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$*$-MAXIMUM LATTICE-ORDERED GROUPS

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ABSTRACT. The category W is archimedean l-groups G with distinguished weak order unit e_G, with unit-preserving l-group homomorphisms. For G in W, G^* denotes the convex sub-l-group generated by e_G. If G satisfies [any isomorphism of H^* with G^* extends to an embedding of H in G], then G is called $*$-maximum. (H^* and G^* are isomorphic if and only if the unit intervals $[0, e_H]$ and $[0, e_G]$ are isomorphic MV-algebras.) This paper analyzes the property "$*$-maximum": Several characterizations are given. It is shown that any $*$-maximum G has quasi-F Yosida space; this applies to prove a conjecture of the author and J. Martinez about "rings of ω_1-quotients." It is shown that each G in W has a $*$-maximum hull, and this hull is described.

1. Introduction. We describe our problem with some points of motivation, background necessary to state the main results, then summarize these main results in subsection 1.4 below.

W is the category of archimedean l-groups G with a distinguished positive weak order unit e_G (meaning $e_G^{\perp} \equiv \{g : |g| \wedge e_G = 0\} = \{0\}$), and morphisms $G \xrightarrow{\varphi} H$ the l-group homomorphisms with $\varphi(e_G) = e_H$. All terms like *embedding*, *isomorphism*, etc., refer to W unless another context is clear.

For $G \in |W|$, $G^* \equiv \{g \in G : \exists n \in \mathbf{N} \ni |g| \leq n e_G\}$, the convex sub-$l$-group generated by e_G; $G^* \in |W|$ also. Let $W^* = \{G \in |W| : G^* = G\}$; this is the full subcategory of W of those G in which e_G is a strong unit. We have a functor $* : W \to W^*$.

Definition 1.1. Let $G \in |W|$. G is $*$-maximum (abbreviated $*$-max) if, whenever $H \in |W|$, and there is an isomorphism $\varphi : H^* \approx G^*$, then there is an embedding $\overline{\varphi} : H \leq G$ with $\overline{\varphi}|H^* = \varphi$.

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This kind of issue is natural for any functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be any (covariant) functor. For $D \in |\mathcal{D}|$, let $F^{-1}(D) = \{C : F(C) \simeq D\}$, and say that C is maximum in $F^{-1}(D)$ if $[C \in F^{-1}(D)$ and $C' \in F^{-1}(D)$ implies that C' embeds into $C]$. Say that C is F-max if C is maximum in $F^{-1}(FC)$. It is not hard to see that: if F has a right adjoint $\mathcal{C} \underset{R}{\leftarrow} \mathcal{D}$, then $[C$ is F-max if and only if $C \in R(\mathcal{D})]$. When F has no right adjoint, there still may be an operator $\mathcal{C} \underset{\rho}{\leftarrow} \mathcal{D}$ with $\rho(\mathcal{D}) = F$-max, representing some sort of "right pseudo-adjoint" to F. This will be seen to be the case for $* : W \to W^*$.

(In passing, we shall also treat the stronger property of an object C with respect to the functor F: C is "F-unique" if $[C' \in F^{-1}FC$ implies $C' \approx C]$).

A somewhat different impetus for Definition 1.1 comes from the theory of MV-categories [**4**]. If A is an abelian l-group with strong unit e_A, then the unit interval $[0, e_A]$ in A with natural operations is an MV-algebra (Chang), and for every MV-algebra M there is a unique A with $[0, e_A] \approx M$ (Mundici). Consequently, for $G \in |W|$, $[0, e_G]$ determines G^*. To say that G is $*$-max is to say that G^* determines G, and thus is to say that $[0, e_G]$ determines G.

Our analysis of $*$-max will be rooted in the Yosida representation for W, which will be coupled with various topological ideas:

Let X be a Tychonoff space, $\mathbf{R}$ the reals, $[-\infty, +\infty] = \mathbf{R} \cup \{\pm\infty\}$ with the obvious topology and order, and let $D(X) = \{f \in C(X, [-\infty, +\infty]) : f^{-1}\mathbf{R}$ dense in $X\}$. $D(X)$ contains all constant functions, is a lattice ($f \leq g$ defined point-wise on X), and has the partial addition ($f + g = h$ means $f(x) + g(x) = h(x)$ if all three are real). The addition is not fully defined, i.e., given f, g there may be no h: for $X = [0, +\infty]$, let $f(x) = x$ and $g(x) = -x + \sin x$.

The space X is a quasi-F space if each dense cozero-set is C^*-embedded in X. (See [**6, 7**] about cozero-sets and C^*-embeddings.) The following simple fact is important to us.

Proposition 1.2 [**17**]. *In $D(X)$, $+$ is fully defined* (*whence $D(X)$ is a group, and an l-group*) *if and only if X is quasi-*F.

In spite of Proposition 1.2, for any X, $G \subseteq D(X)$ is called a W-object in $D(X)$, and we write $G \leq D(X)$, if G is a sublattice, closed under $+$ and $-$ in $D(X)$, and $1 \in G$. For example, $C(X) \leq D(X)$. We follow [**15**] in the description of the Yosida Representation.

For $G \in |W|$, YG denotes the set of values of e_G with the hull-kernel topology. (A value of e_G is a convex sub-l-group M maximal for $e_G \notin M$.)

Theorem 1.3 (Objects). *Let $G \in |W|$. Then YG is compact Hausdorff and there is an isomorphism $G \approx \widehat{G} \leq D(YG)$ for which: $\widehat{e}_G = 1$; if E and F are disjoint closed sets in YG, then there is $g \in G$ with $\widehat{g} = [1$ on $E;\ 0$ on $F]$.*

If $G \approx \overline{G} \leq D(X)$ is another isomorphism with all these properties, then there is a homeomorphism $X \xrightarrow{\tau} YG$ for which $\overline{g} = \widehat{g} \circ \tau$ for all $g \in G$.

(Morphisms). *Let $G \xrightarrow{\varphi} H \in W$. There is a unique continuous $YG \xleftarrow{Y\varphi} YH$, for which $\widehat{\varphi(g)} = \widehat{g} \circ Y\varphi$ for all $g \in G$. If φ is onto, then $Y\varphi$ is one-to-one (and not conversely); φ is one-to-one if and only if $Y\varphi$ is onto.*

This defines a (*contravariant*) *functor $Y : W \to$* Comp (*compact Hausdorff spaces*).

Henceforth, we identify each $G \in |W|$ with its Yosida representation and just write $G \leq D(YG)$. Now the unit e_G becomes 1, and for $n \in \mathbf{Z}$, ne_G becomes n; and G^* becomes $\{g \in G : g \text{ is bounded }\}$. The representation $G \leq D(YG)$ restricts to $G^* \leq D(YG)$ which satisfies Theorem 1.3, so that $YG^* = YG$. Thus, the functor $Y : W \to$ Comp factors as $Y = * \circ (Y|W^*)$. The paradigm "F-max" described in Theorem 1.1 applies to Y (*mutatis mutandis*: Y is contravariant), and evidently, if G is Y-max then G is $*$-max. Y-max will be dealt with in Section 2, on the way to $*$-max.

The following represents a fairly complete trek through the principal constructions and results of the paper. The proof will come in pieces, as indicated.

Theorem 1.4. (I) *The following are equivalent about G.*

(1) *G is* $*$-max.

(2) (Section 3). *If $f \in D(YG)^+$ has the property that $f \wedge k \in G$ for all $k \in \mathbf{N}$, then $f \in G$.*

(3) (Section 4). *If S is a dense cozero-set in YG, and $f \in C(S)$ is locally in G on S, then f extends over YG to a $g \in G$.*

(4) (Section 10). *Any sequence (u_n) in G^+ for which $u_{n+1} \wedge n = u_n$ for all n and $\bigcap_n (u_{n+1} - u_n)^{\perp\perp} = \{0\}$, has the supremum in G.*

(II) (Section 7). *If G is* $*$-max*, then YG is quasi*-F*, and if also G is divisible, then G is uniformly dense in $D(YG)$.*

(III) (Section 8). *Each $G \in |W|$ has an extension lG which is minimum among essential* $*$-max *extensions of G. The Yosida space YlG is the minimum quasi*-F *cover of YG.*

(Essential extensions of l-groups, and covers of spaces, are reviewed in Section 8.)

Condition (I) (4) above is a purely order algebraic characterization of $*$-max (as opposed to (2) and (3)). Similar characterizations of some related classes follow (Section 10).

The proof of (II) above involves some rather delicate theorems (Section 6) about approximating real-valued functions, in roughly the vein of the Stone-Weierstrass theorem. This leads to (III) above, which (it turns out) proves a conjecture about "rings of ω_1-quotients" from [**13**, Section 9].

2. Y-max and Y-unique.

We shall prove the following two theorems, along the way to the main objective, $*$-max.

Theorem 2.1. *G is Y*-max *if and only if $G = D(YG)$ (whose last condition entails YG quasi*-F*).*

Theorem 2.2. *G is Y-unique if and only if $G = \{0\}$.*

The proofs of Theorems 2.1 and 2.2 will employ the following lemma (which also will find use later in the paper). Convenient notation is: in a lattice D with subset B, mB stands for all meets in D of finite subsets of B, and similarly, jB using joins.

Lemma 2.3. *For any X:*

(a) *if A is a group in $C^*(X)$, and $f \in D(X)^+$, then $A + \mathbf{Z} \cdot f$ is a group in $D(X)$.*

(b) [**10**, Theorem 1 (D)]. *If B is a group in $D(X)$, then jmB is an l-group in $D(X)$.*

Proof. (a) Any $a+mf$ $(a \in A)$ is real on $f^{-1}(\mathbf{R})$ (since a is bounded), and extends over $f^{-1}(\mathbf{R}) \cup f^{-1}(+\infty) = X$ with value $+\infty$ (since a is bounded). So $A + \mathbf{Z} \cdot f \subset D(X)$. And, $(a + mf) + (b + nf) = (a + b) + (m + n)f$. So, $A + \mathbf{Z} \cdot f$ is a group in $D(X)$.

(b) See [**10**]. (This is more work than (a).) $\square$

Corollary 2.4. *Let $G \in |W|$ and $f \in D(YG)^+$. Then, $H_f \equiv jm(G^* + \mathbf{Z} \cdot f)$ is a W-object with $YH_f = YG$ and $f \in H$.*

Proof. Use Lemma 2.3 (a), with $X = YG$ and $A = G^*$, then Lemma 2.3 (b) with $B = G^* + \mathbf{Z} \cdot f$. Thus H_f is an l-group in $D(YG)$. Since $1 \in G^* \subseteq H_f$, $H_f \in |W|$, and since G^* separates points of YG, so does H_f. Thus, $YH_f = YG$. $\square$

Proof of Theorem 2.1. If $D(X)$ is an l-group (i.e., X is quasi-F), it is obviously the largest l-group in $D(X)$. Thus, $G = D(YG)$ implies G is Y-max.

