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ESTIMATES OF LARGE EIGENVALUES AND TRACE FORMULA FOR THE VECTORIAL STURM-LIOUVILLE EQUATIONS

CHUAN-FU YANG, ZHEN-YOU HUANG AND XIAO-PING YANG

ABSTRACT. This paper describes the N-dimensional vectorial Sturm-Liouville problem with coupled boundary conditions. We first derive the asymptotic expressions of large eigenvalues for the vectorial Sturm-Liouville operator with smooth coefficients. In addition, the regularized trace formula for the operator is calculated with residue techniques in complex analysis. These formulae are then used to obtain some results of inverse eigenvalue problems in the spirit of Ambarzumyan.

1. Introduction. In a finite-dimensional space, an operator has a finite trace. But, in an infinite-dimensional space, ordinary differential operators do not have a finite trace (the sum of all eigenvalues). But Gelfand and Levitan [**20**] observed that the sum $\sum_n(\lambda_n - \mu_n)$ often makes sense, where $\{\lambda_n\}$ and $\{\mu_n\}$ are the eigenvalues of the "perturbed problem" and "unperturbed problem," respectively. The sum $\sum_n(\lambda_n - \mu_n)$ is called a regularized trace. Gelfand and Levitan first obtained an identity of trace for the Schrödinger operator [**20**].

For scalar Sturm-Liouville problems, there is an enormous amount of literature on estimates of large eigenvalues and the regularized trace formulae which may often be computed explicitly in terms of the coefficients of operators and boundary conditions [**3, 6, 16–18, 20, 22, 24–26, 29, 31–33**]. Their most important application is in solving inverse problems [**19, 23, 33, 37**], namely, given some spectral-related data, how do we reconstruct the unknown potential function?

As a generalization of scalar Sturm-Liouville equations, vectorial Sturm-Liouville equations were found to be important in the study

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of particle physics [**39**]. The vectorial Sturm-Liouville operators raise some interesting new problems. Trace formulae for the vectorial Sturm-Liouville problems were previously considered in [**22, 35**], and recently in [**7, 8**], and they established and analyzed trace formulae for the vectorial Hill's equation and one-dimensional vectorial Schrödinger equation

$$-\phi'' + P(x)\phi = \lambda\phi, \quad \phi \in \mathbf{C}^N,\ \lambda \in \mathbf{C},\ N \in \mathbf{N} \setminus \{0\},$$

which is subject to certain separated boundary conditions.

This paper addresses several related problems in the spectral theory of the N-dimensional vectorial Sturm-Liouville equation

$$-\phi'' + P(x)\phi = \lambda\phi, \quad 0 \le x \le \pi \tag{1.1}$$

with coupled self-adjoint boundary conditions

$$\phi(\pi) = \beta\phi(0), \quad \phi'(0) = \overline{\beta}\phi'(\pi), \tag{1.2}$$

where $P(x)$ is an $N \times N$ real symmetric differentiable matrix-valued function, $\phi(x)$ is a vector-valued function of length N, β is a complex number and $\overline{\beta}$ is the conjugate complex number of β.

Note that problems (1.1) and (1.2) is a self-adjoint eigenvalue problem, especially since $\beta = \pm 1$ (1.2) corresponds to periodic and semi-periodic boundary conditions, respectively. The spectrum for problems (1.1) and (1.2), which consists of eigenvalues, is real and can be determined by the variational principle. Counting multiplicities of the eigenvalues, we can arrange those eigenvalues as $\{\lambda_n^{(i)}\} \triangleq \{\lambda_{1,n}^{(i)}, i = 1, 2, \dots, N\}_{n=0}^{\infty} \bigcup \{\lambda_{2,n}^{(i)}, i = 1, 2, \dots, N\}_{n=1}^{\infty}$, in an ascending order as

$$\lambda_{1,0}^{(1)} \le \lambda_{1,0}^{(2)} \le \cdots \le \lambda_{1,0}^{(N)} \le \cdots \le \lambda_{1,n}^{(1)} \le \lambda_{1,n}^{(2)} \le \cdots \le \lambda_{1,n}^{(N)} \le \cdots,$$

and

$$\lambda_{2,1}^{(1)} \le \lambda_{2,1}^{(2)} \le \cdots \le \lambda_{2,1}^{(N)} \le \cdots \le \lambda_{2,n}^{(1)} \le \lambda_{2,n}^{(2)} \le \cdots \le \lambda_{2,n}^{(N)} \le \cdots.$$

The multiplicity of each eigenvalue of problems (1.1) and (1.2) is at most $2N$.

The main purpose of the present work is to obtain asymptotics of eigenvalues and trace formula for one-dimensional vectorial Schrödinger equations with coupled boundary conditions. In particular, the formulae presented here can be helpful in solving inverse problems. We end this paper with results in the spirit of Ambarzumyan.

2. Main results. Here we give some notational conventions. The imaginary part of λ is denoted by $\operatorname{Im}\lambda$. I_N is an $N\times N$ identity matrix, 0_N is a zero matrix and $\operatorname{tr} A$ denotes the trace of the matrix A. $\det A$ denotes the determinant of the matrix A and $\operatorname{rank} A$ denotes the rank of the matrix A.

Denote the sequence

(2.1)
$$\{\mu_n^{(i)}\} \triangleq \{\mu_{1,n}^{(i)}, i=1,2,\dots,N\}_{n=0}^{\infty} \bigcup \{\mu_{2,n}^{(i)}, i=1,2,\dots,N\}_{n=1}^{\infty} : \\ \mu_{1,n}^{(i)} = (2n+\theta)^2, \quad n=0,1,\dots, \\ \mu_{2,n}^{(i)} = (2n-\theta)^2, \quad n=1,2,\dots,\ i=1,2,\dots,N,$$

where θ satisfies

$$\cos(\theta\pi) = \frac{\beta+\overline{\beta}}{|\beta|^2+1}, \quad 0\le\theta\le 1. \tag{2.2}$$

In the case $P(x)=0_N$ on $[0,\pi]$, we can easily calculate the eigenvalues of problems (1.1) and (1.2) are $\{(2n+\theta)^2\}_{n=0}^{\infty}\bigcup\{(2n-\theta)^2\}_{n=1}^{\infty}$, and each of the eigenvalues is of at least multiplicity N. In particular, when $\theta=0$, the eigenvalues of problem (1.1) and (1.2) are $\{4n^2 : n=0,1,2,\dots\}$, and each of the eigenvalues is of multiplicity $2N$ except that eigenvalue 0 is of multiplicity N; when $\theta=1$, the eigenvalues of problems (1.1) and (1.2) are $\{(2n-1)^2 : n=1,2,\dots\}$, and each of the eigenvalues is of multiplicity $2N$; when $\theta\neq 0,1$, each of the eigenvalues is of multiplicity N exactly.

The main theorems of this paper are as follows.

Theorem 2.1 (Eigenvalue asymptotics). *Suppose that $P(x)$ is a real symmetric differentiable matrix-valued function, and let $\{\lambda_n^{(i)}\}\triangleq$*

$\{\lambda_{1,n}^{(i)}\}_{n=0}^{\infty}\bigcup\{\lambda_{2,n}^{(i)}\}_{n=1}^{\infty}$ be the sequence of eigenvalues of problem (1.1) *and* (1.2)*. Then, for sufficiently large n, the eigenvalue has the following asymptotic expressions*

$$\sqrt{\lambda_{1,n}^{(i)}} = 2n+\theta+\frac{p_i}{2\pi(2n+\theta)}+o\Big(\frac{1}{n}\Big), \tag{2.3}$$

and

$$\sqrt{\lambda_{2,n}^{(i)}} = 2n-\theta+\frac{p_i}{2\pi(2n-\theta)}+o\Big(\frac{1}{n}\Big), \tag{2.4}$$

where p_i, $i=1,2,\dots,N$, are the characteristic values of the $N\times N$ real symmetric matrix $\int_0^\pi P(x)\,dx$, and θ is defined by (2.2)*.*

