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EXISTENCE OF PSEUDO ALMOST AUTOMORPHIC MILD SOLUTIONS TO SOME NONAUTONOMOUS SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we make extensive use of Schauder fixed point principle and exponential stability tools to obtain the existence of pseudo-almost automorphic solutions to some classes of nonautonomous first and second-order abstract differential equations. To illustrate our abstract results, the existence of pseudo almost automorphic solutions to the so called Sine-Gordon boundary value problem will be discussed.

1. Introduction. Fix a Banach space $\mathbf{X}$. This paper is mainly motivated by the paper by Goldstein and N'Guérékata [**21**], in which the existence of *almost automorphic* solutions to the autonomous differential equation

$$\frac{du}{dt} = Au + G(t,u), \quad t \in \mathbf{R} \tag{1.1}$$

where $A : D(A) \subset \mathbf{X} \mapsto \mathbf{X}$ is a closed linear operator on $\mathbf{X}$ which generates an exponentially stable C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ and the function $G : \mathbf{R} \times \mathbf{X} \mapsto \mathbf{X}$ is given by $G(t,u) = P(t)Q(u)$ with P, Q being continuous functions satisfying some additional conditions, was established. The main tools utilized in [**21**] are fractional powers of linear operators and the well-known Schauder fixed point principle.

This paper has two main goals. The first objective consists of generalizing the result obtained in [**21**] by studying the existence of pseudo almost automorphic solutions to the nonautonomous differential

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equation

$$\frac{du}{dt} = A(t)u + F(t,u), \quad t \in \mathbf{R} \tag{1.2}$$

where $A(t)$ for $t \in \mathbf{R}$ is a family of closed linear operators with domains $D(A(t))$ satisfying the so called Acquistapace-Terreni conditions, and the function $F : \mathbf{R} \times \mathbf{X} \mapsto \mathbf{X}$ is compact pseudo almost automorphic in $t \in \mathbf{R}$ uniformly in the second variable. For that, we will make extensive use of ideas and techniques utilized in [**21**], exponential stability tools and the Schauder fixed point theorem.

Let $\mathbf{H}$ be an infinite-dimensional separable Hilbert space over the field of complex numbers. The second goal in this paper consists of making use of the existence results for equation (1.2) to study the existence of pseudo almost automorphic solutions to the class of second-order differential equations

$$\frac{d^2u}{dt^2} + a(t)\frac{du}{dt} + b(t)Au = f(t,u), \quad t \in \mathbf{R}, \tag{1.3}$$

where $A : D(A) \subset \mathbf{H} \mapsto \mathbf{H}$ is a self-adjoint linear operator whose spectrum consists of isolated eigenvalues: $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ with each eigenvalue having a finite multiplicity γ_j equaling the multiplicity of the corresponding eigenspace, the functions $a, b : \mathbf{R} \mapsto (0, \infty)$ are continuous, and $f : \mathbf{R} \times \mathbf{H} \mapsto \mathbf{H}$ is jointly continuous satisfying some additional conditions.

For that, the main idea consists of rewriting equation (1.3) as a nonautonomous first-order differential equation in $D(A) \times \mathbf{H}$ involving the family of 2×2-operator matrices $\{\mathcal{A}(t)\}_{t\in\mathbf{R}}$. Indeed, Setting $z := \binom{u}{u'}$, equation (1.3) can be rewritten in the following form

$$\frac{dz}{dt} = \mathcal{A}(t)z + \mathcal{G}(t,z), \quad t \in \mathbf{R}, \tag{1.4}$$

where $\mathcal{A}(t)$ is the family of 2×2-operator matrices defined by

$$\mathcal{A}(t) = \begin{pmatrix} 0 & I_{\mathbf{H}} \\ -b(t)A & -a(t)I_{\mathbf{H}} \end{pmatrix}$$

whose domain $D(\mathcal{A}(t))$ is constant in $t \in \mathbf{R}$ and is given by

$$D = D(A) \times \mathbf{H}$$

for all $t \in \mathbf{R}$.

Moreover, the semilinear term $\mathcal{G}$ appearing in equation (1.4) is defined on $\mathbf{R} \times D$ by

$$\mathcal{G}(t, z) = \begin{pmatrix} 0 \\ f(t, u) \end{pmatrix}.$$

The concept of pseudo almost automorphy is a powerful generalization of both the notion of almost automorphy due to Bochner [**8**] and that of pseudo almost periodicity due to Zhang (see [**14**]), which had been introduced in the literature a few years ago by Liang, Xiao and Zhang [**27, 37, 38**]. Such a concept, since its introduction in the literature, has recently generated several developments, see, e.g., [**11, 12, 19, 20, 26**].

