

PHASE CALCULATIONS FOR PLANAR PARTITION POLYNOMIALS

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ABSTRACT. In the study of the asymptotic behavior of polynomials from partition theory, the determination of their leading term asymptotics inside the unit disk depends on a sequence of sets derived from comparing certain complex-valued functions constructed from polylogarithms, functions defined as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

These sets we call *phases*. This paper applies complex analytic techniques to describe the geometry of these sets in the complex plane.

1. Introduction. In the study of the asymptotic behavior of polynomials from partition theory, the determination of their leading term asymptotics inside the unit disk depends on a sequence of complex-valued functions constructed from certain polylogarithms, functions defined as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

In [4], the polynomials are given by the generating function

$$\prod_{n=1}^{\infty} \frac{1}{1-xu^n} = 1 + \sum_{m=1}^{\infty} F_m(x)u^m,$$

where $F_m(x) = \sum_{k=1}^m p_k(m)x^k$ with $p_k(m)$ denoting the number of partitions of m with exactly k parts. The corresponding sequence for their asymptotics is $\{\Re\sqrt{Li_2(x^k)/k}\}$.

2010 AMS *Mathematics subject classification.* Primary 11C08, Secondary 11M35, 30C55, 30E15.

Keywords and phrases. Asymptotic, phase, plane partition, polylogarithm, polynomials.

Received by the editors on September 24, 2010, and in revised form on July 31, 2011.

DOI:10.1216/RMJ-2014-44-1-1 Copyright ©2014 Rocky Mountain Mathematics Consortium

In [9], the polynomials $\{Q_m(x)\}$ come from the theory of plane partitions:

$$\prod_{n=1}^{\infty} \frac{1}{(1-xu^n)^n} = 1 + \sum_{m=1}^{\infty} Q_m(x)u^m,$$

where $Q_m(x) = \sum_{k=1}^m pp_k(m)x^k$ with $pp_k(m)$ counting the number of plane partitions of m with trace k . The sequence governing their asymptotics is $\{\Re \sqrt[3]{Li_3(x^k)}/k\}$.

To continue, we adapt a notion out of statistical mechanics (see [2]) which we call a *phase*.

Definition 1. Let $\{L_k(x)\}$ be a sequence of complex-valued functions defined on a domain D in the complex plane. The set $R(m)$ is called the m th phase (or simply phase m) of an “ $\{L_k(x)\}$ -region of dominance” if and only if

(1) If $x \in R(m)$, then for every $k \in \mathbf{N}$, $k \neq m$, implies $\Re L_m(x) > \Re L_k(x)$.

(2) If an open set V has property (1), $V \subset R(m)$.

Remark 2. We give a brief discussion of the interplay of ideas from statistical mechanics, polynomial asymptotics, and the role of m -phases. In the Lee-Yang theory, phase transitions occur when a physical quantity does not depend analytically on some control parameter. The logarithm of the partition function of the system, which depends on all possible configurations of the system, gives the singularities for phase transitions; that is, the zeros of the partition function determine phase transitions. For certain models [10], the partition function is given as a limit $(1/a_n) \ln p_n(x)$, as $n \rightarrow \infty$, where $\{p_n(x)\}$ is a sequence of polynomials and a_n is a sequence of real constants, typically $a_n = (1/n)$. Roughly, if this limit exists on a domain D , then D is almost a union of subdomains $R(j)$, $1 \leq j \leq m$, such that D is the union of $R(j)$ together with their boundaries in D and the limit on each $R(j)$ is a distinct analytic function (that is, they are not analytic continuations of each other). Each subdomain $R(j)$ represents distinct phases of the physical system. Recently, this phenomenon was discussed in much more generality in papers like [2].

To sum up, the question of phase transitions reduces to finding the limit $(1/a_n) \ln p_n(x)$ by using an appropriate asymptotic expansion

for $\{p_n(x)\}$. We have discovered that many polynomials from integer partition theory fall within this framework, such as $F_n(x)$ and $Q_m(x)$. Information such as the dominant term asymptotic are dependent on the m -phases and that the polynomial zeros accumulate on the boundaries between two distinct m -phases.

The problem of determining the m -phases of a sequence $\{L_k(x)\}$ becomes a question in complex analysis. In [3], with $L_k(x) = \sqrt{Li_2(x^k)}/k$ and D the punctured open unit disk, there are three distinct phases: $R(1)$, $R(2)$ and $R(3)$. In other words, $L_k(x) < L_j(x)$ for $x \in R(j)$, $j = 1, 2, 3$, and $k \geq 4$. We note that equality among $\Re L_1$, $\Re L_2$ or $\Re L_3$ occurs at the boundary of the phases $R(j)$.