Suppose G is Y-max. If $f \in D(YG)^+$, and H_f is as in Corollary 2.4, then $f \in H_f \subseteq G$. Thus, for any $f \in D(YG)$, $f = f^+ - f^-$ and f^+ and $f^- \in G$, so $f \in G$. $\square$

Proof of Theorem 2.2. $\{0\}$ is Y-unique, since the only $G \in |W|$ with $YG = Y\{0\} = \varnothing$ is $G = \{0\}$.

Suppose G is Y-unique. First note that $G = C(YG)$ (since $YG = YG^*$. Thus, $G = G^* \leq C(YG)$, and $YC(YG) = YG$). We now show that $G \neq \{0\}$ implies G is not Y-unique, by showing that $C(X)$ is not Y-minimum, thus not Y-unique.

Case (i). X is finite. Then $C(X) = \mathbf{R}^n$, while $Y\mathbf{Z}^n = X$ also.

Case (ii). X is infinite. Then, X has a non-P point p, which means there is $f_0 \in C(X)$ which is constant on no neighborhood of p. (See [**7**].) Let $C_K(X - \{p\})$ be the functions of compact support on the locally compact space $X - \{p\}$, and let $H = jm(C_K(X - \{p\}) + \mathbf{Z} \cdot 1)$ within $C(X)$. By Lemma 2.3, H is an l-group in $C(X)$. Since $1 \in H$, and H evidently separates the points of X, $YH = X$. Visibly, any $h \in H$ is constant on a neighborhood of p, so $f_0 \notin H$, so $H \neq C(X)$. $\square$

Remark 2.5. We just showed that for compact Hausdorff $X \neq \varnothing$, $C(X)$ is not Y-minimal. On the other hand, if such X also has a base of clopen sets ("X is Boolean"), then $C(X, \mathbf{Z})$ is even Y-minimum. (*Proof.* If $G \in |W|$ and U is clopen in YG, then Theorem 1.3 implies that the characteristic function of U is in G.) We have nothing further to say now about Y-min (or $*$-min).

3. $*$-max, I: Truncations.

For $G \in |W|$, we set $\operatorname{Trunc} G = \{f \in D(YG)^+ : f \wedge k \in G$ for all $k \in \mathbf{N}\}$. We shall prove Theorem 3.1 below (which is Theorem 1.4 I (1) $\Leftrightarrow$ (2)). This is one of the principal technical tools of this paper.

Theorem 3.1. *G is $*$-max if and only if $\operatorname{Trunc} G \subseteq G$.*

The following, Proposition 3.2, stands in analogy with Corollary 2.4 and constitutes most of the proof of Theorem 3.1. We shall derive Theorem 3.1 from Proposition 3.2, then prove Proposition 3.2 (which is somewhat involved).

Proposition 3.2. *Let $G \in |W|$ and $u \in D(YG)^+$. Then, $T_u \equiv jm(G^* + \mathbf{Z} \cdot u)$ is a W-object, with $YT_u = YG$ and $u \in T_u$. The following are equivalent.*

(a) $u \in \operatorname{Trunc} G$.

(b) $T_u^+ \subseteq \operatorname{Trunc} G$.

(c) $T_u^* = G^*$.

Proof of Theorem 3.1. Suppose G is $*$-max, and $u \in \operatorname{Trunc} G$. By Proposition 3.2, $T_u^* = G^*$, so by $*$-max, $T_u \subseteq G$. Thus, $u \in G$. Suppose

$\operatorname{Trunc} G \subseteq G$, and H has $H^* = G^*$. Thus, $YH = YG$, so $H \leq D(YG)$. If $h \in H^+$, then for all k $h \wedge k \in H^* = G^*$, so $h \in \operatorname{Trunc} G \subseteq G$. Thus, $H^+ \subseteq G$, and $H \subseteq G$. $\square$

The proof of Proposition 3.2 involves various technical lemmata, which in turn involve the idea "locally in" (which will re-surface in a serious way in Section 5). For $G \in |W|$ and $w \in D(YG)$, we say "w is locally in G^*" if, for each $p \in YG$, there is a neighborhood U_p and $g_p \in G^*$ for which $g \mid U_p = w \mid U_p$.

Lemma 3.3 [**15**, 5.5 (a)]. *Let $G \in |W|$. If $w \in D(YG)$ is locally in G^*, then $w \in G^*$.*

Proof. It suffices to prove this for $w \geq 0$. By compactness of YG, there is a finite open cover $\{U_i\}$ of YG and $\{g_i\} \subseteq G^+$ with $g_i = f_i$ on U_i for all i. So, w is bounded, say $w \leq b \in \mathbf{N}$. By [**6**, page 44], there is another open cover $\{W_i\}$ on the same index set with $\overline{W}_i \subseteq U_i$ for all i. There is, for all i, $u_i \in G_i$ with $0 \leq u_i \leq 1$ and $u_i = [1 \text{ on } \overline{W_i};\ 0 \text{ on } YG - U_i]$. Then $w = \bigvee g_i \wedge (bu_i)$. $\square$

Notation used below: For $u \in D(X)$, $\{u \leq k\} = \{p \in X : u(p) \leq k\}$. Similarly with $<$.

Lemma 3.4. *Suppose $G \in |W|$ and $u \in D(YG)^+$. Then, $u \in$* $\operatorname{Trunc} G$ *if and only if for all k there exists $g \in G$ with $g = u$ on $\{u \leq k\}$.*

Proof. $\Rightarrow$. $u \wedge k = u$ on $\{u \leq k\}$.

$\Leftarrow$. Suppose the condition, take k; we show $u \wedge k \in G$. There is $g \in G$ with $g = u$ on $\{u \leq k\}$. Thus, $g \wedge k = u \wedge k$ on $\{u < k+1\}$ and $g \wedge k \in G^*$. And, $u \wedge k = k$ on $\{u > k\}$ and $k \in G^*$. Thus, $u \wedge k$ is locally in G^*, so in G^*, by Lemma 3.3. $\square$

Lemma 3.5. *Let $G \in |W|$.*

(a) $\operatorname{Trunc} G$ *is a sublattice of $D(YG)^+$.*

(b) *Fix $u \in D(YG)^+$. Then, $L_u \equiv \{f \in D(YG)^+ :$ for all $k \in \mathbf{N}$ there exists $g \in G$ with $f = g$ on $\{u \leq k\}\}$ is a sublattice of $D(YG)^+$, closed under $+$ and multiplication by positive integers.*

Proof. Let $*$ denote either $\wedge$ or $\vee$, or in (b), $+$.

(a) $(f_1 * g_2) \wedge k = (f_1 \wedge k) * (g_2 \wedge k)$ on $\{u \le k\}$.

(b) If $f_i = g_i$ on $\{u \le k\}$, then $f_1 * f_2 = g_1 * g_2$ on $\{u \le k\}$, and if $g_i \in G$, then $g_1 * g_2 \in G$. $\square$

Lemma 3.6. *Let $G \in |W|$. If $u \in \operatorname{Trunc} G$, then $T_u^+ \subseteq L_u$. (T_u is from Proposition* 3.2, *L_u from Lemma* 3.5.)

Proof. $u \in L_u$ by Lemma 3.4. Knowing that, Lemma 3.5 (b) gives the conclusion. $\square$

Proof of Proposition 3.2. T_u is a W-object, etc., by Corollary 2.4. For the equivalence: (b) $\Rightarrow$ (c). If $0 \le h \in T_u^*$, then there is $k \in \mathbf{N}$ with $h \le k$, so $h = h \wedge k$. By (b), $h \in \operatorname{Trunc} G$, so $h \wedge k \in G$.

(c) $\Rightarrow$ (a). $u \in T_u$, so any $u \wedge k \in T_u^*$. Apply (c).

(a) $\Rightarrow$ (b). (This is the hard part, and the point of all the lemmas.) Let $h \in T_u^+$, so $0 \le h = \bigvee_i \bigwedge_j (g_{ij} + z_{ij} u)$ (finite $\vee$ and $\wedge$; g's $\in G$, z's $\in \mathbf{Z}$). Put $h_i = \bigwedge_j (g_{ij} + z_{ij} u)$. So, $h = h \vee 0 = (\bigvee_i h_i) \vee 0 = \bigvee_i (h_i \vee 0)$. By Lemma 3.5 (a), it suffices that each $h_i \vee 0 \in \operatorname{Trunc} G$. Consider such an expression (suppressing some indices) $h \vee 0 = [\bigwedge (g_j + z_j u)] \vee 0 = \bigwedge [(g_j + z_j u) \vee 0]$. By Lemma 3.5 (a), it suffices that each $(g_j + z_j u) \vee 0 \in \operatorname{Trunc} G$. We prove this next.

Consider an expression (suppressing indices) $g + zu$, fix $k \in \mathbf{N}$, and let $w = [(g + zu) \vee 0] \wedge k$. If $z = 0$, then obviously $w \in G$. For $z \ne 0$, we show w is locally in G^*, and apply Lemma 3.3. Suppose $z < 0$. There is n_0 such that, on the set $\{n_0 < u\}$, we have successively, $g + zu < 0$, so $(g + zu) \vee 0 = 0$, so $[(g + zu) \vee 0] \wedge k = 0$. On the other hand, Lemma 3.6 applied to w says: there is $g' \in G$ for which $w = g'$ on $\{u \le n_0 + 1\}$. Thus, w is locally in G^* (via the cover $\{\{u < n_0 + 1\}, \{n_0 < u\}\}$), so $w \in G^*$ by Lemma 3.3. The case $z > 0$ is similar. $\square$

(It is not the case that $u \in \operatorname{Trunc} G$ implies $L_u \subseteq \operatorname{Trunc} G$. If it were, the computation in (a) $\Rightarrow$ (b) above could be avoided. An example of $L_u \not\subseteq \operatorname{Trunc} G$ is: Let $\alpha\mathbf{N}$ be the one-point compactification of discrete $\mathbf{N}$, $G = \{g \in C(\alpha\mathbf{N}) :$ range g is finite$\}$, and let $u \in D(\alpha\mathbf{N})$ be defined by $u(n) = n$. So $u \in \operatorname{Trunc} G$. Then $f(n) = 1/n$ has $f \in L_u$, $f \notin \operatorname{Trunc} G$.)