Theorem 2.2 (Trace formula). *Suppose that $P(x)$ is a real symmetric differentiable matrix-valued function, and let $\{\lambda_n^{(i)}\}\triangleq\{\lambda_{1,n}^{(i)}\}_{n=0}^{\infty}\bigcup\{\lambda_{2,n}^{(i)}\}_{n=1}^{\infty}$ be the sequence of eigenvalues of problem* (1.1) *and* (1.2)*. Then, there is the identity of trace*

$$\begin{aligned}\sum_{n=0}^{\infty}\Big[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-\mu_{1,n}^{(i)})-\frac{1}{\pi}\mathrm{tr}\int_0^\pi P(x)\,dx\Big]&\\ +\sum_{n=1}^{\infty}\Big[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-\mu_{2,n}^{(i)})-\frac{1}{\pi}\mathrm{tr}\int_0^\pi P(x)\,dx\Big]&\\ =\frac{|\beta|^2-1}{|\beta|^2+1}\mathrm{tr}\,\frac{P(\pi)-P(0)}{4}&,\end{aligned} \tag{2.5}$$

where $\mu_n^{(i)}$ is defined by (2.1)*.*

Theorem 2.3 (Ambarzumyan-type theorem). *Suppose that $P(x)$ is a real symmetric continuous matrix-valued function, and let $\{\lambda_n^{(i)}\}$ be the sequence of eigenvalues of problem* (1.1) *and* (1.2)*. In addition, either $\{(2m_j+\theta)^2 : j=1,2,\dots\}\bigcup\{\theta^2\}$ or $\{(2m_j-\theta)^2 : j=1,2,\dots\}\bigcup\{\theta^2\}$ is a subset of the spectrum $\{\lambda_n^{(i)}\}$ of eigenvalue problem* (1.1) *and* (1.2)*. The multiplicity of each eigenvalue $(2m_j+\theta)^2$ or $(2m_j-\theta)^2$ is N* (*in the case $\theta\neq 0,1$*)*, $2N$* (*in the case $\theta=0,1$*)*, where θ^2 is the first eigenvalue* *of problem* (1.1) *and* (1.2), m_j *is a strictly ascending infinite sequence of positive integers.*

(a) *If* $\beta = 1$ (*in this case* $\theta = 0$), *then*

$$P(x) = 0_N \quad \text{on} \quad [0, \pi];$$

(b) *If* $\beta = -1$ (*in this case* $\theta = 1$). *Then* $\int_0^\pi (P(x))_{ii} \cos(2x)\, dx = 0$, $i = 1, 2, \dots, N$, *implies*

$$P(x) = 0_N \quad \text{on} \quad [0, \pi];$$

(c) *If* $\beta \neq \pm 1$ (*in this case* $\theta \neq 0, 1$). *Then*

$$\int_0^\pi (P(x))_{ii} \cos(2\theta x)\, dx = \int_0^\pi (P(x))_{ii} \sin(2\theta x)\, dx = 0, \quad 1 \le i \le N, \tag{2.6}$$

implies

$$P(x) = 0_N \quad \textit{on} \quad [0, \pi],$$

where θ *is defined by* (2.2) *and* $(P(x))_{ii}$ *denotes the entry of the matrix* $P(x)$ *at the ist row and ist array,* $i = 1, 2, \dots, N$.

3. The eigenvalue equation. This section gives an analysis of the eigenvalue equation for problems (1.1) and (1.2) with the help of asymptotic expansions for solutions of (1.1), similar to well-known techniques from the scalar case.

In the space $L_N^2[0, \pi] =: \oplus_{i=1}^N L^2[0, \pi]$, define the inner product and norm

$$(f, g) = \int_0^\pi g^{*t}(x) f(x)\, dx = \sum_{j=1}^N \int_0^\pi f_j(x) \overline{g_j}(x)\, dx,$$

$$||f|| = \left(\sum_{j=1}^N \int_0^\pi |f_j(x)|^2 dx \right)^{1/2}, \quad \text{for all } f, g \in L_N^2[0, \pi],$$

where $f = (f_1, \dots, f_N)^t$, $g = (g_1, \dots, g_N)^t$, and $g^{*t}(x)$ denotes the conjugate transpose of $N \times 1$ vector $g(x)$.

Suppose $P(x)$ is an $N \times N$ symmetric, differentiable matrix-valued function. Let $\Phi_1(x,\lambda)$ satisfy the matrix differential equation

$$\begin{cases} -Y'' + P(x)Y = \lambda Y \\ Y(0) = I_N,\ Y'(0) = 0_N. \end{cases} \tag{3.1}$$

Then, by [**12**], the solution $\Phi_1(x,\lambda)$ can be expressed as

$$\Phi_1(x,\lambda) = \cos(\sqrt{\lambda}x)I_N + \int_0^x K(x,t)\cos(\sqrt{\lambda}t)\,dt, \tag{3.2}$$

where $K(x,t)$ is a symmetric matrix-valued function whose entries have continuous partial derivatives up to order two with respect to t and x. Similarly, let $\Phi_2(x,\lambda)$ satisfy the matrix differential equation

$$\begin{cases} -Y'' + P(x)Y = \lambda Y \\ Y(0) = 0_N,\ Y'(0) = I_N. \end{cases} \tag{3.3}$$

Then the solution $\Phi_2(x,\lambda)$ can be expressed as

$$\Phi_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}I_N + \int_0^x L(x,t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,dt, \tag{3.4}$$

where $L(x,t)$ is a symmetric matrix-valued function whose entries have continuous partial derivatives up to order two with respect to t and x. Furthermore, the kernels of transformation $K(x,t)$ and $L(x,t)$ satisfy the following partial differential equations [**12**]:

(3.5)

$$K''_{xx} - P(x)K = K''_{tt}, \quad K(x,x) = \frac{1}{2}\int_0^x P(x)\,dx, \quad K'_t(x,0) = 0_N;$$

$$L''_{xx} - P(x)L = L''_{tt}, \quad L(x,x) = \frac{1}{2}\int_0^x P(x)\,dx, \quad L(x,0) = 0_N.$$

When $P(x) \in C^1$, (3.5) can be written as Volterra integral equations

$$\begin{aligned} K(x,t) &= \frac{1}{2}\Big[\int_0^{(x+t)/2} P(x)\,dx + \int_0^{(x-t)/2} P(x)\,dx\Big] \\ &\quad + \int_0^{(x-t)/2} d\tau \int_\tau^{(x+t)/2} P(\sigma+\tau)K(\sigma+\tau,\sigma-\tau)\,d\sigma, \\ L(x,t) &= \frac{1}{2}\Big[\int_0^{(x+t)/2} P(x)\,dx - \int_0^{(x-t)/2} P(x)\,dx\Big] \\ &\quad + \int_0^{(x-t)/2} d\tau \int_{(x-t)/2}^{(x+t)/2} P(\sigma+\tau)L(\sigma+\tau,\sigma-\tau)\,d\sigma, \end{aligned} \tag{3.6}$$

which are solvable. By (3.6), a direct computation implies

$$
\begin{aligned}
\frac{\partial K(x,x)}{\partial t} &= \frac{P(x)-P(0)}{4} - \frac{[\int_0^x P(x)\,dx]^2}{8},\\
\frac{\partial K(x,x)}{\partial x} &= \frac{P(x)+P(0)}{4} + \frac{[\int_0^x P(x)\,dx]^2}{8},\\
\frac{\partial L(x,x)}{\partial t} &= \frac{P(x)+P(0)}{4} - \frac{[\int_0^x P(x)\,dx]^2}{8},\\
\frac{\partial L(x,x)}{\partial x} &= \frac{P(x)-P(0)}{4} + \frac{[\int_0^x P(x)\,dx]^2}{8}.
\end{aligned}
\tag{3.7}
$$

Using integration by parts, by (3.2), (3.5) and (3.7) we can compute $\Phi_1(\pi,\lambda)$ as

$$
\begin{aligned}
\Phi_1(\pi,\lambda) &= \cos(\sqrt{\lambda}\pi)I_N + K(\pi,\pi)\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\\
&\quad - \frac{1}{\sqrt{\lambda}}\int_0^\pi K'_t(\pi,t)\sin(\sqrt{\lambda}t)\,dt\\
&= \cos(\sqrt{\lambda}x)I_N + K(\pi,\pi)\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\\
&\quad + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda}K'_t(\pi,\pi)\\
&\quad - \frac{1}{\lambda}\int_0^\pi K''_{tt}(\pi,t)\cos(\sqrt{\lambda}t)\,dt\\
&= \cos(\sqrt{\lambda}\pi)I_N + \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}\int_0^\pi P(t)\,dt\\
&\quad + \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}(P(\pi)-P(0))\\
&\quad - \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}\left(\int_0^\pi P(t)\,dt\right)^2\\
&\quad - \frac{1}{\lambda}\int_0^\pi K''_{tt}(\pi,t)\cos(\sqrt{\lambda}t)\,dt,
\end{aligned}
\tag{3.8}
$$

and

(3.9)

$$
\Phi'_1(\pi,\lambda) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)I_N + K(\pi,\pi)\cos(\sqrt{\lambda}\pi)
$$

$$
\begin{aligned}
&+\int_0^\pi K_x'(\pi,t)\cos(\sqrt{\lambda}t)\,dt\\
&= -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)I_N + K(\pi,\pi)\cos(\sqrt{\lambda}\pi)\\
&+\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}K_x'(\pi,\pi)\\
&-\frac{1}{\sqrt{\lambda}}\int_0^\pi K_{xt}''(\pi,t)\sin(\sqrt{\lambda}t)\,dt\\
&= -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)I_N + \frac{1}{2}\cos(\sqrt{\lambda}\pi)\int_0^\pi P(t)\,dt\\
&+\frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda}}(P(\pi)+P(0))\\
&+\frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda}}\bigg(\int_0^\pi P(t)\,dt\bigg)^2\\
&-\frac{1}{\sqrt{\lambda}}\int_0^\pi K_{xt}''(\pi,t)\sin(\sqrt{\lambda}t)\,dt.
\end{aligned}
$$