The purpose of this paper is to determine the phases for the plane partition polynomials. We need to introduce some notation. Let $L_k(x) = (1/k)\sqrt[3]{Li_3(x^k)}$. Then for the level curve $\gamma: \Re L_1(x) = \Re L_2(x)$, with $\Im x \geq 0$, let θ^* be the solution of $\Re L_1(x) = \Re L_2(x)$ with $x = e^{i\theta^*}$, and let x^* be the solution along the negative real axis. Numerically, $x^* \approx -0.825\dots$ and $\theta^* \approx 0.951\pi$. Figure 1 shows the phases $R(1)$ and $R(2)$ inside the unit disk. We now state our main result.

Theorem 3. *Let D be the punctured open unit disk and $L_k(x) = (1/k)\sqrt[3]{2Li_3(x^k)}$. Then there exist exactly two nonempty phases $R(1)$ and $R(2)$. These phases have the following properties:*

- (1) *The portion of the phase $R(2)$ along the real axis is given by*

$$R(2) \cap (-1, 1) = (-1, x^*).$$

- (2) *$R(2)$ lies inside the open left half plane.*

- (3) *The boundaries of phases $R(1)$ and $R(2)$ consist of the level curve $\gamma(r)$, and its complex conjugate $\bar{\gamma}(r)$, which together represent the set $\{x \in D : \Re L_1(x) = \Re L_2(x)\}$. Both level curves intersect the real axis at $x^* \approx -0.825\dots$ and the unit circle at $\theta^* \approx \pm 0.951\pi$.*

We now give a brief outline of the sections of the paper. In Section 2, we review the conformal mapping properties of the trilogarithm and

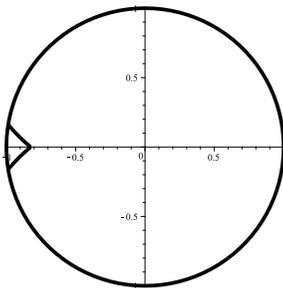


FIGURE 1. The region $R(2)$ is the smaller region enclosed by an arc of the unit circle, $\gamma(r)$ and $\bar{\gamma}(r)$. Its complement inside the unit disk is $R(1)$.

establish inequalities inside the unit disk for $Li_3(x)$ and $Li_3(x^2)$. In Section 3, we study the real part of $L_k(x)$ and establish their basic properties. Section 4 is devoted to the proof of our main theorem. We end the paper with several open problems in Section 5.

2. Classical analysis of the trilogarithm. We begin with analyzing the trilogarithm, defined by the function

$$Li_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3},$$

which is at the heart of $L_k(x)$'s behavior on the open unit disk. From classic Fourier analysis and geometric function theory there are a few theorems that are very useful in understanding the trilogarithm. If you are unfamiliar with any of these, [6] is a good book on geometric function theory and [7] is extremely helpful. We begin with the elementary theorems needed that we cite.

Theorem 4 (Fejer) ([7, page 513]). *For $n = 0, 1, 2, \dots$, let $r \in \mathbf{R}^+$ so that $r^n a_n > 0$ and $\Delta^4 r^n a_n > 0$. $\Delta a_n = a_{n+1} - a_n$ is known as the forward difference operator and $\Delta^{k+1} a_n = \Delta(\Delta^k a_n)$. Define the function*

$$f(x) = \sum_{n=1}^{\infty} a_n x^n.$$

If we let $\theta \in (-\pi, \pi]$, then $|f(re^{i\theta})|$ is a decreasing function for $0 < \theta < \pi$ and increasing on $-\pi < \theta < 0$.

For $0 < r \leq 1$, (r^n/n^3) has a positive fourth forward difference. Therefore the trilogarithm is monotone along circles of radius ≤ 1 .

Theorem 5 ([8, page 1]). *Li₃(x) is a star-shaped univalent map, i.e.,*

$$\Re\left(\frac{x(Li_3(x))'}{Li_3(x)}\right) > 0.$$

Likewise, star-shaped mappings have two useful properties [6, pages 41–42]. If we define $0 < r \leq 1$ and $\theta \in (-\pi, \pi]$, then

$$\frac{\partial \arg Li_3(re^{i\theta})}{\partial \theta} > 0, \quad \frac{\partial |Li_3(re^{i\theta})|}{\partial r} > 0.$$

The geometric properties of the polylogarithm have been studied in previous works, such as [8, 9]. Next, we need an expression for its derivative and the Quadratic Transformation.