See [**14**] for a discussion of the following. $G \in |W|$ is called uniformly complete if, whenever $f \in D(YG)$ and there is (g_n) in G with $g_n \to f$ uniformly over YG, then $f \in G$. (This is easily translated into an intrinsic order-algebraic statement about G of the form "Every Cauchy sequence has a limit.") Using the Stone-Weierstrass theorem (see Section 6 here, if needed), one sees: Divisible G is uniformly complete if and only if $G^* = C(YG)$. Thus,

Corollary 3.7. *G is divisible, uniformly complete, and $*$-max if and only if $G = D(YG)$ (whose last condition entails YG quasi-F, by Proposition 1.2).*

Proof. If $G = D(YG)$, then obviously: G is divisible; G is uniformly complete since $G^* = C(YG)$; G is $*$-max since $\operatorname{Trunc} G \subseteq G$.

If $G^* = C(YG)$ and Trunc $\subseteq G$, then obviously $G = D(YG)$. $\square$

In Section 9, we will convert Corollary 3.7 to a purely order-algebraic characterization of the W-objects of the form $D(Y)$.

4. Ubiquitous truncations; $*$-unique. Our principal observation here is Theorem 4.2, which illustrates how large $\operatorname{Trunc} G$ is, and which is essential to subsequent sections. An immediate corollary is the characterizations of $*$-uniqueness in Corollary 4.3.

Lemma 4.1. *Let S be a cozero-set in the (arbitrary) space X. Then, there are closed sets $K_1, K_2, \dots$ with $K_n \subseteq \operatorname{int} K_{n+1}$ for all n and $S = \bigcup_n K_n$.*

Proof. For $S = \operatorname{coz} f$ with $0 \leq f \leq 1$, let $K_n = f^{-1}[(1/n), 1]$. $\square$

Theorem 4.2. *Let $G \in |W|$. If S is any dense cozero-set in YG, then: whenever $S = \bigcup_n K_n$ as in Lemma 4.1, there is $u \in \operatorname{Trunc} G$ with $u \geq n$ on $Y_G - K_n$ and $u^{-1}\mathbf{R} = S$.*

(Thus, if G is $*$-max, the family $G^{-1}\mathbf{R} \equiv \{g^{-1}\mathbf{R} : g \in G \leq D(YG)\}$ consists of all dense cozero-sets of YG).

Proof. G^* 0-1 separates the closed sets of YG. So, for all $n \geq 2$, there exists $g_n \in G$ with $0 \leq g_n \leq 1$ and $g_n = [0 \text{ on } K_{n-1};\ 1 \text{ on } X - \operatorname{int} K_n]$. Define $u \in D(X)$ as:

$$u(x) = \left[\bigvee_n n g_n(x) \quad \text{if } x \in S;\ +\infty \text{ if } x \notin S \right].$$

We have (+) On any K_n, only $g_2, \dots, g_n$ are non-zero, so on K_n, $u = 2g_2 \vee \cdots \vee ng_n$. This implies $u^{-1}\mathbf{R} = S$, and continuity of u on $S = \bigcup_n \operatorname{int} K_n$. Also, $u \geq ng_n$ for all n, so $u \geq n$ on $YG - K_n$, and u is continuous at points $x \notin S$. So we have $u \in D(YG)$. Finally, (+) implies that on $\{x : u(x) \leq x\}$, $u = 2g_2 \vee \cdots \vee (k+1)g_{k+1} \in G$. By Lemma 3.4, $u \in \operatorname{Trunc} G$. ◻

Recall that a space is called "almost P" if it has no proper dense cozero-sets [**21**]. (A space is P if all cozero-sets are closed [**7**].)

Corollary 4.3. *For $G \in |W|$, the following are equivalent.*

(1) *G is $*$-unique.*

(2) *G is* $*$-max *and $G = G^*$.*

(3) *Every function in* $\operatorname{Trunc} G$ *is bounded.*

(4) *YG is almost P.*

Proof. (1) $\Leftrightarrow$ (2). Clear.

(2) $\Rightarrow$ (3). Use Theorem 3.1.

(3) $\Leftrightarrow$ (4). Use Theorem 4.2.

(4) $\Rightarrow$ (1). Suppose (4). Then $G = G^*$ (since $g \in G - G^*$ would have $g^{-1}\mathbf{R}$ proper dense cozero). Suppose $H^* = G^*$. Then, $YH = YG$, which is almost P, so $H = H^*$, and then $H^* = G^* = G$. ◻

In Section 9, we add an order-algebraic condition to the list in Corollary 4.3.

5. $*$-max, II: local on dense cozeros. The result of this section is Theorem 5.3 (which is Theorem 1.4 I ((1) $\Leftrightarrow$ (3))). This is crucial to the proof in Section 7 that G $*$-max implies YG quasi-F and to the construction in Section 8 of $*$-max hulls.

We formalize some notation.

Definition 5.1. Let $G \in |W|$, and let $\operatorname{dcoz} YG$ stand for the family of all dense cozero-sets of YG. Let $S \in \operatorname{dcoz} YG$.

$f \in \operatorname{loc}(G, S)$ means: $f \in C(S)$ and f is locally in G on S, i.e., for all $p \in S$ there exists an open $U_p \ni p$ and there exists $g_p \in G$ with $g_p = f$ on U_p.

$\operatorname{loc}(G, S) \subseteq G$ means: for all $f \in \operatorname{loc}(G, S)$ there exists $g \in G$ with $g = f$ on S, i.e., f extends over YG to a function $g \in G$.

Proposition 5.2. *Let $G \in |W|$ and $S \in \operatorname{dcoz} YG$. These are equivalent.*

(a) $f \in \operatorname{loc}(G, S)$.

(b) *There are $u_1, u_2 \in \operatorname{Trunc} G$, with each $u_i^{-1}\mathbf{R} \supseteq S$, and $f(x) = u_1(x) - u_2(x)$ for all $x \in S$.*

We shall prove Proposition 5.2 below. It quickly implies our theorem.

Theorem 5.3. *G is $*$-max if and only if $\operatorname{loc}(G, S) \subseteq G$ for each $S \in \operatorname{dcoz} YG$.*

Proof. We use Theorem 3.1, of course.

Suppose $\operatorname{Trunc} G \subseteq G$, $S \in \operatorname{dcoz} YG$, and $f \in \operatorname{loc}(G, S)$. Choose $u_1, u_2 \in \operatorname{Trunc} G$ by Proposition 5.2. Then $u_1, u_2 \in G$, so $u_1 - u_2 \in G$, and f extends from S over YG to $u_1 - u_2$.

Suppose $\operatorname{loc}(G, S) \subseteq G$ for all S, and $u \in \operatorname{Trunc} G$. Then, $f \equiv u|u^{-1}\mathbf{R} \in \operatorname{loc}(G, u^{-1}\mathbf{R})$ by Proposition 5.2 (using $S = u^{-1}\mathbf{R}$, $u_1 = u$ and $u_2 = 0$), so f extends over YG to a function in G and that function is obviously u: $u \in G$. $\square$

We now prove Proposition 5.2.

Lemma 5.4. *Let $G \in |W|$.*

(a) $\operatorname{loc}(G, S)$ *is a sub-W-object of $C(S)$ for all $S \in \operatorname{dcoz} YG$.*

(b) *If* $S_1, S_2 \in \operatorname{dcoz} YG$, $S_1 \supseteq S_2$ *and* $f \in \operatorname{loc}(G, S_1)$, *then* $f \mid S_2 \in \operatorname{loc}(G, S_2)$.

(c) *If* $u \in \operatorname{Trunc} G$, *then* $u \mid u^{-1}\mathbf{R} \in \operatorname{loc}(G, u^{-1}\mathbf{R})$.

Proof. (a) Suppose for $i = 1, 2$, $f_i \in \operatorname{loc}(G, S)$ witnessed by $\{U_p^i\}, \{g_p^i\}$. Then, $\{U_p^1 \cap U_p^2\}, \{g_p^1 \otimes g_p^2\}$ witnesses $f_1 \otimes f_2 \in \operatorname{loc}(G, S)$, for $\otimes = +, -, \vee, \wedge$. Clearly, 1 is a weak unit in $\operatorname{loc}(G, S)$.

(b) If $f \in \operatorname{loc}(G, S_1)$ is witnessed by $\{U_p\}, \{g_p\}$, then $\{U_p \cap S_2\}, \{g_p\}$ witnesses $f|S_2 \in \operatorname{loc}(G, S_2)$.

(c) If $u \in \operatorname{Trunc} G$, then the cover $\{u^{-1}[0, k) : k = 1, 2, \ldots\}$, with the functions $\{u \wedge k\}$ witness the conclusion of (c). $\square$

Proof of Proposition 5.2. ((b) $\Rightarrow$ (a)). If $u_i \in \operatorname{Trunc} G$ and $u_i^{-1}\mathbf{R} \supseteq S$, then $f_i \mid S \in \operatorname{loc}(G, S)$ by Lemma 5.4 (b) and (c). By Lemma 5.4 (a), $f_1 \mid S - f_2 \mid S \in \operatorname{loc}(G, S)$, which is the conclusion. $\square$

More lemmas are needed for Proposition 5.2 ((a) $\Rightarrow$ (b)).

Lemma 5.5. *Let* $G \in |W|$. *If* $u_1, u_2 \in \operatorname{Trunc} G$, *then* $u_1 + u_2 \in \operatorname{Trunc} G$.

Proof. Use Lemma 3.4: Let $k \in \mathbf{N}$. There are $g_1, g_2 \in G$ with $u_i = g_i$ on $\{u_i \leq k\}$ since $u_i \geq 0$, $\{u_1 + u_2 \leq k\} \subset \{u_1 \leq k\} \cap \{u_2 \leq k\}$ and, on the latter, $u_1 + u_2 = g_1 + g_2$. $\square$

Lemma 5.6. *Let* $G \in |W|$, $S \in \operatorname{dcoz} YG$, *and* $f \in \mathbf{R}^S$. *The following are equivalent.*

(a) $f \in \operatorname{loc}(G, S)$.

(b) *There is a countable open cover* $\{U_n\}$ *of* S, *and* $\{g_n\} \subseteq G^*$ *with* $f = g_n$ *on* U_n *for all* n.

(c) *For each compact* $K \subseteq S$, *there is open* $U \supseteq K$ *and* $g \in G^*$ *with* $f = g$ *on* U.