Similarly,

(3.10)

$$
\begin{aligned}
\Phi_2(\pi,\lambda) &= \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}I_N - \frac{\cos(\sqrt{\lambda}\pi)}{\lambda}L(\pi,\pi)\\
&\quad+\frac{1}{\lambda}\int_0^\pi L_t'(\pi,t)\cos(\sqrt{\lambda}t)\,dt\\
&= \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}I_N - \frac{\cos(\sqrt{\lambda}\pi)}{\lambda}L(\pi,\pi)\\
&\quad+\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda^3}}L_t'(\pi,\pi)\\
&\quad-\frac{1}{\sqrt{\lambda^3}}\int_0^\pi L_{tt}''(\pi,t)\sin(\sqrt{\lambda}t)\,dt\\
&= \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}I_N - \frac{\cos(\sqrt{\lambda}\pi)}{2\lambda}\int_0^\pi P(t)\,dt\\
&\quad+\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda^3}}\bigg(\frac{P(\pi)+P(0)}{4}-\frac{(\int_0^\pi P(t)\,dt)^2}{8}\bigg)\\
&\quad-\frac{1}{\sqrt{\lambda^3}}\int_0^\pi L_{tt}''(\pi,t)\sin(\sqrt{\lambda}t)\,dt,
\end{aligned}
$$

and

(3.11)

$$
\begin{aligned}
\Phi_2'(\pi,\lambda) &= \cos(\sqrt{\lambda}\pi)I_N + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}L(\pi,\pi) \\
&\quad + \int_0^\pi L_x'(\pi,t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,dt \\
&= \cos(\sqrt{\lambda}\pi)I_N + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}L(\pi,\pi) \\
&\quad - \frac{\cos(\sqrt{\lambda}\pi)}{\lambda}L_x'(\pi,\pi) \\
&\quad + \frac{1}{\lambda}\int_0^\pi L_{xt}''(\pi,t)\cos\sqrt{\lambda}t\,dt \\
&= \cos(\sqrt{\lambda}\pi)I_N + \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}\int_0^\pi P(t)\,dt \\
&\quad - \frac{\cos(\sqrt{\lambda}\pi)}{\lambda}\left(\frac{P(\pi)-P(0)}{4}\right) \\
&\quad - \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}\left(\int_0^\pi P(t)\,dt\right)^2 \\
&\quad + \frac{1}{\lambda}\int_0^\pi L_{xt}''(\pi,t)\cos(\sqrt{\lambda}t)\,dt.
\end{aligned}
$$

From (3.1) and (3.3), it is easy to verify that Wronskian determinant

$$\det\begin{pmatrix}\Phi_1(x,\lambda) & \Phi_2(x,\lambda)\\ \Phi_1'(x,\lambda) & \Phi_2'(x,\lambda)\end{pmatrix}$$

is a constant independent of x. From

$$\det\begin{pmatrix}\Phi_1(0,\lambda) & \Phi_2(0,\lambda)\\ \Phi_1'(0,\lambda) & \Phi_2'(0,\lambda)\end{pmatrix} = 1,$$

we get that $(\Phi_1(x,\lambda), \Phi_2(x,\lambda))$ is the fundamental matrix of solutions of equation (1.1). Therefore, the general solutions of equation (1.1) have the form

$$\phi(x,\lambda) = (\Phi_1(x,\lambda), \Phi_2(x,\lambda))C,$$

where $C = (c_1, c_2, \dots, c_{2N})^t$, $c_k \in \mathbf{C}$, $k = 1, 2, \dots, 2N$, and A^t denotes transpose of the matrix A.

If $\phi(x,\lambda) = (\Phi_1(x,\lambda), \Phi_2(x,\lambda))C$ is a nontrivial solution of problems (1.1) and (1.2), then there exists a non-vanishing vector C satisfying the matrix equation

$$\begin{pmatrix} \beta I_N - \Phi_1(\pi,\lambda) & -\Phi_2(\pi,\lambda) \\ -\overline{\beta}\Phi_1'(\pi,\lambda) & I_N - \overline{\beta}\Phi_2'(\pi,\lambda) \end{pmatrix} C = 0.$$

Therefore, λ is an eigenvalue of problems (1.1) and (1.2) if and only if the matrix

$$W(\lambda) = \begin{pmatrix} \beta I_N - \Phi_1(\pi,\lambda) & -\Phi_2(\pi,\lambda) \\ -\overline{\beta}\Phi_1'(\pi,\lambda) & I_N - \overline{\beta}\Phi_2'(\pi,\lambda) \end{pmatrix} \tag{3.12}$$

is singular. Furthermore, the multiplicity of λ is equal to $2N$-$\operatorname{rank} W(\lambda)$. Thus, λ is an eigenvalue of multiplicity $2N$ of problems (1.1) and (1.2) if $W(\lambda)$ is a zero-matrix.

Since the matrix $\int_0^\pi P(t)\,dt$ is real symmetric, there exists an orthogonal matrix U, $U^{-1} = U^*$, such that

$$U^*\bigg(\int_0^\pi P(t)\,dt\bigg)U = \operatorname{diag}[p_1, p_2, \dots, p_N], \tag{3.13}$$

where p_i, $1 \le i \le N$, are those characteristic values of the constant matrix $\int_0^\pi P(t)\,dt$. In this case,

$$U^*\bigg(\int_0^\pi P(t)\,dt\bigg)^2 U = \operatorname{diag}[p_1^2, p_2^2, \dots, p_N^2]. \tag{3.14}$$

Denote

$$\begin{aligned} &(U^*(P(\pi) - P(0))U)_{ii} = a_i, \\ &(U^*(P(\pi) + P(0))U)_{ii} = b_i, \quad 1 \le i \le N, \end{aligned}$$

and

$$\begin{aligned} U^*(P(\pi) - P(0))U - \operatorname{diag}[a_1, \dots, a_N] = A, \\ U^*(P(\pi) + P(0))U - \operatorname{diag}[b_1, \dots, b_N] = B. \end{aligned}$$

Obviously, the diagonal entries in the constant matrices A and B are zero. Combining (3.8)–(3.11), (3.13) and (3.14), we have

$$
\begin{aligned}
(3.15)\quad & U^*(\Phi_1(\pi,\lambda))U \\
&= \operatorname{diag}\left[\cos(\sqrt{\lambda}\pi)+\frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i+\frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i-\frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\right] \\
&\quad -\frac{1}{\lambda}\int_0^\pi U^*K''_{tt}(\pi,t)U\cos(\sqrt{\lambda}t)\,dt+\frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}A,
\end{aligned}
$$

$$
\begin{aligned}
(3.16)\quad & U^*\Phi_1'(\pi,\lambda)U \\
&= \operatorname{diag}\left[-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)+\frac{\cos(\sqrt{\lambda}\pi)}{2}p_i+\frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda}}b_i+\frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda}}p_i^2\right] \\
&\quad -\frac{1}{\sqrt{\lambda}}\int_0^\pi U^*K''_{xt}(\pi,t)U\sin(\sqrt{\lambda}t)\,dt+\frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda}}B,
\end{aligned}
$$

$$
\begin{aligned}
(3.17)\quad & U^*\Phi_2(\pi,\lambda)U \\
&= \operatorname{diag}\left[\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}-\frac{\cos(\sqrt{\lambda}\pi)}{2\lambda}p_i+\frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda^3}}b_i-\frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda^3}}p_i^2\right] \\
&\quad -\frac{1}{\sqrt{\lambda^3}}\int_0^\pi U^*L''_{tt}(\pi,t)U\sin(\sqrt{\lambda}t)\,dt+\frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda^3}}B,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\quad & U^*\Phi_2'(\pi,\lambda)U \\
&= \operatorname{diag}\left[\cos(\sqrt{\lambda}\pi)+\frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i-\frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i-\frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\right] \\
&\quad +\frac{1}{\lambda}\int_0^\pi U^*L''_{xt}(\pi,t)U\cos(\sqrt{\lambda}t)dt-\frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}A.
\end{aligned}
$$

To reach our goal, let us get back to (3.12). It follows that

$$
\begin{aligned}
(3.19)\quad & \det W(\lambda) \\
&= \det\begin{pmatrix} U^* & 0_N \\ 0_N & U^*\end{pmatrix}\begin{pmatrix}\beta I_N-\Phi_1(\pi,\lambda) & -\Phi_2(\pi,\lambda) \\ -\overline{\beta}\Phi_1'(\pi,\lambda) & I_N-\overline{\beta}\Phi_2'(\pi,\lambda)\end{pmatrix}\begin{pmatrix} U & 0_N \\ 0_N & U\end{pmatrix} \\
&= \det\begin{pmatrix}\beta I_N-U^*\Phi_1(\pi,\lambda)U & -U^*\Phi_2(\pi,\lambda)U \\ -\overline{\beta}U^*\Phi_1'(\pi,\lambda)U & I_N-\overline{\beta}U^*\Phi_2'(\pi,\lambda)U\end{pmatrix}.
\end{aligned}
$$