Lemma 6 (Derivative and Quadratic Transformation) ([11, page 113]).

$$\frac{dLi_3(x)}{dx} = \frac{Li_2(x)}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

$$Li_3(x) + Li_3(-x) = 2^{-2}Li_3(x^2).$$

Last we need the trilogarithm’s behavior on the real line. The proof of this result is omitted because it is trivial.

Lemma 7. *The trilogarithm on the set $(-1, 1)$ is monotonic, maps positive axis to positive axis, and negative axis to negative axis.*

With that, we prove a few necessary nontrivial facts about the trilogarithm.

Lemma 8. *For $0 < x \leq 1$,*

$$Li_2(x)Li_3(-x) < Li_2(-x)Li_3(x).$$

Proof. Begin with the definitions of $Li_2(x)$ and $Li_3(x)$:

$$\begin{aligned} & Li_2(-x)Li_3(x) - Li_2(x)Li_3(-x) \\ &= \sum_{m=1}^{\infty} \frac{(-x)^m}{m^2} \sum_{n=1}^{\infty} \frac{x^n}{n^3} - \sum_{m=1}^{\infty} \frac{x^m}{m^2} \sum_{n=1}^{\infty} \frac{(-x)^n}{n^3}. \end{aligned}$$

With some trivial simplification, one gets

$$Li_2(-x)Li_3(x) - Li_2(x)Li_3(-x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x^{m+n}}{m^2 n^3} ((-1)^m - (-1)^n).$$

If n and m have the same parity, the relevant term vanishes. If n and m have opposite parity, $(-1)^m - (-1)^n$ will be ± 2 .

Therefore, we can state with more simplification that

$$\begin{aligned} & Li_2(-x)Li_3(x) - Li_2(x)Li_3(-x) \\ &= 2 \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} x^{2j+2l-1} \left(\frac{1}{(2j)^2(2l-1)^3} \right) \\ &\quad - 2 \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} x^{2j+2l-1} \left(\frac{1}{(2l-1)^2(2j)^3} \right) \\ &= 2 \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^{2j+2l-1}}{(2j)^3(2l-1)^3} (2j-2l+1) \\ &= 2^{-2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^{2(j+l)-1}}{j^3(2l-1)^3} (2(j-l)+1). \end{aligned}$$

We now reindex by taking $(j, l) \rightarrow (r, r-j)$ through the standard bijection $j+l=r$ with $j=1 \dots r-1$ and $r=2 \dots \infty$.

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^{2(j+l)-1}}{j^3(2l-1)^3} (2(j-l)+1) &= \sum_{r=2}^{\infty} x^{2r-1} \sum_{j=1}^{r-1} \frac{(2(2j-r)+1)}{j^3(2(r-j)-1)^3} \\ &= \sum_{r=2}^{\infty} x^{2r-1} A_r. \end{aligned}$$

If we demonstrate that $A_r > 0$ for every r , we will show the expression is a power series with all positive coefficients. These are trivially positive

when x is positive and therefore we will be done. Thus, one sees this by examining

$$A_r = \sum_{j=1}^{r-1} \frac{(2(2j-r)+1)}{[j(2(r-j)-1)]^3} = \sum_{j=1}^{r-1} \frac{a_{j,r}}{b_{j,r}^3}$$

$$a_{j,r} = (2(j - [r-j]) + 1)$$

$$b_{j,r} = j(2[r-j] - 1).$$

Now observe the following facts for $1 \leq j < (r/2)$. First, $a_{j,r} + a_{r-j,r} = 2$. Next, $b_{j,r} > b_{r-j,r} > 0$. Then, $a_{r/2,r} > 0$. Last, $a_{j-r,r} > 0$. Using a term δ_r which is one if r is even and zero if r is odd, we can sum by pairing the first term with the last term, the second term with penultimate term, and so on to get

