cap D_\alpha$ and $sx^{-1} \notin T$, or ends with some

$$xt^{-1}$$

such that $x \in G_\beta$, $\beta \neq \alpha$, $t \in R_\alpha \cap T_f$ and $tx^{-1} \notin T$, for some $\alpha, \beta \in J$.

Proof. This is [**1**, Lemma 2] with a general number of factors. ◻

Lemma 2.5 [**1**]. *If H is finitely generated, then T_s and T_{s+} are finite sets.*

Proof. By the proof of Lemma 3 in [**1**], T_s is a finite set. Therefore T_{s+} is also a finite set. □

3. Proof of the theorems. Stallings [**6**] first proved the result for free groups using graph theory. He attributes the origin of his theorem to Greenberg [**3**]. An algebraic proof of the result using Marshall Hall's theorem is given in [**2**] by Burns et al. Another proof is given by Kapovich and Myasnikov in [**4**]. Their argument involves graphs but is algebraic in nature. Our proofs of Theorem 1.1 and Theorem 1.2 follow along the lines of Kapovich and Myasnikov.

Let G be any group and H be a subgroup of G. The commensurator $\mathrm{Comm}_G(H)$ of H in G is defined by

$$\mathrm{Comm}_G(H) = \{g \in G \mid |H : H \cap gHg^{-1}| < \infty \quad \text{and} \quad |H : H \cap g^{-1}Hg| < \infty\}.$$

See [**4**]. It is probably well known to the expert that $\mathrm{Comm}_G(H)$ is a subgroup of G containing H. For completeness, we include a proof here.

Lemma 3.1. $\mathrm{Comm}_G(H)$ *is a subgroup of* G *containing* H*.*

Proof. It suffices to show that if $g_1, g_2 \in \mathrm{Comm}_G(H)$, then $g_1g_2 \in \mathrm{Comm}_G(H)$.

Suppose $g_1, g_2 \in \mathrm{Comm}_G(H)$. By definition,

$$|H : H \cap g_1Hg_1^{-1}| < \infty,$$
$$|H : H \cap g_1^{-1}Hg_1| < \infty,$$

and

$$|H : H \cap g_2Hg_2^{-1}| < \infty.$$

It follows that

$$|g_1^{-1}Hg_1 \cap H : (g_1^{-1}Hg_1 \cap H) \cap (g_2Hg_2^{-1} \cap H)| \leq |H : (g_1^{-1}Hg_1 \cap H) \cap (g_2Hg_2^{-1} \cap H)| < \infty.$$

Hence,

$$\begin{aligned}|g_1^{-1}Hg_1 : g_2Hg_2^{-1} \cap g_1^{-1}Hg_1| \\ &\leq |(g_1^{-1}Hg_1 : (g_1^{-1}Hg_1 \cap H) \cap (g_2Hg_2^{-1} \cap H)| \\ &= |g_1^{-1}Hg_1 : g_1^{-1}Hg_1 \cap H||g_1^{-1}Hg_1 \cap H : \\ &\qquad (g_1^{-1}Hg_1 \cap H) \cap (g_2Hg_2^{-1} \cap H)| \\ &= |H : H \cap g_1Hg_1^{-1}||g_1^{-1}Hg_1 \cap H : \\ &\qquad (g_1^{-1}Hg_1 \cap H) \cap (g_2Hg_2^{-1} \cap H)| < \infty.\end{aligned}$$

Therefore,

$$|H : g_1g_2Hg_2^{-1}g_1^{-1} \cap H| = |g_1^{-1}Hg_1 : g_2Hg_2^{-1} \cap g_1Hg_1^{-1}| < \infty.$$

Similarly,

$$|H : (g_1g_2)^{-1}H(g_1g_2) \cap H| < \infty.$$

Thus, $g_1g_2 \in \mathrm{Comm}_G(H)$, and Lemma 3.1 is proved. □

If H_1 is a subgroup of G containing H such that H is of finite index in H_1, then H_1 is contained in $\mathrm{Comm}_G(H)$. If G is a free group and H is a nontrivial finitely generated subgroup of G, the results of Kapovich and Myasnikov [**4**] show that H is also of finite index in $\mathrm{Comm}_G(H)$. Thus, in this case, $\mathrm{Comm}_G(H)$ is the largest subgroup of G containing H as a subgroup of finite index. We will see that this is also true if $G = \prod_{\alpha \in J} *G_\alpha$ is a free product and H is a nontrivial finitely generated subgroup of G which is not contained in any conjugate of any factor G_α of G. See Lemma 3.4 below.