Also, from (3.15) and (3.18), we have

$$
\begin{aligned}
(3.20)\quad & \beta I_N - U^*\Phi_1(\pi,\lambda)U \\
&= \operatorname{diag}\left[\beta - \cos(\sqrt{\lambda}\pi) - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i + \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\right] \\
&\quad + \frac{1}{\lambda}\int_0^\pi U^*K''_{tt}(\pi,t)U\cos(\sqrt{\lambda}t)\,dt - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}A,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.21)\quad & I_N - \overline{\beta}U^*\Phi_2'(\pi,\lambda)U \\
&= \operatorname{diag}\left[1 - \overline{\beta}\left(\cos(\sqrt{\lambda}\pi) + \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i - \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\right)''\right] \\
&\quad - \frac{\overline{\beta}}{\lambda}\int_0^\pi U^*L''_{xt}(\pi,t)U\cos(\sqrt{\lambda}t)\,dt + \frac{\overline{\beta}\cos(\sqrt{\lambda}\pi)}{4\lambda}A.
\end{aligned}
$$

Thus, the matrix

$$
\begin{pmatrix} \beta I_N - U^*\Phi_1(\pi,\lambda)U & -U^*\Phi_2(\pi,\lambda)U \\ -\overline{\beta}U^*\Phi_1'(\pi,\lambda)U & I_N - \overline{\beta}U^*\Phi_2'(\pi,\lambda)U \end{pmatrix}
$$

can be written as the sum of two $2N \times 2N$ matrices $T = (t_{ij})$ and $S = (s_{ij})$, where
(3.22)

$$
T = \begin{pmatrix} \operatorname{diag}[t_{1,1},\dots,t_{N,N}] & \operatorname{diag}[t_{1,N+1},\dots,t_{N,2N}] \\ \operatorname{diag}[t_{N+1,1},\dots,t_{2N,N}] & \operatorname{diag}[t_{N+1,N+1},\dots,t_{2N,2N}] \end{pmatrix},
$$

$$
t_{ij} = \begin{cases} \beta - \cos\sqrt{\lambda}\pi - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i + \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2 \\ \qquad \text{for } i = j = 1,2,\dots,N; \\ \overline{\beta}\left(\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) - \frac{\cos(\sqrt{\lambda}\pi)}{2}p_i - \frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda}}b_i - \frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda}}p_i^2\right) \\ \qquad \text{for } i - N = j = 1,2,\dots,N; \\ -\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} + \frac{\cos(\sqrt{\lambda}\pi)}{2\lambda}p_i - \frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda^3}}b_i + \frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda^3}}p_i^2 \\ \qquad \text{for } i = j - N = 1,2,\dots,N; \\ 1 - \overline{\beta}\left(\cos(\sqrt{\lambda}\pi) + \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i - \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\right) \\ \qquad \text{for } i - N = j - N = 1,2,\dots,N, \end{cases}
$$

and

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

$$\begin{aligned}
S_1 &= \frac{1}{\lambda}\int_0^\pi U^* K''_{tt}(\pi,t)U\cos(\sqrt{\lambda}t)\,dt - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}A, \\
S_2 &= \frac{1}{\sqrt{\lambda^3}}\int_0^\pi U^* L''_{tt}(\pi,t)U\sin(\sqrt{\lambda}t)\,dt - \frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda^3}}B, \\
S_3 &= \frac{\overline{\beta}}{\sqrt{\lambda}}\int_0^\pi U^* K''_{xt}(\pi,t)U\sin(\sqrt{\lambda}t)dt - \frac{\overline{\beta}\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda}}B, \\
S_4 &= -\frac{\overline{\beta}}{\lambda}\int_0^\pi U^* L''_{xt}(\pi,t)U\cos(\sqrt{\lambda}t)\,dt + \frac{\overline{\beta}\cos(\sqrt{\lambda}\pi)}{4\lambda}A.
\end{aligned}$$

Using the Laplace expansion of determinants, from (3.19) and (3.22), we obtain

$$\det W(\lambda) = \sum_{i_1,\dots,i_{2N}} (-1)^{\tau(i_1,\dots,i_{2N})}(t_{1,i_1}+s_{1,i_1})\cdots(t_{2N,i_{2N}}+s_{2N,i_{2N}}),$$

and

(3.23)

$$\begin{aligned}
\det T = &\sum_{i_1,\dots,i_{2N}} (-1)^{\tau(i_1,\dots,i_{2N})} t_{1,i_1}\cdots t_{2N,i_{2N}} \\
&= \Pi_{i=1}^N \Bigg\{\Bigg[\beta - \cos(\sqrt{\lambda}\pi) - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i + \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\Bigg] \\
&\quad\times\Bigg[1-\overline{\beta}\Bigg(\cos(\sqrt{\lambda}\pi) + \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}p_i - \frac{\cos(\sqrt{\lambda}\pi)}{4\lambda}a_i - \frac{\cos(\sqrt{\lambda}\pi)}{8\lambda}p_i^2\Bigg)\Bigg] \\
&\quad - \overline{\beta}\Bigg[-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + \frac{\cos\sqrt{\lambda}\pi}{2}p_i + \frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda}}b_i + \frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda}}p_i^2\Bigg] \\
&\quad\times\Bigg[\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} - \frac{\cos(\sqrt{\lambda}\pi)}{2\lambda}p_i + \frac{\sin(\sqrt{\lambda}\pi)}{4\sqrt{\lambda^3}}b_i - \frac{\sin(\sqrt{\lambda}\pi)}{8\sqrt{\lambda^3}}p_i^2\Bigg]\Bigg\} \\
&= \Pi_{i=1}^N \Bigg[(\beta+\bar{\beta}) - (|\beta|^2+1)\cos(\sqrt{\lambda}\pi) - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}(|\beta|^2+1)p_i \\
&\quad + \frac{\cos\sqrt{\lambda}\pi}{\lambda}\Bigg(\frac{|\beta|^2-1}{4}a_i + \frac{|\beta|^2+1}{8}p_i^2\Bigg) + \frac{1}{\lambda^2}\Bigg(-\frac{\cos^2(\sqrt{\lambda}\pi)}{16}a_i^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\sin^2(\sqrt{\lambda}\pi)}{16} b_i^2 + \frac{p_i^4}{64}\Big)\overline{\beta}\Big] \\
&= (|\beta|^2+1)^N \Pi_{i=1}^N \bigg\{ \cos(\theta\pi) - \cos(\sqrt{\lambda}\pi) - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}} p_i \\
&\quad + \frac{\cos\sqrt{\lambda}\pi}{\lambda}\bigg[\frac{|\beta|^2-1}{|\beta|^2+1} \times \frac{a_i}{4} + \frac{p_i^2}{8}\bigg] + \frac{1}{\lambda^2}\bigg(- \frac{\cos^2(\sqrt{\lambda}\pi)}{16} a_i^2 \\
&\quad - \frac{\sin^2(\sqrt{\lambda}\pi)}{16} b_i^2 + \frac{p_i^4}{64}\bigg) \times \frac{\overline{\beta}}{|\beta|^2+1}\bigg\},
\end{aligned}
$$

where θ is defined by (2.2). Write $\det W(\lambda)$ as

$$\det W(\lambda) = \det T + R(\lambda), \tag{3.24}$$

where each term in $R(\lambda)$ contains at least one element of the matrix S. Moreover, the remainder $R(\lambda)$ of (3.24) possesses the following characteristics:

If a term in $R(\lambda)$ has an element of the matrix S_1 at the ith row and ith column, $i = 1, 2, \ldots, N$, then it lacks a factor $t_{i,i}$, exactly.

If a term in $R(\lambda)$ has an off-diagonal element of the matrix S_1, then it contains a product factor of an off-diagonal element of the matrix S_1 and a $t_{i,N+i}$ for some i or two off-diagonal elements of the matrix S_1 or an off-diagonal element of the matrix S_1 and an off-diagonal element of the matrix S_2; for the other cases the arguments are similar.

We see that λ is an eigenvalue of problems (1.1) and (1.2) if and only if $\det W(\lambda) = 0$. Thus, we say $\det W(\lambda) = 0$ is the eigenvalue equation for problems (1.1) and (1.2).

4. The eigenvalue asymptotics.

This section gives the asymptotic expressions of the large eigenvalues for problems (1.1) and (1.2) with the help of the Rouché theorem.

Proof of Theorem 2.1. Write $\det W(\lambda)$ as

$$\det W(\lambda) = w_0(\lambda) + \mathcal{E}(\lambda), \tag{4.1}$$

where

$$w_0(\lambda) = (|\beta|^2+1)^N [\cos(\theta\pi) - \cos(\sqrt{\lambda}\pi)]^N \tag{4.2}$$

and $\mathcal{E}(\lambda)$ is the remainder.