$$\begin{aligned} A_r &= \left(\frac{a_{1,r}}{b_{1,r}^3} + \frac{a_{r-1,r}}{b_{r-1,r}^3} \right) + \left(\frac{a_{2,r}}{b_{2,r}^3} + \frac{a_{r-2,r}}{b_{r-2,r}^3} \right) \dots \\ &= \sum_{j=1}^{\lfloor (r-1)/2 \rfloor} \left(\frac{a_{j,r}}{b_{j,r}^3} + \frac{a_{r-j,r}}{b_{r-j,r}^3} \right) + \delta_r \frac{a_{r/2,r}}{b_{r/2,r}^3} \\ &= \sum_{j=1}^{\lfloor (r-1)/2 \rfloor} \left(\frac{-a_{j-r,r}}{b_{j,r}^3} + \frac{2}{b_{j,r}^3} + \frac{a_{r-j,r}}{b_{r-j,r}^3} \right) + \delta_r \frac{a_{r/2,r}}{b_{r/2,r}^3} \\ &= \sum_{j=1}^{\lfloor (r-1)/2 \rfloor} \left(a_{j-r,r} \left(\frac{1}{b_{r-j,r}^3} - \frac{1}{b_{j,r}^3} \right) + \frac{2}{b_{j,r}^3} \right) + \delta_r \frac{a_{r/2,r}}{b_{r/2,r}^3} \\ &> \sum_{j=1}^{\lfloor (r-1)/2 \rfloor} \left(\frac{2}{b_{j,r}^3} \right) + \delta_r \frac{a_{r/2,r}}{b_{r/2,r}^3} \\ &> 0. \quad \square \end{aligned}$$

Lemma 9. For $0 < x \leq 1$, $(Li_3(x^2))/(Li_3(x))$ is a positive, increasing function of x .

Proof. Positivity is a direct corollary of Lemma 7. To prove that the ratio is increasing, apply the Quadratic Transformation to simplify the problem:

$$\frac{1}{4}Li_3(x^2) = Li_3(x) + Li_3(-x).$$

$$\frac{Li_3(x^2)}{Li_3(x)} = 4 \frac{Li_3(x) + Li_3(-x)}{Li_3(x)} = 4 \left(1 + \frac{Li_3(-x)}{Li_3(x)} \right).$$

Therefore showing that $Li_3(-x)/Li_3(x)$ is increasing proves our lemma. Taking the derivative shows

$$\left(\frac{Li_3(-x)}{Li_3(x)} \right)' = \frac{Li_2(-x)Li_3(x) - Li_2(x)Li_3(-x)}{xLi_3(x)^2}.$$

And, because of Lemma 8 and $x > 0$, $Li_2(-x)Li_3(x) > Li_2(x)Li_3(-x)$ and we are done. \square

If we observe that $\lim_{x \rightarrow 0} Li_3(x^2)/Li_3(x) = 0$ and $\lim_{x \rightarrow 1} Li_3(x^2)/Li_3(x) = 1$ the next fact follows.

Theorem 10. *There exists a unique $0 < r < 1$ such that, for $r < x < 1$,*

$$Li_3(x^2) > \frac{4}{5}Li_3(x)$$

for $r > x > 0$,

$$Li_3(x^2) < \frac{4}{5}Li_3(x)$$

and $x = r$,

$$Li_3(x^2) = \frac{4}{5}Li_3(x).$$

3. Facts about $\Re L_k(x)$. Because the notation of $\Re L_k(x)$ gets rather bulky and crude, we trim it by working with the function $f_k(x)$ defined as

$$f_k(x) = \frac{1}{k} \Re \sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3, \dots$$

The cube root here is the principal branch on $(-\pi, \pi]$. Thus $f_k(x)$ can be written in polar form as

$$f_k(x) = \frac{1}{k} |Li_3(x^k)|^{1/3} \cos \left[\frac{\arg Li_3(x^k)}{3} \right], \quad |\arg Li_3(x^k)| \leq \pi.$$

Note $f_k(x)$ and $\Re L_k(x)$ differ by a positive constant multiplication, thus $\Re L_k(x) > \Re L_{k'}(x)$ if and only if $f_k(x) > f_{k'}(x)$. With those definitions, the following statements are proven just by inspection.

Lemma 11. *Define*

$$f_k(x) = \frac{1}{k} \Re \sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3 \dots$$

Then,

- (1) $f_k(x) = f_k(\bar{x})$,
- (2) $f_k(x) = (1/k)f_1(x^k)$,
- (3) $f_k(x) \geq (1/k)|Li_3(x^k)|^{1/3} \cos(\pi/3) \geq 0$
- (4) $f_k(x) = (1/k)|Li_3(x^k)|^{1/3} \cos(\pi/3)$ if and only if $x^k \leq 0$.

Lemma 12. *Define D as the open punctured unit disk, and let $x \in \bar{D}$. Define*

$$f_k(x) = \frac{1}{k} \Re \sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3 \dots$$

Then $f_k(x)$ is harmonic inside the unit disk minus the set $\{x : x^k \leq 0\}$. Furthermore, the maximum modulus principle applies, i.e., for $k \neq j$,

$$\sup_{x \in D} [f_k(x) - f_j(x)] = \sup_{\{|x|=1\} \cup \{x^k \leq 0\} \cup \{x^j \leq 0\}} [f_k(x) - f_j(x)],$$

and this supremum can only be attained on $\{|x| = 1\} \cup \{x^k \leq 0\} \cup \{x^j \leq 0\}$.