Lemma 3.2. *Let $G = \prod_{\alpha \in J} *G_\alpha$ be a free product. Let H be an infinite subgroup of G. If H is contained in a conjugate of a factor G_α of G, then* $\mathrm{Comm}_G(H)$ *is contained in the same conjugate.*

Proof. If $1 \neq h \in G$ is in some free factor G_α and $g \in G$ such that ghg^{-1} is also in G_α, then it is easy to see that $g \in G_\alpha$. It follows that if H is contained in some G_α, and $g \in G$ such that $H \cap gHg^{-1}$ is nontrivial, then $g \in G_\alpha$.

Now suppose H is in some conjugate $fG_\alpha f^{-1}$ of a free factor G_α. Let $g \in \mathrm{Comm}_G(H)$. Then $H \cap gHg^{-1}$ is of finite index in H, hence nontrivial. It follows that $f^{-1}Hf \cap f^{-1}gf(f^{-1}Hf)f^{-1}g^{-1}f$ is nontrivial. Since $f^{-1}Hf$ is contained in G_α, the above shows that $f^{-1}gf$ is also in G_α, hence g is in $fG_\alpha f^{-1}$. Thus, $\mathrm{Comm}_G(H)$ is contained in $fG_\alpha f^{-1}$. This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let $G = Z_2 * Z_2$, where Z_2 is the cyclic group of order 2. Any subgroup of G either*

(i) *contains an infinite cyclic subgroup of G and is of finite index in G, or*

(ii) *is a cyclic subgroup of order 2.*

Proof. $G = Z_2 * Z_2$ is the infinite dihedral group and the result is standard. □

Lemma 3.4. *Let G be as in Lemma 3.2 and H be a nontrivial finitely generated subgroup of G. If H is not contained in any conjugate of any factor G_α of G, then H is of finite index in $\mathrm{Comm}_G(H)$.*

Proof. Let $G_1 = \mathrm{Comm}_G(H)$. Clearly $G_1 = \mathrm{Comm}_{G_1}(H)$. G_1 contains H, so G_1 is not contained in any conjugate of any factor G_α of G. By the subgroup theorem, H and G_1 are both nontrivial free products. In particular, H is not a torsion group.

Replacing G by G_1, if necessary, we may assume that $G = \mathrm{Comm}_G(H)$ and H is a nontrivial finitely generated subgroup of G that is not torsion. We want to prove that H is of finite index in G. We consider two cases.

Case 1. $G = Z_2 * Z_2$. By Lemma 3.3, H is either an infinite subgroup of G of finite index or a cyclic group of order 2. Since H is not torsion, it must be infinite and of finite index in G.

Case 2. $G \neq Z_2 * Z_2$. Seeking a contradiction we suppose that H is of infinite index in G.

Let $h \in H$ be any element with infinite order. Let $x = \lambda(h)$ and $y = \varepsilon(h)$. Since $G \neq Z_2 * Z_2$, we can choose $z \neq 1$ in some factor G_α such that $zx \neq 1$ and $yz^{-1} \neq 1$. This is trivial if there are at least three factors. If there are only two factors, then at least one of the factors is different from Z_2 and z can be chosen as follows. If x and y are in the same factor, we choose $z \neq 1$ from the other factor. If x and y are in different factors, without loss of generality, assume that the factor containing x is different from Z_2. Then there is at least one nontrivial element in the factor that is different from x^{-1} and we choose z to be such an element.