It is easy to obtain the zeros $\mu_{1,n}^{(i)}$, $\mu_{2,n}^{(i)}$ of the function $w_0(\lambda)$, $i = 1, 2, \dots, N$, counting multiplicities of zero:

$$\sqrt{\mu_{1,n}^{(i)}} = 2n + \theta \ (n = 0, 1, \dots), \\ \sqrt{\mu_{2,n}^{(i)}} = 2n - \theta \ (n = 1, 2, \dots), \tag{4.3}$$

where θ is defined by (2.2) and all the zeros in $\{(2n+\theta)^2\}_{n=0}^{\infty} \bigcup \{(2n-\theta)^2\}_{n=1}^{\infty}$ are at least of order N.

Since the zeros of $\det W(\lambda)$, eigenvalues for self-adjoint problems (1.1) and (1.2), are real, we may suppose $|\mathrm{Im}\,\lambda| < \kappa$ for some fixed constant $\kappa > 0$.

Now it follows from (3.23), (3.24), (4.1), (4.2) and the properties of $R(\lambda)$ defined in (3.24) that there exists a constant $c > 0$ such that

$$|\mathcal{E}(\lambda)| = |\det W(\lambda) - w_0(\lambda)| < \frac{c}{\sqrt{\lambda}}$$

for all $|\mathrm{Im}\,\lambda| < \kappa$ for some fixed constant $\kappa > 0$. Since $w_0(\lambda) = (|\beta|^2 + 1)^N [\cos(\theta\pi) - \cos(\sqrt{\lambda}\pi)]^N$, for every $r \in (0, \varepsilon)$, we can find $d > 0$ such that $|w_0(\lambda)| > d$ for all $\lambda \in \mathbf{C} \backslash \bigcup_n C_n$, where C_n are circles of radii r with the centers at the points $\mu_{1,n}^{(i)}, \mu_{2,n}^{(i)}$, $i = 1, 2, \dots, N$. Thus, for all $\lambda \in \{\lambda | \lambda \in \mathbf{C} \backslash \bigcup_n C_n, \sqrt{\lambda} > c/d\}$, we have

$$|\det W(\lambda) - w_0(\lambda)| < \frac{c}{\sqrt{\lambda}} < d < |w_0(\lambda)|. \tag{4.4}$$

Let $\{\lambda_n^{(i)}\} = \{\lambda_{1,n}^{(i)}\} \bigcup \{\lambda_{2,n}^{(i)}\}$ be the eigenvalues of problems (1.1) and (1.2), namely, zeros of $\det W(\lambda)$. By Rouché's theorem and taking arbitrarily small r, we obtain the following results. For a sufficiently large integer n, there lie exactly N zeros of $\det W(\lambda)$ in a suitable neighborhood of $\mu_{1,n}^{(i)}, \mu_{2,n}^{(i)}$, $i = 1, 2, \dots, N$, respectively. Denote

$$\sqrt{\lambda_{1,n}^{(i)}} = 2n + \theta + \alpha_{n,i}, \tag{4.5}$$

$$\sqrt{\lambda_{2,n}^{(i)}} = 2n - \theta + \beta_{n,i}, \quad i = 1, 2, \dots, N, \tag{4.6}$$

where $\alpha_{n,i} = o(1)$, $\beta_{n,i} = o(1)$ as $n \to \infty$. It is not difficult to see that $\alpha_{n,i} = O(1/(2n+\theta))$, $\beta_{n,i} = O(1/(2n-\theta))$.

From (3.24) and the properties of $R(\lambda)$, a calculation implies

$$R(\lambda_{1,n}^{(i)}),\ R(\lambda_{2,n}^{(i)}) = \begin{cases} o(1/n^{2N}) & \theta = 0,1, \\ o(1/n^{N}) & \theta \neq 0,1. \end{cases} \tag{4.7}$$

Substituting $\lambda_{1,n}^{(i)}$ into $\det W(\lambda) = 0$, then, from (3.23) and (3.24), we have

$$\prod_{j=1}^{N} [\sin(\theta + \alpha_{n,i}/2)\pi \sin \frac{\alpha_{n,i}\pi}{2} - \frac{\sin(\theta + \alpha_{n,i})\pi}{4(2n+\theta)} p_j + O(1/n^2)] = \begin{cases} o(1/n^{2N}) & \theta = 0,1, \\ o(1/n^{N}) & \theta \neq 0,1, \end{cases}$$

which implies

$$\sin(\alpha_{n,i}\pi/2) = O\left(\frac{1}{2n+\theta}\right) + O(1/n^2).$$

Using the Lagrange inversion formula, then we get

$$\alpha_{n,i} = \frac{c_i^{(1)}}{2n+\theta} + \frac{\gamma_{i,n}^{(1)}}{n}. \tag{4.8}$$

Similarly,

$$\beta_{2,n}^{i} = \frac{c_i^{(2)}}{2n-\theta} + \frac{\gamma_{i,n}^{(2)}}{n}, \tag{4.9}$$

where $c_i^{(1)}, c_i^{(2)}$, $1 \leq i \leq N$, are constants depending on $P(x)$, and $\gamma_{i,n}^{(1)}, \gamma_{i,n}^{(2)} \to 0$ as $n \to \infty$.

Substituting (4.5) and (4.8) into the equation

$$\det W(\lambda) = 0,$$

then

$$\begin{aligned}\prod_{i=1}^{N}\Bigg[\cos(\theta\pi)-\cos\left(\theta+\frac{c_i^{(1)}}{2n+\theta}+o(1/n)\right)\pi\\ -\frac{\sin(\theta+(c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{2(2n+\theta)}p_i\\ +\frac{\cos(\theta+(c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{(2n+\theta)^2}\\ \times\left(\frac{|\beta|^2-1}{|\beta|^2+1}\times\frac{a_i}{4}+\frac{p_i^2}{8}\right)+O(1/n^3)\Bigg]\\ =\begin{cases}o(1/n^{2N}) & \theta=0,1,\\ o(1/n^N) & \theta\neq 0,1.\end{cases}\end{aligned}\tag{4.10}$$

Case I. If $\theta=0,1$ (in this case $\beta=\pm1$), then we have

$$\begin{aligned}\prod_{i=1}^{N}\Bigg[\pm1\mp\cos\left(\frac{c_i^{(1)}}{2n+\theta}+o(1/n)\right)\pi\mp\frac{\sin((c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{4n+2\theta}p_i\\ \pm\frac{\cos((c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{8(2n+\theta)^2}p_i^2+O(1/n^3)\Bigg]+o(1/n^{2N})=0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\prod_{i=1}^{N}\Bigg[\pm(2n+\theta)^2\mp(2n+\theta)^2\cos\left(\frac{c_i^{(1)}}{2n+\theta}+o(1/n)\right)\pi\\ \mp(2n+\theta)\frac{\sin((c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{2}p_i\\ \pm\frac{\cos((c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{8}p_i^2+O(1/n)\Bigg]+o(1)=0.\end{aligned}$$

Expanding the left-hand side of the resulting equation in power series and letting $n\to\infty$, we have

$$\prod_{i=1}^{N}\left[\frac{1}{2}\left(\pi c_i^{(1)}\right)^2-\frac{p_i\pi c_i^{(1)}}{2}+\frac{p_i^2}{8}\right]=0.$$

Thus, for $1 \le i \le N$,

$$\left(\pi c_i^{(1)}\right)^2 - p_i \pi c_i^{(1)} + \frac{p_i^2}{4} = 0,$$

and we obtain for $1 \le i \le N$,

$$c_i^{(1)} = \frac{p_i}{2\pi}. \tag{4.11}$$

Similarly, we have for $1 \le i \le N$,

$$c_i^{(2)} = \frac{p_i}{2\pi}. \tag{4.12}$$

Case II. If $\theta \neq 0, 1$ (in this case $\beta \neq \pm 1$), then we have

$$\Pi_{i=1}^{N} \Bigg[\cos(\theta\pi) - \cos\left(\theta + \frac{c_i^{(1)}}{2n+\theta} + o(1/n)\right)\pi \\ - \frac{\sin(\theta + (c_i^{(1)})/(2n+\theta) + o(1/n))\pi}{2(2n+\theta)} p_i + O(1/n^2)\Bigg] + o(1/n^N) = 0,$$

which is equivalent to

$$\begin{aligned}\Pi_{i=1}^{N} \Bigg[& \cos(\theta\pi) - \cos(\theta\pi)\cos\left(\frac{c_i^{(1)}}{2n+\theta} + o(1/n)\right)\pi \\ & + \sin(\theta\pi)\sin\left(\frac{c_i^{(1)}}{2n+\theta} + o(1/n)\right)\pi \\ & - \frac{\sin(\theta\pi)\cos((c_i^{(1)})/(2n+\theta) + o(1/n))\pi}{2(2n+\theta)} \\ & + \frac{\cos(\theta\pi)\sin((c_i^{(1)})/(2n+\theta) + o(1/n))\pi}{2(2n+\theta)} p_i + O(1/n^2)\Bigg] \\ & + o(1/n^N) \\ = 0, &\end{aligned}$$

that is,

$$\Pi_{i=1}^{N}\Bigg[(2n+\theta)\cos(\theta\pi)-(2n+\theta)\cos(\theta\pi)\cos\left(\frac{c_i^{(1)}}{2n+\theta}+o(1/n)\right)\pi$$
$$+(2n+\theta)\sin(\theta\pi)\times\sin\left(\frac{c_i^{(1)}}{2n+\theta}+o(1/n)\right)\pi$$
$$-\frac{\sin(\theta\pi)\cos((c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{2}$$
$$+\frac{\cos(\theta\pi)\sin((c_i^{(1)})/(2n+\theta)+o(1/n))\pi}{2}p_i+O(1/n)\Bigg]+o(1)=0,$$

expanding the left-hand side of the resulting equation in power series and letting $n\to\infty$, we have

$$\sin(\theta\pi)\prod_{i=1}^{N}\left[c_i^{(1)}\pi-\frac{p_i}{2}\right]=0.$$

Since $\sin(\theta\pi)\neq 0$ in this case, for $1\le i\le N$,

$$c_i^{(1)}=\frac{p_i}{2\pi}. \tag{4.13}$$

Similarly, we have for $1\le i\le N$,

$$c_i^{(2)}=\frac{p_i}{2\pi}. \tag{4.14}$$

Together with (4.5), (4.6), (4.8), (4.9), (4.11)–(4.14), the proof of Theorem 2.1 is finished. □