We remark from [1, pages 245–247] that if two functions $f_i(x)$, $f_j(x)$ are harmonic on D , then $\max(f_i(x), f_j(x))$ is subharmonic on D and Lemma 12 still applies to $\max(f_i(x), f_j(x))$.

Proposition 13. *For $0 < r \leq 1$ and for $0 < \arg((e^{i\theta})^k) < \pi$, $f_k(re^{i\theta})$ is a decreasing function of θ and for $-\pi < \arg((e^{i\theta})^k) < 0$ $f_k(re^{i\theta})$ is an increasing function of θ .*

Proof. Write $f_k(x)$ in its polar form:

$$f_k(x) = \frac{1}{k} |Li_3(x^k)|^{1/3} \cos\left(\frac{\arg Li_3(x^k)}{3}\right).$$

Recall that Theorem 5 demonstrates $\arg Li_3(re^{i\theta})$ is increasing with θ . Therefore $\cos(\arg Li_3(re^{i\theta})/3) \geq \cos(\pi/3) > 0$ and $\cos(\arg Li_3(re^{i\theta})/3)$ is going to be decreasing on $\theta \in (0, \pi)$. Combining these facts with Theorem 4 and $f_k(x)$ will be a composition of positive decreasing functions when $(e^{i\theta})^k$ maps to the upper half plane. And, because $f_k(\bar{x}) = f_k(x)$, we attain the result for the lower half plane. \square

Corollary 14.

$$\frac{1}{k}f_1(-r^k) \leq f_k(re^{i\theta}) \leq \frac{1}{k}f_1(r^k) \leq \frac{1}{k}f_1(r).$$

As much of this paper suggests, we will be comparing these $f_k(x)$ functions on the open punctured unit disk. However, a big nuisance for us is that the maximum modulus principle requires us to check points along rays at which $\arg(x^k) = \pi$. This can become quite tedious for comparing, say $f_1(x)$ and $f_{200}(x)$, where one technically has to check about 200 rays. But, with this infrastructure in place, we may now prove that checking any ray where $\arg(x^k) = \pi$ other than $\arg(x) = \pi$ is redundant.

Lemma 15. *For every $k \in \mathbf{N}$, $k > 1$, $0 < r \leq 1$ and $\theta \in 2\pi\mathbf{Q}$ such that $(e^{i\theta})^k = -1$, then*

$$f_1(re^{i\theta}) > f_k(re^{i\theta}).$$

Proof. Recall that $Li_3(x)$ maps negative axis to negative axis and is star shaped thus radially increasing in modulus. Therefore,

$$f_1(-r) = |Li_3(-r)|^{1/3} \cos\left(\frac{\pi}{3}\right)$$

must be radially increasing on the negative axis. Thus, for $0 < r \leq 1$,

$$f_1(re^{i\theta}) \geq f_1(-r) \geq f_1(-r^k) > \frac{1}{k}f_1(-r^k) = f_k(re^{i\theta}). \quad \square$$

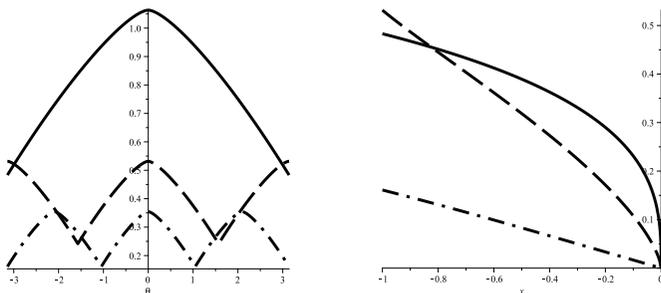


FIGURE 2. The two plots represent the curves $f_1(x)$, (solid) $f_2(x)$, (dashed) and $f_3(x)$ (dash-dotted) on the upper half unit circle (on the left) and negative real axis (on the right). As the proposition suggests, $f_1(x)$ dominates $f_k(x)$ for $k > 2$ on the unit circle and the maximum of $f_1(x)$ and $f_2(x)$ dominate $f_k(x)$ for $k > 2$. Also observe that $f_2(x) > f_1(x)$ on $(-1, x^*)$ and $f_1(x) > f_2(x)$ for $(x^*, 0)$.