Proof. (c) $\Rightarrow$ (b). For each $p \in S$, apply (c) to $K = \{p\}$, creating $\{U_p\}$ and $\{g_p\}$, and take a countable subcover of $\{U_p\}$. (S is σ-compact, thus Lindelöf.)

(b) $\Rightarrow$ (a). Obvious.

(a) $\Rightarrow$ (c). Suppose $f \in \operatorname{loc}(G, S)$, witnessed by $\{U_p\}$ and g_p. Since f is continuous with $f(p) \in \mathbf{R}$, there is open V_p with $p \in V_p$ and compact $\overline{V}_p \subseteq U_p$, and $n(p) \in \mathbf{N}$ with $|f| \leq n(p)$ on V_p. Thus $h_p \equiv g_p \wedge n(p) \in G^*$ and $f = h_p$ on V_p. Now take compact $K \subseteq S$. Then, $K \subseteq \bigcup V_{p(i)}$ for finite $\{V_{p(i)}\}$. Let $L = \bigcup \overline{V}_{p(i)}$ (which is compact), and consider $H \equiv$ the set of restrictions $G^* \mid K : H \in |W|$, $H = H^*$ and $YH = K$. We have $f = h_{p(i)}$ on $V_{p(i)} \cap K$ for all i.

This shows $f \mid L$ is, *per* Lemma 3.3, locally in H, and thus in H, which means there exists $g \in G^*$ with $f = g$ on L. So, $U = \bigcup V_{p(i)}$ has $f = g$ on U. $\square$

Lemma 5.7. *Let $G \in |W|$, $S \in \operatorname{dcoz} YG$, and $f \in \operatorname{loc}(G, S)^*$. There is $u \in \operatorname{Trunc} G$ with $u^{-1}\mathbf{R} = S$ and $u + f \in \operatorname{Trunc} G$.*

Proof. We apply Proposition 5.2: As there, write $S = \bigcup K_n$ and take $u \in \operatorname{Trunc} G$ with $u^{-1}\mathbf{R} = S$ and $u \geq n$ on $YG - K_n$. We claim $u + f \in \operatorname{Trunc} G$.

Let $n \in \mathbf{N}$. We show that there is open $U_n \supseteq K_n$ and $g_n \in G^*$ for which $(u + f) \wedge n = [g_n \text{ on } U_n;\ n \text{ on } YG - K_n]$. Then $(u + f) \wedge n \in G^*$ will follow by Lemma 3.3. Note that $u \in \operatorname{loc}(G, S)$ by Lemma 5.4 (c). Apply Lemma 5.6 (c) to each of f and u: there are open $U_n \supseteq K_n$, and $g_n', g_n'' \in G^*$ with $g_n' = f$, and $g_n'' = u$ on U_n; let $g_n = g_n' + g_n''$. Now, on $YG - K_n$, we have $u_n \geq n$, so $u + f \geq n$, and $(u + f) \wedge n = n$. $\square$

Proof of Proposition 5.2. ((a) $\Rightarrow$ (b)). Let $f \in \operatorname{loc}(G, S)$, write $f = f^+ - f^-$ ($f^+ = f \vee 0$, $f^- = (-f) \vee 0$), so $f^+, f^- \in \operatorname{loc}(G, S)^+$ by Lemma 5.4 (a). By Lemma 5.7, there are $u, v \in \operatorname{Trunc} G$ with $u^{-1}\mathbf{R} = v^{-1}\mathbf{R} = S$ and each of $u + f^+ \equiv u_0$, $v + f^- \equiv v_0$ are in $\operatorname{Trunc} G$. So, $u_0 + v, v_0 + u \in \operatorname{Trunc} G$ by Lemma 5.5. Note that $u_0^{-1}\mathbf{R}$, $v_0^{-1}\mathbf{R} \supseteq S$. So, on S,

$$f = f^+ - f^- = (u_0 - u) - (v_0 - v) = (u_0 + v) - (v_0 + u). \qquad \square$$

6. Local/global approximations of real-valued functions. The purpose for this paper of the material in this section is to prove: if G is $*$-max, then YG is quasi-F. We get to that in the next section; the approach seems sufficiently interesting to isolate it here in Section 6.

For $h \in C^*(S)$, $\beta h \in C(\beta S)$ is the unique continuous extension of h. For $H \subseteq C^*(S)$, $\beta H = \{\beta h : h \in H\}$. Let $H \leq C^*(S)$. We repeat part of Lemma 5.6: $f \in \operatorname{loc}(H,S)$ means $f \in C(S)$ and, there is a countable open cover $\{U_n\}$ of S and countable $\{h_n\} \subseteq H$ with $[h_n = f$ on U_n for all $n]$. Then by Lemma 5.4, $\operatorname{loc}(H,S) \leq C(S)$, and $\operatorname{loc}(H,S)^* \leq C^*(S)$. We emphasize "countable" here.

We shall state the results, then prove them. Various remarks follow the proofs.

Theorem 6.1. *Suppose S is locally compact and σ-compact. If $H \leq C^*(S)$ separates points and closed sets in S, then $\beta\operatorname{loc}(H,S)^*$ separates points in βS.*

Theorem 6.2. *For any S: If $H \leq C^*(S)$, and H is uniformly dense in $C^*(S)$, then* $\operatorname{loc}(H,S)$ *is uniformly dense in $C(S)$.*

Theorem 6.3 (Stone-Weierstrass [**24**]). *Suppose S is compact. If $K \leq C(S)$ separates points of S and K is divisible, then K is uniformly dense in $C(S)$.*

Corollary 6.4. *Suppose S is locally compact and σ-compact. If $H \leq C^*(S)$ separates points and closed sets of S, and H is divisible, then* $\operatorname{loc}(H,S)$ *is uniformly dense in $C(S)$.*

Proof of Theorem 6.1. Write $S = u^{-1}\mathbf{R}$ for $u \in D(\beta S)$. (Since S is locally compact, S is open in βS, and since S is σ-compact, S is F_σ in βS. Thus, S is cozero in βS, since βS is normal. Write $S = \operatorname{coz} f$ for $f \in C(\beta_S)^+$, and let $u = [1/f$ on $S; +\infty$ on $\beta S - S]$. (See [**6**].)

For $n \in \mathbf{N}$, let $U_n = \{x \in S : n-4 < u(x) < n\}$ and $E_n = \{x \in S : n-3 \leq u(x) \leq n-1\}$. So $E_n \subseteq U_n$, E_n is compact, U_n is open, $S = \bigcup_n E_n = \bigcup_n U_n$, and $|m-n| \geq 4$ implies $U_m \cap U_n = \varnothing$.

Thus, suppose $\{w_n\}_n \subseteq C(S)^+$ has $\operatorname{coz} w_n \subseteq U_n$, and we define $w(x) = \bigvee_n w_n(x)$ for all $x \in S$. Then, for all n, we have [for $x \in U_n$, $w(x) = \bigvee_{n-3}^{n+3} w_i(x)$]. Thus,

(1) $w \in C(S)$, and

(2) whenever $H \leq C(S)$ and $\{w_n\} \subseteq H$, then $w \in \operatorname{loc}(H,S)$, and if $0 \leq w_n \leq 1$ for each n, then $0 \leq w \leq 1$ so $w \in \operatorname{loc}(H,S)^*$.

For the proof proper: First suppose $p \neq q$ in βS. Let A and B be open in βS with $p \in A \subseteq \overline{A} \subseteq B$ and $q \notin \overline{B}$ (closures in βS). We then have

$$\overline{A} \cap S = \overline{A} \cap \left(\bigcup_n E_n \right) = \bigcup_n \left(\overline{A} \cap E_n \right),$$

and for all n, $(\overline{A} \cap E_n) \cap (S - U_n) = \varnothing$, and $(\overline{A} \cap E_n) \cap ((\beta S - S) \cap S) = \varnothing$. Set $F_n = (S - U_n) \cup ((\beta S - B) \cap S)$, so $[\overline{A} \cap E_n] \cap F_n = \varnothing$ for all n. Note that $\overline{A} \cap E_n$ is compact and F_n is closed in S. Now suppose $H \leq C^*(S)$ separates points and closed sets of S thus separates compact sets and closed sets in S. Then, for all n, choose $w_n \in H$ with $0 \leq w_n \leq 1$ and $u_n = [1 \text{ on } A \cap E_n;\ 0 \text{ on } F_n]$. As noted above, it follows that $w(x) = \bigvee_n w_n(x)$ for all $x \in S$ defines $w \in \operatorname{loc}(H, S)^*$.

$\beta w(p) = 1$: $w = 1$ on $\bigcup_n (\overline{A} \cap E_n) = \overline{A} \cap S$, so by continuity, $\beta w = 1$ on $\overline{\overline{A} \cap S}$. But $\overline{\overline{A} \cap S} = \overline{A}$, since in βS A is open and S is dense. $\beta w(q) = 0$: $w = 0$ on $(\beta S - B) \cap S$ (since for all n, $w_n = 0$ on F_n), so by continuity, $\beta w = 0$ on $(\beta S - B) \cap S$. But $\overline{(\beta S - B) \cap S} \supseteq \overline{(\beta S - \overline{B}) \cap S} = \overline{\beta S - \overline{B}} \supseteq \beta S - \overline{B}$ (since in βS, $\beta S - \overline{B}$ is open and S is dense). $\square$

Proof of Theorem 6.2. In $C(S)$, a locally finite partition of unity is a family $\{g_\alpha\} \subseteq C(S, [0,1])$ with $\{\operatorname{coz} g_\alpha\}$ locally finite, and for all $x \in S$, $\sum_\alpha g_\alpha(x) = 1$. Each $x \in S$ has a neighborhood U for which $F(U) \equiv \{\alpha : \operatorname{coz} g_\alpha \cap U \neq \varnothing\}$ is finite, so on U, $\sum_\alpha g_\alpha =$ the finite sum $\sum\{g_\alpha : \alpha \in F(U)\}$. Thus, whenever $\{r_\alpha\} \subseteq \mathbf{R}$, $\sum_\alpha r_\alpha g_\alpha = \sum\{r_\alpha g_\alpha : \alpha \in F(U)\}$ on U and $\sum_\alpha r_\alpha g_\alpha \in C(S)$ (and locally belongs to any group for which the $r_\alpha g_\alpha$'s belong).

We shall use $(*)$ [**3**, 2.1]. If $\{C_n\}$ is a countable cover of S by cozero-sets, then there is a (countable) locally finite partition of unity $\{g_n\}$ with $\operatorname{coz} g_n \subset C_n$ for all n.