Now let $h_1 = zhz^{-1}$ be in reduced form, and let $x_1 = \lambda(h_1)$ and $y_1 = \varepsilon(h_1)$. The choice of z implies that x_1, y_1 and z are in the same factor. Let $h_n = h_1^n = zh^nz^{-1}$ be in reduced form. By the remark following Lemma 2.2, $\lambda(h_n) = \lambda(h_1) = x_1$ and $\varepsilon(h_n) = \varepsilon(h_1) = y_1$ for any $n > 0$.

Choose a uniform Schreier system $\{R_\alpha | \alpha \in J\}$ for H in G as in Section 2. Since H is of infinite index in G, T is an infinite set. Let T_s and T_{s+} be the subsets of T as in Section 2. Since H is finitely generated, by Lemma 2.5, T_{s+} is a finite subset of T. Let g_0 be a transversal element that is not in T_{s+}. By Lemma 2.4, g_0^{-1} is not a terminal segment of any element of H.

Now choose a w in some factor of G as follows. If $\varepsilon(g_0)$ and z (hence also x_1 and y_1) are in the same factor, choose $w \neq 1$ in any other factor. Otherwise, choose $w = 1$. Then g_0wz, g_0wx_1 and $y_1w^{-1}g_0^{-1}$ are all reduced as written. It follows that, for any $n > 0$, $g_0wh_nw^{-1}g_0^{-1}$ is reduced as written and is the reduced form of $g_0wzh^nz^{-1}w^{-1}g_0^{-1}$ and, in particular, g_0^{-1} is a terminal segment of it. Therefore, $g_0wzh^nz^{-1}w^{-1}g_0^{-1}$ cannot be an element of H.

On the other hand, for any $g \in G$, since $H \cap gHg^{-1}$ is of finite index in gHg^{-1}, there is a number $n > 0$ such that $(gHg^{-1})^n$ is in $H \cap gHg^{-1}$, hence also in H. In particular, $gh^ng^{-1} \in H$. Let $g = g_0wz$. We get a contradiction. Therefore, H is of finite index in G. This completes the proof of Lemma 3.4. □

Proof of Theorem 1.1. The "only if" part of Theorem 1.1 is obvious. We prove the other part.

Let $G = \prod_{j \in J} *G_j$ be a free product, and let U and V be finitely generated subgroups of G such that $U \cap V$ is nontrivial and is of finite index in both U and V.

First note that U and V are contained in $\mathrm{Comm}_G(U \cap V)$. It follows that $\langle U \cup V \rangle$ is also contained in $\mathrm{Comm}_G(U \cap V)$. Also, as a subgroup of finite index of a finitely generated group, $U \cap V$ is also finitely generated. We consider two cases.

Case 1. $U \cap V$ is contained in some conjugate of a free factor of G. We consider two subcases.

Subcase 1.1. $U \cap V$ is a finite subgroup. Then U and V are both finite. And, by Lemma 2.3, each is in some conjugate of a free factor of G. They must be in the same conjugate, since they both contain the nontrivial subgroup $U \cap V$. It follows that $\langle U \cup V \rangle$ is also in the conjugate. By the assumption of Theorem 1.1 on the factors of G, $U \cap V$ is of finite index in $\langle U \cup V \rangle$ in this subcase.

Subcase 1.2. $U \cap V$ is an infinite subgroup. By Lemma 3.2, $\mathrm{Comm}_G(U \cap V)$ is contained in the same conjugate. Thus, again, all the subgroups $U \cap V$, U, V, and $\langle U \cup V \rangle$ are contained in the same conjugate, hence $U \cap V$ is of finite index in $\langle U \cup V \rangle$.

Case 2. $U \cap V$ is not contained in any conjugate of any factor G_α of G. By Lemma 3.4, $U \cap V$ is of finite index in $\mathrm{Comm}_G(U \cap V)$, hence also in $\langle U \cup V \rangle$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. It is simply Case 2 above. □

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