5. Trace formula. In this section, we try to develop the regularized trace formula for problems (1.1) and (1.2) by explicit expressions in the coefficients of the operator. Let the contour Γ_{N_0}, integer $N_0 = 0,1,2,\ldots\to\infty$, denote the following sequences of circular contours, traversed counterclockwise:

(i) The contour Γ_{N_0} is the circle of radius $(2N_0+1)^2$ with its center at the origin (for $0\le\theta<1$);

(ii) The contour Γ_{N_0} is the circle of radius $(2N_0+2)^2$ with its center at the origin (for $\theta = 1$).

Obviously, $\mu_{1,n}^{(i)}, \mu_{2,n}^{(i)}$, $i = 1, 2, \dots, N$, defined in (2.1), zeros of the function $w_0(\lambda)$, don't lie in the contour Γ_{N_0}. To obtain the trace formula we need the following lemma in complex analysis.

Lemma 5.1 [**1, 6**]. *Suppose $\omega(\lambda)$ and $\omega_0(\lambda)$ are two entire functions and $\omega_0(\lambda)$ has no zeros on a closed contour Γ_{N_0} of λ-complex plane. If these functions satisfy the estimate*

$$\frac{\omega(\lambda)}{\omega_0(\lambda)} = 1 + \frac{\alpha_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\alpha_2(\sqrt{\lambda})}{\lambda} + O(1/\sqrt{\lambda^3}) \quad \text{on} \quad \Gamma_{N_0},$$

where the functions $\alpha_k(\sqrt{\lambda})/(\sqrt{\lambda})^k$ are single valued and analytic on Γ_{N_0} and $\alpha_k(\sqrt{\lambda})$ are uniformly bounded on Γ_{N_0}, then, on Γ_{N_0},

$$\begin{aligned}\sum_{\Gamma_{N_0}}(\lambda_n - \mu_n) &= -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \log\frac{\omega(\lambda)}{\omega_0(\lambda)}\, d\lambda \\ &= -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\left[\frac{\alpha_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\alpha_2(\sqrt{\lambda}) - \alpha_1^2(\sqrt{\lambda})/2}{\lambda}\right] d\lambda \\ &\quad + O(1/N_0),\end{aligned} \tag{5.1}$$

where λ_n and μ_n are zeros of entire functions $\omega(\lambda)$, $\omega_0(\lambda)$ inside the contour Γ_{N_0} listed with multiplicity, respectively.

Proof of Theorem 2.2. Applying Lemma 5.1 and the methods of asymptotic analysis, we can obtain the regularized trace formula for problems (1.1) and (1.2).

From (3.24) and (4.2), we have the estimate

$$\frac{R(\lambda)}{w_0(\lambda)} = O(1/\sqrt{\lambda^3}) \quad \text{on} \quad \Gamma_{N_0}. \tag{5.2}$$

Let $\Psi(\lambda) = \cos(\theta\pi) - \cos(\sqrt{\lambda}\pi)$. By (3.23), (3.24) and (4.2), integration by parts then gives, on the contour Γ_{N_0},

$$\frac{\det W(\lambda)}{w_0(\lambda)} = \Pi_{i=1}^N\left[1 - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}\Psi(\lambda)}p_i + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}\left(\frac{|\beta|^2-1}{|\beta|^2+1}\times\frac{a_i}{4} + \frac{p_i^2}{8}\right)\right.$$

$$
\begin{aligned}
&\qquad\qquad + O(1/\lambda^2)\Big] + O(1/\sqrt{\lambda^3})\\
&= 1 - \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}\Psi(\lambda)}\sum_{i=1}^{N} p_i\\
&\quad + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}\frac{|\beta|^2-1}{|\beta|^2+1}\frac{\sum_{i=1}^N a_i}{4} + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}\frac{\sum_{i=1}^N p_i^2}{8}\\
&\quad + \frac{\sin^2(\sqrt{\lambda}\pi)}{4\lambda\Psi^2(\lambda)}\sum_{1\le i<j\le N} p_i p_j + O(1/\sqrt{\lambda^3}).
\end{aligned}
$$

Taylor's expansion tells us

(5.3)

$$
\begin{aligned}
\log\frac{\det W(\lambda)}{w_0(\lambda)} &= -\frac{\sin\sqrt{\lambda}\pi}{2\sqrt{\lambda}\Psi(\lambda)}\sum_{i=1}^N p_i\\
&\quad + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}\frac{|\beta|^2-1}{|\beta|^2+1}\frac{\sum_{i=1}^N a_i}{4}\\
&\quad + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}\frac{\sum_{i=1}^N p_i^2}{8} + \frac{\sin^2(\sqrt{\lambda}\pi)}{4\lambda\Psi^2(\lambda)}\sum_{1\le i<j\le N} p_i p_j\\
&\quad - \frac{\sin^2(\sqrt{\lambda}\pi)}{8\lambda\Psi^2(\lambda)}\left(\sum_{i=1}^N p_i\right)^2 + O(1/\sqrt{\lambda^3})\\
&= -\frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}\Psi(\lambda)}\sum_{i=1}^N p_i\\
&\quad + \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}\frac{|\beta|^2-1}{|\beta|^2+1}\frac{\sum_{i=1}^N a_i}{4}\\
&\quad + \left[\frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)} - \frac{\sin^2(\sqrt{\lambda}\pi)}{\lambda\Psi^2(\lambda)}\right]\frac{\sum_{i=1}^N p_i^2}{8} + O(1/\sqrt{\lambda^3}).
\end{aligned}
$$

For $\theta = 0$, we have

(5.4)

$$
\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}\Psi(\lambda)}\,d\lambda = -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}(\cos(\sqrt{\lambda}\pi)-1)}\,d\lambda
$$

$$= \frac{1}{4\pi i}\oint_{\Gamma_{N_0}} \frac{\cot(\sqrt{\lambda}\pi)/2}{\sqrt{\lambda}}\,d\lambda$$
$$= \frac{2N_0+1}{\pi}.$$

For $\theta = 1$, we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}\Psi(\lambda)}\,d\lambda &= -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}(\cos(\sqrt{\lambda}\pi)+1)}\,d\lambda \\ &= -\frac{1}{4\pi i}\oint_{\Gamma_{N_0}} \frac{\tan\frac{\sqrt{\lambda}\pi}{2}}{\sqrt{\lambda}}\,d\lambda \\ &= \frac{2N_0+2}{\pi}.\end{aligned} \tag{5.5}$$

For $\theta \neq 0, 1$, we have

$$\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \frac{\sin(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}\Psi(\lambda)}d\lambda = \frac{2N_0+1}{\pi}, \tag{5.6}$$

$$\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)}d\lambda = -1 + O\left(\frac{1}{N_0}\right), \tag{5.7}$$

and

$$\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \left[\frac{\cos(\sqrt{\lambda}\pi)}{\lambda\Psi(\lambda)} - \frac{\sin^2\sqrt{\lambda}\pi}{\lambda\Psi^2(\lambda)}\right]d\lambda = O\left(\frac{1}{N_0}\right). \tag{5.8}$$

From the above arguments, it follows that the zeros $\lambda_n^{(i)}$ of $\det W(\lambda)$ are the eigenvalues of problems (1.1) and (1.2). By Rouché's theorem, the number of zeros of $\det W(\lambda)$ and $w_0(\lambda)$ inside the contour Γ_{N_0} is just the same for sufficiently large N_0.