Now suppose $H \leq C^*(S)$ is uniformly dense, $f \in C(S)$, and $\varepsilon > 0$. For $n \in \mathbf{Z}$, let $C_n = \{x : n - 1 < (1/\varepsilon) f(x) < n + 1\}$. These are cozero sets, and $(*)$ applies to produce $\{g_n\}$. Note that, for each $x \in S$, there is $n(x)$ such that $x \in C_n$ implies $n = n(x)$ or $n(x) + 1$; thus, $g_n(x) > 0$ implies $n = n(x)$ or $n(x) + 1$. Thus, $1 = \sum g_n(x) = g_{n(z)}(x) + g_{n(x)+1}(x)$.

(i) Define $g = \sum n \varepsilon g_n \in C(S)$. In the sum, we can suppose $n \neq 0$ (since $0 \cdot \varepsilon \cdot g_0 = 0$ contributes 0). A short calculation ([**3**, page 43]) shows $|f(x) - g(x)| < \varepsilon$ for all $x \in S$.

(ii) Now we approximate the g_n's by functions $h_n \in H$, as follows. For each $n \neq 0$, choose $h_n \in H$ with $0 \leq h_n \leq g_n$ and $|g_n - h_n| \leq 1/n$. Then, $\operatorname{coz} h_n \subseteq \operatorname{coz} g_n$ for all n, so, for all $x \in S$, $h_n(x) > 0$ implies $n = n(x)$ or $n(x) + 1$.

Let $h = \sum n\varepsilon h_n$ $(n \neq 0)$. On $\operatorname{coz} h_n$, h coincides with $(n-1)\varepsilon h_{n-1} + n\varepsilon h_n + (n+1)\varepsilon h_{n+1}$, so $h \in \operatorname{loc}(G, S)$. Now take, with sets index $n(x) = k$, $g(x) = \sum\{n\varepsilon g_n(x) : n = k, k+1\}$ and $h(x) = \sum\{n\varepsilon h_n(x) : n = k, k+1\}$, so $|g(x) - h(x)| = |\sum\{n\varepsilon(g_n(x) - h_n(x)) : n = k, k+1\}| \leq \sum\{n\varepsilon|g_n(x) - h_n(x)| : n = k, k+1\} \leq \sum\{n\varepsilon(1/n) : n = k, k+1\} = 2\varepsilon$.

(iii) For each $x \in S$, $|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \leq \varepsilon + 2\varepsilon = 3\varepsilon$. $\square$

Proof of Corollary 6.4. Assume the hypotheses. By Theorem 6.1, $K = \beta\operatorname{loc}(H, S)^*$ separates points of βS. Since H is divisible, so is K. By Theorem 6.3, K is uniformly dense in $C(\beta S)$, which means $K|S = \operatorname{loc}(H, S)^*$ is uniformly dense in $C^*(S)$. By Theorem 6.2, $\operatorname{loc}(H, S)$ is uniformly dense in $C(S)$. $\square$

Remarks 6.5. (a) Corollary 6.4 includes the Stone-Weierstrass Theorem 6.3: If S is compact and $K \leq C(S)$ separating points, then $\operatorname{loc}(K, S) = K$ by Theorem 1.3 and Lemma 3.3.

(b) While Theorems 6.1, 6.2 and 6.3 imply Corollary 6.4, and Corollary 6.4 implies Theorem 6.3 ((a) above), I am unable to see that (or if) Corollary 6.4 implies Theorem 6.1. We need Theorem 6.1 here.

(c) In [**9**] (written subsequent to the present paper), a more direct proof of Corollary 6.4 is given (avoiding Theorems 6.1 and 6.2), and further, the conclusion of Corollary 6.4 is shown to hold for Lindelöf S. It seems inappropriate to go into all that here.

7. Quasi-F Yosida space. The present point of Section 6 is the following.

Theorem 7.1. *Suppose G is $*$-*max. *Then,*

(a) *YG is quasi-*F*, and*

(b) *if also G is divisible, then G is uniformly dense in $D(YG)$.*

Proof. (a) (This will use Theorem 6.1.) For any S dense in YG, YG is a compactification of S so the inclusion $S \hookrightarrow YG$ has a continuous extension $\beta S \overset{\pi}{\twoheadrightarrow} YG$. For each $g \in G^* \leq C(YG)$, we have $g|S \in C^*(S)$ and its extension $\beta(g|S) \in C(\beta S)$ for which $\beta(g|S) = g \circ \pi$. Then, S is C^*-embedded in YG if and only if π is one-to-one if and only if $\{\beta(g|S) \mid g \in G^*\}$ separates points of βS.

Now suppose G is $*$-max and $S \in \operatorname{dcoz} YG$, so that $H = \operatorname{loc}(G, S)^* \subseteq G$ (meaning, $h \in H$ extends to $g \in G^* \subseteq C(YG)$). Theorem 6.1 says βH separates points of βS, so S is C^*-embedded by the previous paragraph.

(b) (This will use Corollary 6.4.) Let $u \in D(YG)$ and $\varepsilon > 0$. Let $S = u^{-1}\mathbf{R} \in \operatorname{dcoz} YG$ and $f = u \mid S \in C(S)$. Let $H = G^* \mid S \leq C^*(S)$. Since G^* separates points and closed sets of YG, H does so also in S, i.e., H does. Assuming G divisible, it is too, so $\operatorname{loc}(H, S)$ is uniformly dense in $C(S)$ by Corollary 6.4, so there is $h \in \operatorname{loc}(H, S)$ with $|f - h| \leq \varepsilon$ on S. Assuming G is $*$-max, $\operatorname{loc}(H, S) \subseteq G$, which means h extends to $g \in G \leq D(YG)$. By density of S, $|f - h| \leq \varepsilon$ on S implies $|u - g| \leq \varepsilon$ on YG. $\square$

Remark 7.2. I do not know if (a) G $*$-max implies the divisible hull dG is uniformly dense in $D(YG)$. (A model of dG is $\{rg : r \in \mathbf{Q}, g \in G\} \leq D(YG)$.) It is not true that (b) G $*$-max implies dG $*$-max. (If (b) were true, applying Corollary 6.4 would give (a).) A trivial example showing (b) false is $G = C(\mathbf{N}, \mathbf{Z})$. Here $YG = \beta\mathbf{N}$, $\operatorname{dcoz} YG$ has minimum member $\mathbf{N}$, and $\operatorname{loc}(G, \mathbf{N}) = G$ obviously; this shows G is $*$-max. But $dG = \{rg : r \in \mathbf{Q}, g \in G\}$, while $\operatorname{loc}(dG, \mathbf{N})$ contains $f(n) = 1/n$, and $f \notin dG$.

8. $*$-max hulls. An extension $G \leq H$ is essential if a morphism $H \overset{\phi}{\to} K$ is one-to-one whenever $\phi \mid G$ is one-to-one. Let $\mathcal{C} \subseteq |W|$. A $\mathcal{C}$-hull of $G \in |W|$ is an essential extension $c_G : G \leq cG$ in W, with $cG \in \mathcal{C}$, for which, whenever $G \leq C$ is an essential extension in W with $C \in \mathcal{C}$, then cG W-embeds in C over G. Such (cG, c_G) is essentially unique. If every $G \in |W|$ has a $\mathcal{C}$-hull, then $\mathcal{C}$ is called a hull class, and the resulting function $|W| \overset{c}{\to} \mathcal{C}$ is the hull operator. (See the survey [**22**].)

A function $\gamma : Y \to X$ between compact spaces is irreducible if γ is a continuous surjection and [F proper closed in Y implies $\gamma(F) \neq X$].

For such γ, S dense open in X implies $\gamma^{-1}S$ dense in Y. In this case, (Y,γ) is called a cover of X. For two covers of X, $(Y_2,\gamma_2) \geq (Y_1,\gamma_1)$ means there is irreducible $\delta : Y_2 \to Y_1$ with $\gamma_1\delta = \gamma_2$; such δ is unique. If $(Y_2,\gamma_2) \geq (Y_1,\gamma_1)$ and $(Y_1,\gamma_1) \geq (Y_2,\gamma_2)$, the two δ's are inverse, each is a homeomorphism, and the two covers are viewed as the same. (See the surveys [**8, 23**].)

The Yosida representation yields the close connection between hulls in W, and covers in compact spaces:

Proposition 8.1 [**16**, 4.1]. *Suppose $\phi : G \leq K$ is a W-embedding. ϕ is essential if and only if $Y\phi : YK \to YG$ is irreducible.*

Of particular present importance is the (minimum) quasi-F cover of X, which is $\mathrm{QF}\,(X) \equiv \varprojlim\{\beta S : S \in \mathrm{dcoz}\,X\} \xrightarrow{\pi} X$. ($\varprojlim_S \beta S$ is a subset of $\prod_S \beta S$, so there is the projection onto each βS, in particular the projection π onto X.) (See [**5, 25**].)

The class $\{D(Y) : Y \text{ quasi-F}\}$ is a hull class in W: for $G \in |W|$, the hull is $G \leq D(\mathrm{QF}\,YG)$, with embedding given from $\mathrm{QF}\,YG \xrightarrow{\pi} YG$, as $g \mapsto g \circ \pi$ (which is in $D(\mathrm{QF}\,YG)$ because π is irreducible). The Y-image of $G \leq D(\mathrm{QF}\,YG)$ is π.

We now exhibit the $*$-max hulls.

Let $G \in |W|$. By Corollary 6.4, $\{\mathrm{loc}\,(G,S) : S \in \mathrm{dcoz}\,YG\}$ is a direct system in W. (For $S_2 \subseteq S_1$, the bonding map $\mathrm{loc}\,(G,S_1) \to \mathrm{loc}\,(G,S_2)$ is restriction; this is one-to-one.) We consider the direct limit in W,

$$\ell G = \varinjlim \mathrm{loc}\,(G,S) \quad (\text{over } S \in \mathrm{dcoz}\,YG).$$

ℓG is $\cup_S \mathrm{loc}\,(G,S)$ modulo the equivalence of point-wise equality of functions on intersections of domains. G embeds into ℓG as: $g \mapsto g|g^{-1}\mathbf{R} \in \mathrm{loc}\,(G, g^{-1}\mathbf{R})$. We label this embedding $\ell_G : G \leq \ell G$. Obviously, G is $*$-max if and only if $G = \ell G$ (Theorem 5.3).