4. Proof of main theorem. To prove our theorem, we prove these facts in the order that they are listed. First, we prove that there exist at most two nonempty phases $k = 1, 2$. Then, we prove two facts about these phases that serve as a basis for our analysis: $R(2) \cap (-1, 1) = (-1, x^*)$ and the right half disk is a subset of $R(1)$. Using these two facts we construct two level curves $\gamma(r)$ and $\bar{\gamma}(r)$ in the upper and lower half planes, respectively, that represent all points on the open punctured unit disk where $f_1(x) = f_2(x)$.

4.1. Nonexistence of phases with $k > 2$. We may now prove that there exists *at most* two phases.

Proposition 16. *Define D as the punctured open unit disk, and let $x \in \bar{D}$. Let*

$$f_k(x) = \frac{1}{k} \Re \sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3 \dots$$

For $k > 4/\sqrt[3]{6} = 2.201 \dots > 2$, $f_k(x) < \max(f_1(x), f_2(x))$.

Proof. If one shows that, for $k > 2$:

- (1) $f_k(x) < \max(f_1(x), f_2(x))$ on the unit circle, and
- (2) $f_k(-r) < \max(f_1(-r), f_2(-r))$ on $r \in (0, 1]$,

then Lemmas 12 and 15 demonstrate $f_k(x) < \max(f_1(x), f_2(x))$. Corollary 14 states that, for every $k \in \mathbf{N}$,

$$\frac{1}{k}f_1(-1) \leq f_k(e^{i\theta}) \leq f_k(1).$$

Therefore,

$$|Li_3(-1)|^{1/3} \cos \frac{\pi}{3} \leq f_1(e^{i\theta}) \leq |Li_3(1)|^{1/3}.$$

For those interested, $|Li_3(1)| = \zeta(3) = 1.20\dots$ is the so-called Apéry's Constant. Now we apply the Quadratic Transformation (Proposition 6) to the left hand side noting that the trilogarithm is positive on the positive axis and is negative on the negative axis. This proves

$$Li_3(1)^{1/3} \frac{\sqrt[3]{6}}{4} \leq f_1(e^{i\theta}) \leq Li_3(1)^{1/3}.$$

Now if we observe $f_k(x) = (1/k)f_1(x^k)$ and restrict $k > 2$, the string of inequalities now follows:

$$f_k(e^{i\theta}) \leq \frac{1}{k}Li_3(1)^{1/3} < \frac{\sqrt[3]{6}}{4}Li_3(1)^{1/3} \leq f_1(e^{i\theta}) \leq \max(f_1(e^{i\theta}), f_2(e^{i\theta})).$$

For (2), observe that $f_1(-r)$ and $f_1(r)$ are both positive increasing functions of r on $(0, 1)$. Split these into two cases $k = 2j$ and $k = 2j - 1$ with $j > 1$

$$f_1(-r) > f_1(-r^{(2j-1)}) > \frac{1}{(2j-1)}f_1((-r)^{(2j-1)}) = f_{2j-1}(-r),$$

and if $k = 2j$,

$$f_2(-r) = \frac{1}{2}f_1(r^2) > \frac{1}{2j}f_1((-r)^{2j}) > f_{2j}(-r). \quad \square$$

4.2. Distribution of phases on the real line and closed right half disk.

Proposition 17. *Define*

$$f_k(x) = \frac{1}{k} \Re \sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3, \dots$$

Then there exists a unique $-1 < x^* < 0$ so that $f_1(x^*) = f_2(x^*)$, $f_2(x) > f_1(x)$ for $-1 \leq x < x^*$, and $f_2(x) < f_1(x)$ for $x^* < x < 0$ and $0 < x < 1$.

Proof. Because of monotonicity on the positive real axis, if $0 < x < 1$, then

$$f_2(x) = \frac{1}{2}f_1(x^2) < f_1(x^2) < f_1(x).$$

Assume $0 < r < 1$, and write $x = -r$. Because $f_1(x)$ and $f_2(x)$ are uniformly nonnegative it is sufficient to work with comparing $f_1(x)^3$ and $f_2(x)^3$ rather than $f_1(x)$ and $f_2(x)$. We know that $Li_3(-r) < 0$ (Lemma 7). Via the use of the Quadratic Transformation (Proposition 6) we may use the polar form of $f_1(r)$ to write

$$f_1(-r)^3 = \cos\left(\frac{\arg(Li_3(-r))}{3}\right)^3 |2^{-2}Li_3(r^2) - Li_3(r)|,$$