By Lemma 5.1, (5.3)–(5.8), we obtain, for $\theta \neq 1$,

$$\begin{aligned}\sum_{n=0}^{N_0}\left[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)} - (2n+\theta)^2)\right] &+ \sum_{n=1}^{N_0}\left[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)} - (2n-\theta)^2)\right] \\ &= -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \log\frac{\det W(\lambda)}{w_0(\lambda)}\,d\lambda \\ &= \frac{|\beta|^2-1}{|\beta|^2+1}\operatorname{tr}\frac{P(\pi)-P(0)}{4} + \frac{2N_0+1}{\pi}\sum_{i=1}^{N}p_i + O(1/N_0),\end{aligned}$$

that is,

$$\sum_{n=0}^{N_0}\bigg[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-(2n+\theta)^2)-\frac{1}{\pi}\sum_{i=1}^{N}p_i\bigg] + \sum_{n=1}^{N_0}\bigg[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-(2n-\theta)^2)-\frac{1}{\pi}\sum_{i=1}^{N}p_i\bigg] = \frac{|\beta|^2-1}{|\beta|^2+1}\mathrm{tr}\,\frac{P(\pi)-P(0)}{4}+O\bigg(\frac{1}{N_0}\bigg).$$

Letting $N_0\to\infty$, by (3.13), we have

$$\sum_{n=0}^{\infty}\bigg[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-(2n+\theta)^2)-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,dx\bigg] + \sum_{n=1}^{\infty}\bigg[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-(2n-\theta)^2)-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,dx\bigg] = \frac{|\beta|^2-1}{|\beta|^2+1}tr\frac{P(\pi)-P(0)}{4};$$

therefore,

$$\sum_{n=0}^{\infty}\bigg[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-\mu_{1,n}^{(i)})-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,dx\bigg] + \sum_{n=1}^{\infty}\bigg[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-\mu_{2,n}^{(i)})-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,dx\bigg] = \frac{|\beta|^2-1}{|\beta|^2+1}\mathrm{tr}\,\frac{P(\pi)-P(0)}{4}.$$

Similarly, we obtain, for $\theta=1$ (in this case $\beta=-1$),

$$\sum_{n=0}^{N_0}\bigg[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-(2n+1)^2)\bigg] + \sum_{n=1}^{N_0+1}\bigg[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-(2n-1)^2)\bigg]=\frac{2N_0+2}{\pi}\sum_{i=1}^{N}p_i+O(1/N_0),$$

that is,

$$\sum_{n=0}^{N_0}\left[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-(2n+1)^2)-\frac{1}{\pi}\sum_{i=1}^{N}p_i\right] + \sum_{n=1}^{N_0+1}\left[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-(2n-1)^2)-\frac{1}{\pi}\sum_{i=1}^{N}p_i\right]=O(1/N_0).$$

Letting $N_0\to\infty$, we have

$$\sum_{n=0}^{\infty}\left[\sum_{i=1}^{N}(\lambda_{1,n}^{(i)}-(2n+1)^2)-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,dx\right] + \sum_{n=1}^{\infty}\Big[\sum_{i=1}^{N}(\lambda_{2,n}^{(i)}-(2n-1)^2)-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,dx\Big] = 0. \qquad \square$$

6. The inverse problems. From an historical viewpoint, the paper [**2**] of Ambarzumyan may be thought to be the starting point of the inverse spectral theory aiming to reconstruct the potential from the spectrum (or spectra). Ambarzumyan proved the following theorem:

If $q\in C[0,\pi]$, *and* $\{n^2 : n=0,1,2,\dots\}$ *is the spectra set of the boundary value problem*

$$-y''+q(x)y=\lambda y,\quad y'(0)=y'(\pi)=0,$$

then $q\equiv 0$ *on* $[0,\pi]$.

Theorem 2.1 (or Theorem 2.2) enables us to attack the inverse problems associated with problems (1.1) and (1.2). We follow the approach used in papers [**12, 28**].

Proof of Theorem 2.3. From Theorem 2.1, we see that large eigenvalue λ_n for problems (1.1) and (1.2):

$$\sqrt{\lambda_{1,n}^{(i)}}=2n+\theta+\frac{p_i}{2\pi(2n+\theta)}+o\left(\frac{1}{n}\right)$$

and

$$\sqrt{\lambda_{2,n}^{(i)}} = 2n - \theta + \frac{p_i}{2\pi(2n-\theta)} + o\left(\frac{1}{n}\right), \quad i = 1, 2, \dots, N.$$

Since $\{(2m_j + \theta)^2 : j = 1, 2, \dots\}$ or $\{(2m_j - \theta)^2 : j = 1, 2, \dots\}$ is a subset of the spectrum $\{\lambda_n^{(i)}\}$ of eigenvalue problems (1.1) and (1.2), m_j is a strictly ascending infinite sequence of positive integers, and the multiplicity of each eigenvalue $(2m_j + \theta)^2$ or $(2m_j - \theta)^2$ is N (in the case $\theta \neq 0, 1$), $2N$ (in the case $\theta = 0, 1$), a simple asymptotic analysis implies

$$p_i = 0, \quad i = 1, 2, \dots, N. \tag{6.1}$$

Since p_i, $i = 1, 2, \dots, N$, are the characteristic values of the $N \times N$ real symmetric matrix $\int_0^\pi P(x)\,dx$, we get

$$\int_0^\pi P(x)\,dx = 0_N. \tag{6.2}$$

Next, we consider the associated Rayleigh quotient, namely,

$$\begin{gathered} R[Y] = \frac{\int_0^\pi (-Y^{*t}Y'' + Y^{*t}P(x)Y)\,dx}{\int_0^\pi Y^{*t}Y\,dx}, \\ Y(\pi) = \beta Y(0), Y'(0) = \overline{\beta} Y'(\pi). \end{gathered} \tag{6.3}$$

Since θ^2 is the first eigenvalue of problems (1.1) and (1.2), we have that $\inf_Y R[Y] = \theta^2$.

Case I. $\theta = 0$ (in this case $\beta = 1$). We verify that $Y_i(x) = 1/\sqrt{\pi} e_i$, which satisfies boundary condition (1.2), is the eigenfunction corresponding to the first eigenvalue 0, where e_i is the unit vector whose ith component is 1 ($i = 1, 2, \dots, N$). In fact, by (6.2) and (6.3),

$$0 \le \frac{\int_0^\pi (-Y_i^{*t}Y_i'' + Y_i^{*t}P(x)Y_i)\,dx}{\int_0^\pi Y_i^{*t}Y_i\,dx} = Y_i^{*t}\left(\int_0^\pi P(x)\,dx\right)Y_i = 0, \tag{6.4}$$

that is, the above equality holds. Therefore, the test function $Y_i(x) = 1/\sqrt{\pi} e_i$ is the eigenfunction corresponding to the first eigenvalue 0.

Substituting $Y_i = 1/\sqrt{\pi}e_i$ $(i = 1, 2, \ldots, N)$ into equation (1.1), we have $P(x)e_i = 0$ on $[0, \pi]$, $i = 1, 2, \ldots, N$, which is equivalent to

$$P(x)I_N = 0_N \quad \text{on} \quad [0, \pi]. \tag{6.5}$$

Thus, $P(x) = 0_N$ on $[0, \pi]$ arises, as we asserted.

Case II. $\theta = 1$ (in this case $\beta = -1$). We verify that $Y_i(x) = (\sqrt{2/\pi}\sin x)e_i$ $(i = 1, 2, \ldots, N)$, which satisfies the boundary condition (1.2) and is the eigenfunction corresponding to the first eigenvalue 1. In fact, by (6.3),

$$\begin{aligned} 1 &\le \frac{\int_0^\pi (-Y_i^{*t}Y_i'' + Y_i^{*t}P(x)Y_i)\,dx}{\int_0^\pi Y_i^{*t}Y_i\,dx} \\ &= \frac{2}{\pi}\int_0^\pi \sin^2 x\,dx + \frac{2}{\pi}\int_0^\pi (P(x))_{ii}\sin^2 x\,dx \\ &= 1 + \frac{1}{\pi}\int_0^\pi (P(x))_{ii}\,dx - \frac{1}{\pi}\int_0^\pi (P(x))_{ii}\cos(2x)\,dx, \end{aligned} \tag{6.6}$$

by assumption of Theorem 2.2 and (6.2), the equality holds. Therefore, $Y_i(x) = (\sqrt{2/\pi}\sin x)e_i$ $(i = 1, 2, \ldots, N)$ is the eigenfunction corresponding to the first eigenvalue 1. Substituting $Y_i = (\sqrt{2/\pi}\sin x)e_i$ into equation (1.1), we get

$$\left(\sqrt{\frac{2}{\pi}}\sin x\right)e_i + \left(\sqrt{\frac{2}{\pi}}\sin x\right)P(x)e_i = \left(\sqrt{\frac{2}{\pi}}\sin x\right)e_i$$

on $[0, \pi]$, $i = 1, 2, \ldots, N$; thus, $P(x) = 0_N$ on $[0, \pi]$.

Case III. $\theta \neq 0, 1$ (in this case $\beta \neq \pm 1$). We verify that $Y_i(x) = (\cos(\theta x) + a\sin(\theta x))e_i$ $(i = 1, 2, \ldots, N)$ is the eigenfunction corresponding to the first eigenvalue θ^2, and the complex number

$$a = \frac{\beta - \cos(\theta\pi)}{\sin(\theta\pi)}, \tag{6.7}$$

which ensures $Y_i(x)$ satisfies the boundary condition (1.2).