Lemma 8.2. *Suppose $\phi : G \leq K$ is essential in W, with Y-image $Y\phi : YK \to YG$.*

(a) *Let $S \in \mathrm{dcoz}\,YG$. Then,* $\mathrm{loc}\,(G,S)$ *embeds into* $\mathrm{loc}\,(K,(Y\phi)^{-1}S)$ *via $f \mapsto f \circ \phi$.*

(b) *If K is $*$-*max*, then ℓG embeds into K over G.*

Proof. (a) By Proposition 8.1, $Y\phi$ is irreducible, so $(Y\phi)^{-1}$ is dense, and obviously cozero. If $f \in \operatorname{loc}(G,S)$, and, for example, $g \in G$ has $f = g$ on open $U \subset S$, then $f \circ Y\phi - g \circ Y\phi = \phi(g)$ on $(Y\phi)^{-1}U \subseteq (Y\phi)^{-1}S$.

(b) By (a) and the condition Theorem 5.3 that K is $*$-max. $\square$

Theorem 8.3. *For* $G \in |W|$,

(a) *The* Y*-image of* ℓ_G *is the quasi-*F *cover of* YG; $Y\ell G = \mathrm{QF}\, YG$.

(b) $(\ell G, \ell_G)$ *is the* $*$*-max hull of* G.

Proof. For brevity, let (Y,π) denote the quasi-F cover of YG. We have the embeddings of G and ℓG into $D(Y)$ discussed above *via* compositions with π.

(a) $D(Y)$ is $*$-max (by Theorem 2.1 and a remark after Theorem 1.3), so $\ell G \leq D(Y)$ by Lemma 8.2. It suffices to see that ℓG separates points of Y. As noted above, $Y = \varprojlim \beta S$ ($S \in \operatorname{dcoz} YG$). If $p \neq q$ in $\varprojlim \beta S$, there is S with $\pi_{\beta S}(p) \neq \pi_{\beta S}(q)$ in βS. But $\beta\operatorname{loc}(G,S)$ separates the points of βS by Theorem 6.1.

(b) In view of Lemma 8.2 (b), we need only show that ℓG is $*$-max. It seems easiest to show that $\operatorname{Trunc} \ell G \subseteq \ell G$ and apply Theorem 3.1. (We could show $\ell(\ell G) = \ell G$ and apply Theorem 5.3. This seems a bit more tedious.)

By (a), $Y\ell G = Y$. For $f \in D(Y)$, the meaning of "$f \in \ell G$" is: there is $S \in \operatorname{dcoz} YG$ with $f \in \operatorname{loc}(G \circ \pi, \pi^{-1}S)$. So, suppose $f \in D(Y)^+$ with $f \wedge k \in \ell G$ for all $k \in \mathbf{N}$. Then for all k, there is $S_k \in \operatorname{dcoz} YG$ with $f \wedge k \in \operatorname{loc}(G \circ \pi, \pi^{-1}S_k)$. We have a cover $\{V_n^k\}_n$ of S_k, and $\{g_n^k\} \subseteq G$, with $f \wedge k = g_n^k \circ \pi$ on V_n^k. Let $U_k = \{y \in Y : f(y) < k+1\}$, so $f^{-1}\mathbf{R} = \cup_n U_k$, and $f = f \wedge k$ on U_k. Since S_k is dense, $U_k \cap S_k$ is dense in U_k, so $S \equiv \cup_k (U_k \cap S_k)$ is dense in $\cup_k U_k = f^{-1}\mathbf{R}$, while $\cup_k (U_k \cap S_k) = \cup_k (U_k \cap \cup_n V_n^k) = \cup_{k,n} (U_k \cap V_n^k)$. Thus, $\{U_k \cap V_n^k\}$ is a cover of S, and on $U_k \cap V_n^k$, $f = f \wedge k = g_k^n$. $\square$

9. The ring of ω_1-quotients.

We now show how a conjecture in [**13**] is resolved by the present development.

Let Φ be the category of archimedean f-rings with identity, with the natural morphisms. For $A \in |\Phi|$, the identity of A is a weak unit, so forgetting the multiplication creates a W object. We abuse notation by writing $\Phi \subseteq W$. The Henriksen-Johnson representation [**17**] says A is a Φ-object of extended functions on the space of maximal ℓ-ideals. That space can be identified with YA as explained in [**14**], so we have $A \leq D(YA)$ as a Φ-object.

For $S \in \operatorname{dcoz} YG$, we have the W-object $\ell(A, S)$ as before; it's easy to see that $\ell(A, S) \in |\Phi|$. We say that f is a local quotient from A on S if there is a countable open cover $\{U_n\}$ of S, and $\{(a_n, b_n)\} \in A \times A^+$ for which, for each n, $U_n \subseteq \{x \in YG : 0 < b_n(x) < +\infty\}$ and $[f = a_n/b_n$ on $U_n]$. Let $\ell q(A, S)$ consist of all such f. It's easy to see that $\ell q(A, S) \in |\Phi|$ and $\ell(A, S) \leq \ell q(A, S)$ (in Φ).

Let $Q_{\omega_1} A \equiv \varinjlim\{\ell q(A, S) : S \in \operatorname{dcoz} YG\} \in |\Phi|$. In [**13**], this is called the ring of ω_1-quotients of A, for reasons explained there, and it is shown that Q_{ω_1} is a hull operator in Φ. We have $A \leq \ell A \leq Q_{\omega_1} A$ (in Φ). The following is conjectured in [**13**, subsection 8.1].

Corollary 9.1. *$YQ_{\omega_1}A$ is the quasi*-F *cover of YA.*

Proof. As in Section 8, let $YA \xleftarrow{\pi} Y$ denote the quasi-F cover of YA, so that $Y = Y\ell A$, and in the Yosida representation the embedding $A \leq \ell A$ is $a \mapsto a \circ \pi$. But it's obvious that $Q_{\omega_1} A \leq \varinjlim\{C(S) : S \in \operatorname{dcoz} YG\} \leq D(Y)$ (with the latter embedding realized as $C(S) \ni f \mapsto \beta(f \circ \pi) \in D(Y)$). So the inclusions $A \leq \ell A \leq Q_{\omega_1} A \leq D(Y)$ have Yosida "duals,"

$$\begin{tikzcd} YA & Y = Y\ell A \arrow[l, "\pi"'] & Y\ell q A \arrow[l, "\alpha"'] & Y. \arrow[l, "\beta"'] \arrow[lll, bend left, "\pi"] \end{tikzcd}$$

Since α and β are irreducible, they are homeomorphisms (as noted before Proposition 8.1). $\square$

Let b denote the "bounded inversion" hull operator in $\Phi : bA$ is the ring of quotients of A obtained by inverting all elements ≥ 1. A quotient a/d, with $d \geq 1$, can be identified with $f(x) = a(x)/d(x)$ for $x \in S = a^{-1}\mathbf{R} \cap d^{-1}\mathbf{R}$. Thus, $bA \leq Q\omega_1 A$. It is noted in [**13**] that always $bQ_{\omega_1} A = Q_{\omega_1} A$, and thus $Q_{\omega_1} bA = Q_{\omega_1} A$.

Theorem 9.2. *For $A \in |\Phi|$, $Q_{\omega_1}A = \ell bA$ (as hull operators, $Q_{\omega_1} = \ell b$).*

Proof. 1. $\ell bA \leq Q_{\omega_1}A$: We have $\ell A \leq Q_{\omega_1}A$ for all A; thus $\ell bA \leq Q_{\omega_1}bA$ (for all A). But $Q_{\omega_1}bA = Q_{\omega_1}A$ (noted above).

2. $Q_{\omega_1}A \leq \ell bA$: In fact, for all $S \in \operatorname{dcoz} YA$, $\ell q(A,S) \leq \ell(bA,S)$: if $f \in \ell q(A,S)$, witnessed by $\{U_n\}$ and $\{(a_n,b_n)\}$ as described before Corollary 9.1, let $U_{n_k} = U_n \cap \{x : a/k < b_n(x) < +\infty\}$, so $U_n = \cup_k U_{n_k}$. Let $a_{n_k} = ka_n$ and $b_{n_k} = (kb_n) \vee 1$, so $a_{n_k}/b_{n_k} \in bA$ and $f = a_{n_k}/b_{n_k}$ on U_n. Thus, $Q_{\omega_1}A = \varinjlim \ell q(A,S) \leq \varinjlim \ell(bA,S) = \ell bA$. $\square$

Corollary 9.3. *Let $A \in |\Phi|$, and let Y be the quasi-*F *cover of YA. Then, $Q_{\omega_1}A$ is* $*$-max*, and is uniformly dense in $D(Y)$.*

Proof. By Corollary 9.1, $Y = YQ_{\omega_1}A$. By Theorems 5.3 and 9.2, $Q_{\omega_1}A$ is $*$-max. Obviously, $Q_{\omega_1}A$ is divisible. Apply Theorem 7.1 (b). $\square$

There is a question here: call A ($\in |\Phi|$) $*$-max in Φ if [$B \in |\Phi|$ and B^* Φ-isomorphic to A^* imply B Φ-embeds in A]. Evidently, A is $*$-max in W implies A $*$-max in Φ (just because $\Phi \subseteq W$). I do not know if the converse holds.

10. $*$-max**, III: Expanding sequences.** The two previous characterizations of $*$-max ($\operatorname{Trunc} G \subset G$; $\ell G = G$) are in terms of the representation $G \leq D(YG)$. We now provide a purely order-algebraic description, Theorem 10.2. At the end of this section, we indicate some consequences. We formalize as a definition a condition in Theorem 1.4.

Definition 10.1. Let $G \in |W|$. A sequence (u_n) in G^+ (indexed by $\mathbf{N}$) is called an expanding sequence in G, written $(u_n) \in \operatorname{Ex} G$, if

(e_1) $u_{n+1} \wedge n = u_n$ for all n, and

(e_2) $\cap_n (u_{n+1} - u_n)^{\perp\perp} = \{0\}$.

G is Ex-complete if each $(u_n) \in \operatorname{Ex} G$ has the supremum $u = \bigvee^G u_n$ in G.

Theorem 10.2. *G is* $*$-max *if and only if G is* Ex-*complete.*

The condition Ex-complete is a translation of the condition $\operatorname{Trunc} G \subseteq G$. The process of translation is the content of the next several lemmas, after which we prove Theorem 10.2.