$$f_1(-r)^3 = \cos\left(\frac{\pi}{3}\right)^3 (Li_3(r) - 2^{-2}Li_3(r^2)).$$

Since $r^2 > 0$, $Li_3(r^2) > 0$ (Lemma 7) we can then write $f_2(r)^3$ as

$$f_2(-r)^3 = 2^{-3}Li_3(r^2).$$

Write the difference of the two quantities:

$$f_1(-r)^3 - f_2(-r)^3 = \frac{1}{8}Li_3(r) - \frac{1}{4}Li_3(r^2)\left(\frac{1}{8} + \frac{1}{2}\right)$$

$$= \frac{4}{40}\left(\frac{4}{5}Li_3(r) - Li_3(r^2)\right).$$

Our proposition is now proved by Theorem 10. \square

Corollary 18. *The portion of the phase $R(2)$ along the real axis is given by $R(2) \cap (-1, 1) = (-1, x^*)$.*

Proposition 19. *Define*

$$f_k(x) = \frac{1}{k}\Re\sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3, \dots$$

For every $\theta \in [0, \pi/2]$ and $0 < r \leq 1$,

$$f_1(re^{i\theta}) > f_2(re^{i\theta}).$$

Proof. If one shows that

(1) $f_1(r) > f_2(r)$ for $r \in (0, 1]$,

(2) $f_1(ir) > f_2(ir)$ for $r \in (0, 1]$, and

(3) $f_1(e^{i\theta}) > f_2(e^{i\theta})$ (i.e., $f_1(e^{i\theta}) > \frac{1}{2}f_1(e^{i2\theta})$) for $\theta \in [0, (\pi/2)]$

then we have, by Lemma 12, that on the right quarter circle minus zero $f_1(x) > f_2(x)$. (1) is satisfied by Proposition 17. (2) is an application of Lemma 15

$$f_1(ir) > f_1(-r) \geq f_1(-r^2) > \frac{1}{2}f_1(-r^2) = f_2(ir).$$

(3) is satisfied because $f_1(e^{i\theta})$ is decreasing on the interval $[0, \pi]$. Thus, for $\theta \in [0, (\pi/2)]$,

$$f_2(e^{i\theta}) = \frac{1}{2}f_2(e^{i2\theta}) < \frac{1}{2}f_1(e^{i2\theta}) < f_1(e^{i2\theta}). \quad \square$$

The invariance principle $f_k(x) = f_k(\bar{x})$ allows us to conclude a second desired fact.

Corollary 20. *$R(2)$ lies inside the open left half plane.*

4.3. Existence of level curves $f_1(x) = f_2(x)$ in upper and lower left half plane and proof that they are the boundaries of $R(1)$ and $R(2)$.

Proposition 2.1. *Define*

$$f_k(x) = \frac{1}{k} \Re \sqrt[3]{Li_3(x^k)}, \quad k = 1, 2, 3 \dots$$

There exists $x^ < 0$ so that for every $0 < r < r^* := |x^*|$ and every θ , $f_2(re^{i\theta}) < f_1(re^{i\theta})$. However, for every $r^* \leq r \leq 1$, there exists exactly one $\theta(r) \in (\pi/2, \pi]$ satisfying*

$$f_1(re^{i\theta(r)}) = f_2(re^{i\theta(r)}).$$

If $\theta \in (\pi/2, \theta(r))$, then

$$f_2(re^{i\theta}) < f_1(re^{i\theta}).$$

If $\theta \in (\theta(r), \pi]$, then

$$f_1(re^{i\theta}) < f_2(re^{i\theta}).$$

Proof. Define

$$g(x) = f_1(x) - f_2(z) = f_1(x) - \frac{1}{2}f_1(x^2).$$

This lemma follows naturally from three observations on $\theta \in ((\pi/2), \pi]$.

(1) $f_1(re^{i\theta})$ is decreasing in θ (Proposition 13) and $f_2(re^{i\theta})$ is increasing in θ (Proposition 13) and therefore $g(re^{i\theta})$ is decreasing in θ .

(2) $g(ir)$ is positive for every $r \in (0, 1]$ (Proposition 19).

(3) $g(-r)$ is negative only on $r \in (r^*, 1]$, zero at r^* , and positive on $r \in (0, r^*)$ (Proposition 17). \square

To be clear, when we mention $[r, \theta(r)]$, these are polar coordinates that define a set in the upper left half plane. On this set, $f_1(re^{i\theta(r)}) = f_2(re^{i\theta(r)})$. The uniqueness/nonexistence of $\theta(r)$ for every r demonstrates that $\gamma(r) = [r, \theta(r)]$ is a bijection to the points in the punctured upper half disk where $f_1(re^{i\theta(r)}) = f_2(re^{i\theta(r)})$.