The existence of almost periodic, almost automorphic, pseudo-almost periodic, and pseudo-almost automorphic constitute some of the most attractive topics in qualitative theory of differential equations due to their applications. Some contributions on pseudo-almost automorphic solutions to abstract differential and partial differential equations have recently been made; among them are [**11, 12, 19, 20, 26, 27, 37, 38**]. However, the use of the Schauder fixed point theorem to deal with the existence of pseudo-almost automorphic (mild) solutions to evolution equations of the form (1.3) in the nonautonomous setting is an untreated original question, which in fact is the main motivation of the present paper.

The paper is organized as follows. Section 2 is devoted to preliminary facts related to the existence of an evolution family. Some preliminary results on intermediate spaces are also stated there. In addition, basic definitions and results on the concept of pseudo-almost automorphic, respectively, compact pseudo-almost automorphic, functions are given. In Sections 3 and 4, we first state and prove the main result. In Section 5, we give a few examples to illustrate our main result.

2. Preliminaries. Let $\mathbf{H}$ be an infinite-dimensional separable Hilbert space over the field of complex numbers equipped with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. In this paper, $A : D(A) \subset \mathbf{H} \mapsto \mathbf{H}$ stands for a self-adjoint (possibly unbounded) linear operator on $\mathbf{H}$ whose spectrum consists of isolated eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_l \to \infty \quad \text{as } l \to \infty,$$

with each eigenvalue having a finite multiplicity γ_j equaling the multiplicity of the corresponding eigenspace. Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j\geq 1}$.

Clearly, for each $u \in D(A)$ where $D(A) := \{u \in \mathbf{H} : \sum_{j=1}^{\infty} \lambda_j^2 \|E_j u\|^2 < \infty\}$, we have

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u,$$

where $E_j u = \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k$. Note that $\{E_j\}_{j\geq 1}$ is a sequence of orthogonal projections on $\mathbf{H}$. Moreover, each $u \in \mathbf{H}$ can written as follows:

$$u = \sum_{j=1}^{\infty} E_j u.$$

It should also be mentioned that the operator $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, which is explicitly expressed in terms of those orthogonal projections E_j by, for all $u \in \mathbf{H}$,

$$T(t)u = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j u.$$

In addition, the fractional powers A^r $(r \geq 0)$ of A exist and are given by

$$D(A^r) = \left\{ u \in \mathbf{H} : \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j u\|^2 < \infty \right\}$$

and

$$A^r u = \sum_{j=1}^{\infty} \lambda_j^{2r} E_j u, \quad \text{for all } u \in D(A^r).$$

Let $(\mathbf{X}, \|\cdot\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbf{X}$, then, $D(L)$, $\rho(L)$, $\sigma(L)$, $N(L)$ and $R(L)$ stand respectively for its domain, resolvent, spectrum, null-space or kernel; and range. If $L : D = D(L) \subset \mathbf{X} \mapsto \mathbf{X}$ is a closed linear operator, one denotes its graph norm by $\|\cdot\|_D$. Clearly, $(D, \|\cdot\|_D)$ is a Banach space. Moreover, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(A)$. If $\mathbf{Y}, \mathbf{Z}$ are Banach spaces, then the space $B(\mathbf{Y}, \mathbf{Z})$ denotes the collection of all bounded linear operators from $\mathbf{Y}$ into $\mathbf{Z}$ equipped with its natural topology. This is simply denoted by $B(\mathbf{Y})$ when $\mathbf{Y} = \mathbf{Z}$. If P is a projection, we set $Q = I - P$.

2.1. Evolution families. (H.1). The family of closed linear operators $A(t)$ for $t \in \mathbf{R}$ on $\mathbf{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfies the so called Acquistapace-Terreni conditions, that is, there exist constants $\omega \geq 0$, $\theta \in ((\pi/2), \pi)$, $K, L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$S_\theta \cup \{0\} \subset \rho(A(t) - \omega) \ni \lambda, \qquad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|}$$

and

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \\ \leq L\,|t - s|^\mu\,|\lambda|^{-\nu}$$

for $t, s \in \mathbf{R}$, $\lambda \in S_\theta := \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

It should be mentioned that (H.1) was introduced in the literature by Acquistapace and Terreni in [**2, 3**] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family

$$\mathcal{U} = \{U(t, s) : t, s \in \mathbf{R} \quad \text{such that } t \geq s\}$$

on $\mathbf{X}$ associated with $A(t)$ such that $U(t, s)\mathbf{X} \subset D(A(t))$ for all $t, s \in \mathbf{R}$ with $t \geq s$, and

(a) $U(t, s)U(s, r) = U(t, r)$ for $t, s, r \in \mathbf{R}$ such that $t \geq s \geq r$;

(b) $U(t, t) = I$ for $t \in \mathbf{R}$ where I is the identity operator of $\mathbf{X}$;

(c) $(t, s) \mapsto U(t, s) \in B(\mathbf{X})$ is continuous for $t > s$;

(d) $U(\cdot, s) \in C^1((s,\infty), B(\mathbf{X}))$,

$$\frac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$$

and

$$\left\|A(t)^k U(t,s)\right\| \leq K\,(t-s)^{-k},$$

for $0 < t-s \leq 1$, $k = 0, 1$; and

(e) $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

It should also be mentioned that the above-mentioned properties were mainly established in [**1**, Theorem 2.3] and [**40**, Theorem 2.1], see also [**3, 39**]. In that case, we say that $A(\cdot)$ generates the evolution family $U(\cdot,\cdot)$.