Lemma 10.3. *Suppose X is a set*

(a) *Let $u : X \to [0,+\infty]$ be a function. Set $u_n = u \wedge n$ for all $n \in \mathbf{N}$. Then, $u_{n+1} \wedge n = u_n$ for all n, and $u(x) = \bigvee_n u_n(x)$ for all $x \in X$.*

(b) *Suppose that $u_n : X \to [0,+\infty]$ $(n \in \mathbf{N})$ are functions, with $u_{n+1} \wedge n = u_n$ for all n. Then*:

(i) *If $x \in X$, and $u_{n_0}(x) < n_0$, then $n \geq n_0$ implies $u_n(x) = u_{n_0}(x)$.*

(ii) *Define $u : X \to [0,+\infty]$ as $u(x) \equiv \bigvee_n u_n(x)$ for all $x \in X$. Then, $u \wedge n = u_n$ for all n; $u^{-1}(+\infty) = \bigcap_n \{x : u_n(x) = n\}$.*

(c) *Suppose the situation of* (b), *assuming further that X is a space, and the u_n are continuous. Then, u is continuous, and $u^{-1}(+\infty) = \bigcap_n \{x : u_n(x) = n\} = \bigcap_n \overline{\operatorname{coz}(u_{n+1} - u_n)}$.*

Proof. (a) is obvious.

(b) (i) For $n \geq n_0$, $u_{n_0} = u_n \wedge n_0$. For $a, b, c \in \mathbf{R}$, $a < b$ and $a = c \wedge b$ implies $a = c$.

(ii) For any $x \in X$, and $k \in \mathbf{N}$:

$$u(x) \wedge k = \left(\bigvee_n u_n(x)\right) \wedge k = \bigvee_n (u_n(x) \wedge k)$$
$$= \bigvee_{n \leq k} (u_n(x) \wedge k) \vee \bigvee_{k<n} (u_n(x) \wedge k)\,.$$

But $\bigvee_{n \leq k}(u_n(x) \wedge k)$ is its largest term, which is $u_k(x) \wedge k = u_k(x)$, and for $k < n$, $u_n(x) \wedge k = u_k(x)$. So $u \wedge k = u_k$.

If $u_n(x) = n$ for all n, then $u(x) = \bigvee_n n = +\infty$. If there is n_0 with $u_{n_0}(x) < n_0$, then, using (i),

$$u(x) = \bigvee_n u_n(x) = \bigvee_{n \geq n_0} u_n(x) = u_{n_0}(x) < n_0 < +\infty.$$

(c) We show u is continuous at each $x \in X$. First, if $u(x) \in \mathbf{R}$, then $u(x) <$ some n_0, so $u_{n_0}(x) < n_0$. Thus, $x \in \{y : u_{n_0}(y) <$

$n_0\} \equiv U$, which is open because u_{n_0} is continuous. If $y \in U$, then $u(y) = \bigvee_{n \geq n_0} u_n(y) = u_{n_0}(y)$ (using (i)). So, on open U, u agrees with the continuous u_{n_0}, and u is continuous at x. Next, suppose $u(x) = +\infty$, and let $m \in \mathbf{N}$. By (ii), $u_{m+1}(x) = m + 1 > m$. There is a neighborhood U of x with $[y \in U$ implying $m < u_{m+1}(y) \leq u(y)]$ (since u_{m+1} is continuous). So u is continuous at x.

Now, if $u_n(x) = n$ for all n, then $u_{n+1}(x) - u_n(x) = 1$ for all n, and $x \in \operatorname{coz}(u_{n+1} - u_n) \subseteq \overline{\operatorname{coz}(u_{n+1} - u_n)}$ for all n. If n is such that $u_n(x) < n$, then x has a neighborhood U with $u_n(y) < n$ for $y \in U$ (by continuity of u_n). Thus, for $y \in U$, $u_{n+1}(y) = u_n(y)$ (by (b) (i)), and $U \cap \operatorname{coz}(u_n + 1 - u_n) = \varnothing$. So $x \notin \overline{\operatorname{coz}(u_{n+1} - u_n)}$. $\square$

Lemma 10.4. *Let $G \in |W|$.*

(a) *For $d \in G$, $d^\perp \equiv \{g \in G : |g| \wedge d = 0\}$ is, viewed in $D(YG)$, $d^\perp = \{g \in G : \operatorname{coz} g \cap \operatorname{coz} d = \varnothing\}$, and $d^{\perp\perp} = \{g \in G : \operatorname{coz} g \subseteq \overline{\operatorname{coz} d}\}$.*

(b) *For $\{d_\alpha\} \subseteq G$, $\bigcap_\alpha d_\alpha^{\perp\perp} = \{0\}$ if and only if $\bigcap_\alpha \overline{\operatorname{coz} d_\alpha}$ is a nowhere dense subset of YG.*

Proof. (a) is easy (and true in any representation $G \subseteq D(X)$).

(b) By (a), $\bigcap_\alpha d_\alpha^{\perp\perp} = \bigcap_\alpha \{g : \operatorname{coz} g \subseteq \bigcap_\alpha \overline{\operatorname{coz} d_\alpha}\}$. For any closed set K in YG, we have $\{g : \operatorname{coz} g \subseteq K\} = \{0\}$ if and only if K is nowhere dense: if K is nowhere dense, then $[\operatorname{coz} g \subseteq K$ implies $g = 0]$ by continuity. If K is not nowhere dense, there is non-void open $U \subseteq K$ with $p \in U$ and $g \in G$ such that $g(p) = 1$ and $g = 0$ on $YG - U$; so $\operatorname{coz} g \subseteq K$. $\square$

Corollary 10.5. *Let $G \in |W|$.*

(a) *If $u \in \operatorname{Trunc} G$, then $u \wedge n \in \operatorname{Ex} G$.*

(b) *If $(u_n) \in \operatorname{Ex} G$, then $[u(x) \equiv \bigvee_n u_n(x)$ for all $x \in X]$ defines $u \in \operatorname{Trunc} G$.*

Proof. (a) Let $u \in \operatorname{Trunc} G$. Set $u_n = u \wedge n$. Lemma 10.3 (a) gives (e_1). Now, $u \in D(YG)$ so that $u^{-1}(+\infty)$ is nowhere dense, so, combining Lemma 10.3 (c) and Lemma 10.4 gives (e_2).

(b) Let $(u_n) \in \operatorname{Ex} G$. Then u, as defined above, has $u \wedge n = u_n \in G$ for all n by Lemma 10.3 (b). We require just $u \in D(YG)$. This follows by (e_2), Lemma 10.3 (c) and Lemma 10.4. $\square$

We now convert Corollary 10.5 to a proof of Theorem 10.2. The following is needed because, in a lattice $D(X)$, or in a $G \leq D(X)$, finite suprema are pointwise, but not infinite suprema. The symbol $\bigvee^D$ denotes supremum in the appropriate $D(X)$.

Lemma 10.6. (a) *For any X, suppose f and the f_α's $\in D(X)$. If $f(x) = \bigvee_\alpha f_\alpha(x)$ for all $x \in X$, then $f = \bigvee^D_\alpha f_\alpha$.*

(b) *Suppose $G \in |W|$, $\{g_\alpha\} \subseteq G$ and $f \in D(YG)$. If $f = \bigvee^D_\alpha g_\alpha$ and $f \in G$, then $f = \bigvee^G_\alpha f_\alpha$.*

(c) [**2**, 4.1 (c)]. *Suppose $G \in |W|$, g and the g_α's $\in G$. Then, $g = \bigvee^G_\alpha g_\alpha$ if and only if $\{x \in YG : g(x) = \bigvee_\alpha g_\alpha(x)\}$ is dense in YG.*

Here (a) and (b) are obvious. The relevant part of (c) is the implication ($\Rightarrow$), which is not so hard but uses the Baire Category theorem in compact YG.

Finally:

Proof of Theorem 10.2. We show G Ex-complete if and only if $G \supseteq \operatorname{Trunc} G$, and apply Theorem 3.1.

Suppose G is Ex-complete, and $u \in \operatorname{Trunc} G$. By Corollary 10.5 (a), $(u \wedge n) \in \operatorname{Ex} G$, so there exists $g = \bigvee^G_n (u \wedge n)$ since G is Ex-complete. Then, $\{x : g(x) = \bigvee_n (u \wedge n)(x)\}$ is dense in YG by Lemma 10.3 (a). So, g and u agree on a dense set, so by continuity, $u = g \in G$.

Suppose $G \supseteq \operatorname{Trunc} G$, and $(u_n) \in \operatorname{Ex} G$. By Corollary 10.5 (b) $[u(x) \equiv \bigvee_n u_n(x)$ for all $x \in YG]$ defines $u \in \operatorname{Trunc} G$, so $u \in G$. Then $u = \bigvee^D u_n$ by Lemma 10.6 (a), so $u = \bigvee^G_n u_n$ by Lemma 10.6 (b). $\square$

The following order-algebraic characterizations now become available.

Corollary 10.7. *G is divisible, uniformly complete, and* Ex-*complete if and only if $G = D(YG)$ (whose last condition entails YG quasi*-F).

Proof. Theorems 2.1, 10.2 and a remark after Theorem 1.3. $\square$

Corollary 10.8. *The following are equivalent.*

(1) *G is* Ex-*complete and $G = G^*$.*

(2) *If $(u_n) \in \mathrm{Ex}\, G$, there is n_0 with $u_n \leq n_0$ for all n.*

(3) *If $(u_n) \in \mathrm{Ex}\, G$, there is n_0 with $u_n = u_{n_0}$ for all $n \geq n_0$.*

(4) *YG is almost P.*

Proof. (1) $\Leftrightarrow$ (4) just combines Lemma 4.1 and Theorem 10.2.

(2) and (3) are easily seen to be equivalent.

(1) $\Rightarrow$ (2). If $(u_n) \in \mathrm{Ex}\, G$ and $\bigvee^G u_n \in G^*$, then (2) holds.

(3) $\Rightarrow$ (1). If (u_n) satisfies (3), then $u_{n_0} = \bigvee^G u_n$: (3) implies Ex-complete. Also, (3) implies $G = G^*$: if $g \in G^{\perp}$, then $(g \wedge n) \in \mathrm{Ex}\, G$ so with (3), $g = g \wedge n_0$ for some n_0. $\square$

Corollary 10.9. *Let X be compact. $C(X)$ is* Ex-*complete if and only if X is almost P.*

Proof. For $G = C(X), YG = X$. Apply Corollary 10.8 ((1) $\Leftrightarrow$ (4)). $\square$

Remarks 10.10. (a) Our definitions of expanding sequence, and Ex-complete, resemble those from [**25**] of tower, and tower-complete, respectively, and a result similar to Corollary 10.7 is implicit there (combine Theorems 1, 2 and Proposition 1).