However, because $f_k(x) = f_k(\bar{x})$, there is an equivalent set, $\bar{\gamma}(r) = [r, -\theta(r)]$ which is a bijection to the points in the punctured lower half disk where $f_1(re^{i\theta(r)}) = f_2(re^{i\theta(r)})$. From this, we get our last fact about these phases.

Theorem 22. *If $R(1)$ denotes the first phase and $R(2)$ denotes the second phase on the domain D , then*

$$\partial R(1) = \partial R(2) = \gamma(r) \cup \bar{\gamma}(r).$$

Proof. Recall phases three and higher are empty (Proposition 16). Therefore, we may write

$$\begin{aligned} R(1) &= \{x \in D : f_1(x) > f_2(x)\}, \\ R(2) &= \{x \in D : f_2(x) > f_1(x)\}. \end{aligned}$$

Now $f_1(x)$ and $f_2(x)$ are continuous; therefore, we may state

$$\partial R(1) = \partial R(2) = \{x \in D : f_2(x) = f_1(x)\}.$$

But, as demonstrated in the previous comments,

$$\gamma(r) \bigcup \bar{\gamma}(r) = \{x \in D : f_2(x) = f_1(x)\}. \quad \square$$

5. Concluding open questions and remarks about the generalized setting. This paper has given an analysis of the phase structure of the plane partition polynomials. But, with the cases of Boyer and Goh, it should be clear that both cases relate to a possible generalization. This generalization appears to belong to a family of polynomials $Q_{n,s}(x)$ with $s > 1$ generated by

$$1 + \sum_{n=1}^{\infty} Q_{n,s}(x)u^n = \prod_{m=1}^{\infty} \frac{1}{(1 - xu^m)^{m^{s-2}}},$$

of which the phase structure is described by the sequence of functions given by the real parts of $\{\Re \sqrt[s]{\Gamma(s) Li_s(x^k)/k}\}$.

The reader should note that the case of Boyer and Goh is $s = 2$ and the case discussed here is $s = 3$. The particular case discussed here is relevant because of plane partition theory, while other cases may be more complex. If one lets $s > 2$, this paper can serve as a framework for doing phase calculations. Many theorems can be altered just by replacing 3's with s 's. If one alters these results, one can observe that, as s gets larger, $R(1)$ is getting larger at the expense of the other phases. This ends with the existence of a cutoff s^* (around π but not precisely π) such that for $s > s^*$, $R(1)$ is the only nonempty phase. However, there are potential problems involved with the general case. We present two open questions in the remarks below:

Remark 23. If one discusses the case of $1 < s < 2$, much of this analysis begins to break down, as the real component of the principal root is not uniformly positive. Furthermore, the number of distinct phases in this case appears to grow large, particularly as $s \rightarrow 1^+$. With both of these daunting challenges, the question arises as “How do you alter this analysis to deal with this situation?”

Remark 24. An important fact for calculating the zero distribution of these polynomials asymptotically is that $L'_k(x) \neq L'_j(x)$ for any $x \in D$. This fact is omitted in this paper and is proven in [3] when $s = 2$. Below is a theorem that suggests a method of proof.

Theorem 25. *We let $L_1(x)$ be a convex function and define $L_k(x) = (1/k)L_1(x^k)$ for k a positive integer. If x is in the open punctured unit disk, then for every $k \neq j$,*

$$L'_k(x) \neq L'_j(x).$$

Proof. Prove by contrapositive. As a consequence of Alexander’s theorem ([5, Theorem 2.12]), $xL'_1(x)$ is univalent. Therefore, we suppose there are k, j distinct positive integers so that $L'_k(x) = L'_j(x)$ for some $x \in \mathbf{C}$. Therefore, x also solves the equation $xL'_k(x) = xL'_j(x)$. And so, $xL'_k(x) = x^k L'_1(x^k)$. Thus,

$$x^k L'_1(x^k) = x^j L'_1(x^j).$$

If we demonstrate that $xL'_1(x)$ is univalent, then $x^k = x^j$. Therefore, x would be either zero or on the unit circle. \square

When $L_1(x) = \sqrt[3]{Li_3(x)}$, $L_1(x)$ of course is not analytic and thus not convex. However, the main analytic condition for convexity that the real part of $[1 + zL''_1(z)/L'_1(z)]$ is positive ([6, Theorem 2.11]) can be used as a substitute and we can still show that $xL'_1(x)$ is univalent. Further, it appears that this argument may hold for the general case.

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