From (6.7), together with $\cos(\theta\pi) = (\beta + \overline{\beta})/|\beta|^2 + 1$, we have

$$(a + \overline{a})(1 - \cos(2\theta\pi)) = (|a|^2 - 1)\sin(2\theta\pi). \tag{6.8}$$

From (6.8), we obtain

$$\int_0^\pi |\cos(\theta x) + a\sin(\theta x)|^2\,dx = \frac{(|a|^2+1)\pi}{2}. \tag{6.9}$$

Moreover, by (6.3) and (6.9), we have

$$\begin{aligned}\theta^2 &\le \frac{\int_0^\pi (-Y_i^{*t}Y_i'' + Y_i^{*t}P(x)Y_i)\,dx}{\int_0^\pi Y_i^{*t}Y_i\,dx} = \theta^2 + \frac{2}{(|a|^2+1)\pi} \\ &\quad \times \int_0^\pi (P(x))_{ii}|\cos(\theta x) + a\sin(\theta x)|^2\,dx \\ &= \theta^2 + \frac{1}{\pi}\int_0^\pi (P(x))_{ii}\,dx \\ &\quad + \frac{1-|a|^2}{(|a|^2+1)\pi}\int_0^\pi (P(x))_{ii}\cos(2\theta x)\,dx \\ &\quad + \frac{a+\overline{a}}{(|a|^2+1)\pi}\int_0^\pi (P(x))_{ii}\sin(2\theta x)\,dx,\end{aligned} \tag{6.10}$$

by assumptions (2.6) and (6.2), the equality holds. Therefore, $Y_i(x) = (\cos(\theta x) + a\sin(\theta x))e_i$ $(i = 1, 2, \ldots, N)$ is the eigenfunction corresponding to the first eigenvalue θ^2. Substituting $Y_i(x) = (\cos(\theta x) + a\sin(\theta x))e_i$ $(i = 1, 2, \ldots, N)$ into equation (1.1), we get

$$(\cos(\theta x) + a\sin(\theta x))P(x)e_i = 0 \quad \text{on} \quad [0,\pi],\ i = 1, 2, \ldots, N;$$

thus, $P(x) = 0_N$ on $[0,\pi]$. This finishes the proof of Theorem 2.3. ◻

Remark 6.1. When $\beta = \cos(\theta\pi) \pm i\sin(\theta\pi)$ $(0 < \theta < 1)$ in (1.2), it follows from (6.7) that $|a| = 1$ and $a + \overline{a} = 0$. Thus, by (6.10), when $\beta = \cos\theta\pi \pm i\sin\theta\pi$ $(0 < \theta < 1)$, condition (2.6) in Theorem 2.3 is redundant.

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REFERENCES

1. L. Ahlfors, *Complex analysis*, McGraw-Hill, New York, 1966.

2. V.A. Ambarzumian, *Über eine Frage der Eigenwerttheorie*, Z. Phys. **53** (1929), 690–695.

3. F.V. Atkinson, *Discrete and continuous boundary problems*, Academic Press, New York, 1964.

4. G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta Math. **78** (1946), 1–96.

5. ———, *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, in Proc. 11th Scand. Congr. Mathematicians (Oslo: Johan Grundt Tanums Forlag), (1952), 276–287.

6. C.W. Cao, *Asymptotic traces for non-self adjoint Sturm-Liouville operators*, Acta Math. Sinica **24** (1981), 84–94.

7. R. Carlson, *Large eigenvalues and trace formulas for matrix Sturm-Liouville problems*, SIAM J. Math. Anal. **30** (1999), 949–962.

8. ———, *Eigenvalue estimates and trace formulas for the matrix Hill's equation*, J. Differential Equations **167** (2000), 211–244.

9. N.K. Chakravarty and S.K. Acharyya, *On an extension of the theorem of V.A. Ambarzumyan*, Proc. Roy. Soc. Edinb. **110** (1988), 79–84.

10. H.H. Chern, *On the eigenvalues of some vectorial Sturm-Liouville eigenvalue problems*, eprint arXiv: math/990219, 1999.

11. H.H. Chern, C.K. Law and H.J. Wang, *Extension of Ambarzumyan's theorem to general boundary conditions*, J. Math. Anal. Appl. **309** (2005), 764–768 (corrigendum).

12. H.H. Chern and C.L. Shen, *On the n-dimensional Ambarzumyan's theorem*, Inverse Prob. **13** (1997), 15–18.

13. S. Clark and F. Gesztesy, *Weyl-Titchmarsh M-function asymptotics for matrix-valued Schröinger operators*, Proc. Lond. Math. Soc. **82** (2001), 701–724.

14. S. Clark, F. Gesztesy, H. Holden and B.M. Levitan, *Borg-type theorems for matrix-valued Schröinger operators*, J. Differential Equations **167** (2000), 181–210.

15. R. Courant and D. Hilbert, *Methods of mathematical physics* 1, Interscience, New York, 1953.

16. L.A. Dikiy, *On a formula of Gelfand-Levitan*, Usp. Mat. Nauk **8** (1953), 119–123.

17. ———, *Trace formulas for Sturm-Liouville differential operators*, Amer. Math. Soc. Trans. **18** (1958), 81–115.

18. C. Fulton and S. Pruess, *Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems*, J. Math. Anal. Appl. **188** (1994), 297–340.

19. J. Garnett and E. Trubowitz, *Gaps and bands of one-dimensional periodic Schrödinger operators*, Comment Math. Helv. **59** (1984), 258–321.

20. I.M. Gelfand and B.M. Levitan, *On a formula for eigenvalues of a differential operator of second order*, Dokl. Akad. Nauk **88** (1953), 593–596.

21. F. Gesztesy and S. Clark, *Weyl-Titchmarsh M-function asymptotic, local uniqueness results, trace formulas and Borg-type theorems for Dirac-operators*, Trans. Amer. Math. Soc. **354** (2002), 3475–3534.

22. F. Gesztesy and H. Holden, *On trace formulas for Schrödinger-type operators*, in *Multiparticle quantum scattering with applications to nuclear, atomic and molecular physics*, IMA Vol. Math. Appl. **89** (1977), 121–145.

23. F. Gesztesy and B. Simon, *The Xi function*, Acta Math. **176** (1996), 49–71.

24. G.S. Guseynov and B.M. Levitan, *On the trace formulas for Sturm-Liouville operator*, Vestnik MGU **1** (1978), 40–49.

25. C.J. Halberg and V.A. Kramer, *A generalization of the trace concept*, Duke Math. J. **27** (1960), 607–618.

26. H. Hochstadt, *Asymptotic estimates of the Sturm-Liouville spectrum*, Comm. Pure Appl. Math. **4** (1961), 749–764.

27. M. Jodeit and B.M. Levitan, *Isospectral vector-valued Sturm-Liouville problems*, Lett. Math. Phys. **43** (1997), 117–122.

28. N.V. Kuznetsov, *Generalization of a theorem of V. A. Ambarzumian*, Dokl. Akad. Nauk **146** (1962), 1259–1262 (in Russian).

29. B.M. Levitan, *The computation of the regularized trace of Sturm-Liouville operator*, Usp. Mat. Nauk **19** (1964), 161–164.

30. B.M. Levitan and M.G. Gasymov, *Determination of a differential equation by two of its spectra*, Usp. Mat. Nauk **19** (1964), 3–63.

31. V.B. Lidskiy and V.A. Sadovniciy, *The regularized sum of roots of complete functions belonging to a class*, Funk. Pril. **1** (1967), 52–59.

32. V.A. Marchenko, *Sturm-Liouville operators and applications*, in *Operator theory*: *Advances and applications*, 22, Basel, Birkhäuser, 1986.

33. H.P. McKean and P. Van Moerbeke, *The spectrum of Hill's equation*, Invent. Math. **30** (1975), 217–274.

34. M.A. Naimark, *Linear differential operators*, I, II, Frederick Ungar Publishing Co., New York, 1967, 1968.

35. V.G. Papanicolaou, *Trace formulas and the behaviour of large eigenvalues*, SIAM J. Math. Anal. **26** (1995), 218–237.

36. C.T. Shieh, *Isospectral sets and inverse problems for vector-valued Sturm-Liouville equations*, Inverse Prob. **23** (2007), 2457–2468.

37. E. Trubowitz, *The inverse problem for periodic potentials*, Comm. Pure Appl. Math. **30** (1977), 321–337.

38. C.F. Yang, Z.Y. Huang and X.P. Yang, *The multiplicity of spectral of a vectorial Sturm-Liouville differential equation of dimension two and some applications*, Rocky Mountain J. Math. **37** (2007), 1379–1398.

39. B.N. Zakhariev and A.A. Suzko, *Direct and inverse problems*, Springer, Berlin, 1990.

Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P.R. China
Email address: chuanfuyang@njust.edu.cn

Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P.R. China
Email address: zyhuangh@yahoo.com

Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P.R. China
Email address: yangxp@njust.edu.cn