(b) Corollary 10.8 has the form: An order-algebraic property of G is equivalent to a topological property of YG. Of course, for G *a priori* of the form $C(X)$, there are many such theorems, e.g., Corollary 10.9, but abundantly in [**7**].

11. ∗-max and lateral σ-completeness. The characterization of ∗-max as Ex-complete suggests comparison with other "σ-completeness" properties. The results are Theorems 11.2 and 11.6.

Definitions 11.1. Let $G \in |W|$.

G is laterally σ-complete ($L(\omega_1)$) if for all countable disjoint $\{g_n\} \in G^+$, $\bigvee^G g_n$ exists. (See [**13**], *et al.*)

G is densely laterally σ-complete ($\delta L(\omega_1)$) if for all countable disjoint $\{g_n\} \subseteq G^\perp$ with $\bigcap g_n^\perp = \{0\}$, $\bigvee^G g_n$ exists. ($\bigcap g_n^\perp = \{0\}$ means $\bigcup \operatorname{coz} g_n$ is dense in YG.)

($\delta L(\omega_1)$ seems to be a new definition. Of course, one may make these definitions for arbitrary ℓ-groups. This won't concern us here.)

A space is called zero-dimensional if it has a base of clopen sets.

Theorem 11.2. (a) *If G is* $*$-max*, then G is $\delta L(\omega_1)$. The converse fails.*

(b) *If G is $\delta L(\omega_1)$ and YG is zero-dimensional, then G is* $*$-max*.*

Proof. (a) Suppose G is $*$-max and $\{g_n\} \subseteq G^\perp$ is disjoint. with $\bigcup \operatorname{coz} g_n$ dense in YG. Then, $S \equiv \bigcup_n \operatorname{coz} g_n \cap g_n^{-1}\mathbf{R} \in \operatorname{dcoz} YG$ and $f = [g_n \text{ on } \operatorname{coz} g_n \cap g_n^{-1}\mathbf{R}]$ is in $\operatorname{loc}(G, S)$; this extends to $g \in G$ since G is $*$-max. By Lemma 10.6 (c), $g = \bigvee^G g_n$.

Here is an example of G which is $\delta L(\omega_1)$, not $*$-max. In [**18**, 3.13], we find a space X which is quasi-F, connected, locally compact, σ-compact, not compact, and $[S \in \operatorname{dcoz} \beta X \Rightarrow S \supseteq X]$. Let $G = C(\beta X)$, so $YG = \beta X$. Now, βX is quasi-F (since X is), $X \in \operatorname{dcoz} \beta X$ (since X is locally compact and σ-compact), $D(\beta X) \approx C(X) \neq C^*(X)$ (since X is not compact)), and $C^*(X) \approx C(\beta X)$. So, G is not $*$-max (e.g., since $\operatorname{Trunc} G = D(\beta X)^+$). But G is $\delta L(\omega_1)$: if $\{g_n\} \subseteq G^\perp$ is disjoint with $S \equiv \bigcup \operatorname{coz} g_n$ dense, then $S \supseteq X$ and $X = X \cap S = \bigcup (X \cap \operatorname{coz} g_n)$, so there is n_0 with $X \cap \operatorname{coz} g_{n_0} \neq \varnothing$. Then, $(X \cap \operatorname{coz} g_{n_0}) \cap (\bigcup_{n \neq n_0} (X \cap \operatorname{coz} g_n)) = \varnothing$ (since $\{g_n\}$ is disjoint). Since X is connected, $\bigcup_{n \neq n_0} (X \cap \operatorname{coz} g_n) = \varnothing$, so $g_n = 0$ for $n \neq n_0$. Thus, $g_{n_0} = \bigvee^G g_n$.

(b) Suppose the hypotheses. We show $*$-max using Theorem 5.3. Let $S \in \operatorname{dcoz} YG$, and let $f \in \operatorname{loc}(G, S)^+$. We can suppose $f \geq 1$. Write $S = \bigcup_n U_n$ for disjoint $\{U_n\} \subseteq \operatorname{clop} YG$. (Cover S by clopen sets contained in S, take a countable subcover, then disjointify by induction). Let c_n be the characteristic function of U_n; $c_n \in G$ by Theorem 1.3. For each n, there is $m(n) \in \mathbf{N}$ with $[f \leq m(n) \text{ on } U_n]$. Let $g_n = f \wedge (m(n) c_n)$, so $g_n = [f \text{ on } U_n;\ 0 \text{ off } U_n]$. We have $\operatorname{coz} g_n = U_n$ (since $f \geq 1$), so $\{g_n\}$ is disjoint with $\bigcup_n \operatorname{coz} g_n = S$. Thus, there is $\bigvee^G g_n = g$, and obviously $g \mid S = f$. $\square$

Discussion of $L(\omega_1)$ requires further ideas.

Definitions 11.3. (I) Let $G \in |W|$.

G is complemented if for all $g_1 \in G^+$ there exists $g_2 \in G^+$ with $g_1 \wedge g_2 = 0$ and $\{g_1 \vee g_2\} = \{0\}$ (which means $\operatorname{coz} g_1 \cup \operatorname{coz} g_2$ is dense in YG). (See [**11**, and its bibliography].)

G is σ-complemented if for all countable $\mathcal{G}_1 \subseteq G^+$ there exists countable $\mathcal{G}_2 \subseteq G^+$ with $g_1 \wedge g_2 = 0$ for all $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$, and $(\mathcal{G}_1 \cup \mathcal{G}_2)^\perp = \{0\}$.

(II) Let Y be compact Hausdorff (or, more generally).

Y is basically disconnected (BD) if each cozero-set has open closure (see [**7**]).

Y is cozero-complemented if for all cozero-set U there exists a cozero-set V with $U \cup V$ dense, i.e., if $C(Y)$ is complemented. (See [**11**].)

("σ-complemented" seems to be a new definition. Of course, one may make these definitions for arbitrary l-groups, where, in fact, one could define for infinite cardinals α, β, (α, β)-complemented: for all $\mathcal{G}_1 \subseteq G^+$ with $|\mathcal{G}_1| \leq \alpha$ $\exists$ disjoint $\mathcal{G}_2 \subseteq G^+$ with $|\mathcal{G}_2| < \beta$ with $(\mathcal{G}_1 \cup \mathcal{G}_2)^\perp = \{0\}$. This won't concern us here.)

We have several facts, some easy and some known.

Proposition 11.4. [**18**, 2.13]. (1) *Y is* BD *if and only if Y is quasi-*F *and cozero-complemented.*

(2) *Let $G \in |W|$.* (a) *U is a cozero-set of YG if and only if there exists countable $\mathcal{G} \subseteq G^{*+}$ with $U = \bigcup\{\operatorname{coz} g : g \in \mathcal{G}\}$.*

(b) *YG is cozero-complemented if and only if G is σ-complemented.*

(c) *YG is* BD *if and only if YG is quasi-*F *and G is σ-complemented.*

Proof. (1) See [**18**].

(2) (a) $\{\operatorname{coz} g : g \in G^*\}$ is an open base in YG (by Theorem 1.3), the family $\{\operatorname{coz} f : f \in C(YG)\}$ is closed under countable union [**7**], and every member is Lindelöf since YG is compact.

(b) Use (a).

(c) Use (b) and (1). □

Proposition 11.5. *Let $G \in |W|$. Then $(n) \Rightarrow (n{+}1)$ for $n = 1, 2, 3$.*

(1) *G is $L(\omega_1)$.*

(2) *YG is* BD.

(3) *YG is zero-dimensional, and G is complemented and σ-complemented.*

(4) *YG is zero dimensional and cozero-complemented.*

Proof. (1) $\Rightarrow$ (2). [**12**, 2.4 and 3.2].

(2) $\Rightarrow$ (3). Suppose YG is BD. [**7**] shows zero-dimensional. If $\mathcal{G}_1 \subseteq \mathcal{G}^+$ is countable, $\bigcup\{\operatorname{coz} g : \mathcal{G}_1\}$ is cozero in YG (as noted above), so has open closure U. Then $V = YG - U$ is clopen, and its characteristic function, say g_2, has $g_2 \in G$ (by Theorem 1.3). This shows G is both complemented (use $\mathcal{G}_1 = \{g_1\}$) and σ-complemented (even "(ω_1, ω)-complemented").

(3) $\Rightarrow$ (4). Use Proposition 11.4 (2) (b). □

Assembling all this creates the following (perhaps excessive) list.

Theorem 11.6. *For $G \in |W|$, the following are equivalent.*

(1) *G is $L(\omega_1)$.*

(2) *G is* $*$-max *and YG is* BD.

(3) *G is* $*$-max, *complemented and σ-complemented.*

(4) *G is* $*$-max *and σ-complemented.*

(5) *G is $\delta L(\omega_1)$ and YG is* BD.

(6) *G is $\delta L(\omega_1)$, complemented and σ-complemented, and YG is zero-dimensional.*

(7) *G is $\delta L(\omega_1)$ and σ-complemented, and YG is zero-dimensional.*

Proof. We show $[1 \Rightarrow 5 \Rightarrow 2 \Rightarrow 1]$, $[2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2]$, $[5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 4]$.

$1 \Rightarrow 5$. Obvious.

$5 \Rightarrow 2$. By Theorem 11.2.

$2 \Rightarrow 1$. Suppose $\{g_n\} \subseteq G^+$ is disjoint, Let $r\mathrm{coz}\, g_n = \{x : 0 < g_n(x) < +\infty\}$. Then, $U = \overline{\bigcup r\mathrm{coz}\, g_n}$ is clopen, $S = \bigcup r\mathrm{coz}\, g_n \cup (YG - U) \in \mathrm{dcoz}\, YG$, and $f \equiv [g_n \text{ on } r\mathrm{coz}\, g_n;\ 0 \text{ on } YG - U] \in \mathrm{loc}\,(G, S)$. So, there is $g \in G$ with $g = f$ on S. Obviously, $g = \bigvee^G g_n$.

$2 \Rightarrow 3$. Proposition 11.5 ($2 \Rightarrow 3$).

$3 \Rightarrow 4$. Obvious.

$4 \Rightarrow 2$. YG is quasi-F (Theorem 7.1) and cozero-complemented (Proposition 11.4 (2)). Apply Proposition 11.4 (1).

$5 \Rightarrow 6$. Apply Proposition 11.4 (1).

$6 \Rightarrow 7$. Obvious.

$7 \Rightarrow 4$. Theorem 11.2. $\square$

The situation, especially $\delta L(\omega_1)$, probably deserves further study. We leave the subject for now.

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