Definition 2.1. One says that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ ($t \in \mathbf{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;

(g) the restriction $U_Q(t,s) : Q(s)\mathbf{X} \to Q(t)\mathbf{X}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$); and

(h) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|\widetilde{U}_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbf{R}$.

This setting requires some estimates related to $U(t,s)$. For that, we make extensive use of the real interpolation spaces of order (α,∞) between $\mathbf{X}$ and $D(A(t))$, where $\alpha \in (0,1)$. We refer the reader to the excellent books [**4, 18, 28**] for proofs and further information on these interpolation spaces.

Let A be a sectorial operator on $\mathbf{X}$ (for that, in assumption (H.1), replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbf{X}_\alpha^A := \{x \in \mathbf{X} : \|x\|_\alpha^A := \sup\nolimits_{r>0} \|r^\alpha (A-\omega)R(r, A-\omega)x\| < \infty\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience, we further write

$$\mathbf{X}_0^A := \mathbf{X}, \qquad \|x\|_0^A := \|x\|, \qquad \mathbf{X}_1^A := D(A)$$

and

$$\|x\|_1^A := \|(\omega - A)x\|.$$

Moreover, let $\widehat{\mathbf{X}}^A := \overline{D(A)}$ of $\mathbf{X}$. In particular, we have the following continuous embedding

$$D(A) \hookrightarrow \mathbf{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbf{X}_\alpha^A \hookrightarrow \widehat{\mathbf{X}}^A \hookrightarrow \mathbf{X}, \tag{2.1}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbf{X}_\alpha^A$ and $\mathbf{X}$. However, we have the following continuous injection

$$\mathbf{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbf{R}$, satisfying (H.1), we set

$$\mathbf{X}_\alpha^t := \mathbf{X}_\alpha^{A(t)}, \quad \widehat{\mathbf{X}}^t := \widehat{\mathbf{X}}^{A(t)}$$

for $0 \le \alpha \le 1$ and $t \in \mathbf{R}$, with the corresponding norms. Then the embedding in equation (2.1) holds with constants independent of $t \in \mathbf{R}$. These interpolation spaces are of the class $\mathcal{J}_\alpha$ ([**28**, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \le c(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)).$$

We have the following fundamental estimates for the evolution family $U(t,s)$.

Proposition 2.2 [**5**]. *Suppose the evolution family $U = U(t,s)$ has exponential dichotomy. For $x \in \mathbf{X}$, $0 \le \alpha \le 1$ and $t > s$, the following hold*:

(i) *There is a constant* $c(\alpha)$, *such that*

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\delta(t-s)/2}(t-s)^{-\alpha}\|x\|. \tag{2.2}$$

(ii) *There is a constant* $m(\alpha)$, *such that*

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.3}$$

It should be mentioned that, if $U(t,s)$ is exponentially stable, then $P(t) = I$ and $Q(t) = I - P(t) = 0$ for all $t \in \mathbf{R}$. In that case, (2.2) still holds and can be rewritten as follows: for all $x \in \mathbf{X}$,

$$\|U(t,s)x\|_\alpha^t \le c(\alpha)e^{-\delta(t-s)/2}(t-s)^{-\alpha}\|x\|. \tag{2.4}$$

In addition to the above we also assume that the following assumptions hold:

(H.2). The evolution family $U = U(t,s)$ is exponentially stable, that is, there exist constants $M, \delta > 0$ such that $\|U(t,s)\| \le Me^{-\delta(t-s)}$ for all $t \ge s$.

(H.3). There exist α, β with $0 < \alpha < \beta < 1$ and such that

$$\mathbf{X}_\alpha^t = \mathbf{X}_\alpha \quad \text{and} \quad \mathbf{X}_\beta^t = \mathbf{X}_\beta,$$

for all $t \in \mathbf{R}$, with uniform equivalent norms.

2.2. Pseudo-almost automorphic functions. Let $BC(\mathbf{R}, \mathbf{X})$ (respectively, $BC(\mathbf{R} \times \mathbf{Y}, \mathbf{X})$) denote the collection of all $\mathbf{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbf{R} \times \mathbf{Y} \mapsto \mathbf{X}$). The space $BC(\mathbf{R}, \mathbf{X})$ equipped with its natural norm, that is, the sup norm defined by

$$\|u\|_\infty = \sup_{t \in \mathbf{R}} \|u(t)\|,$$

is a Banach space. Furthermore, $C(\mathbf{R}, \mathbf{Y})$ (respectively, $C(\mathbf{R} \times \mathbf{Y}, \mathbf{X})$) denotes the class of continuous functions from $\mathbf{R}$ into $\mathbf{Y}$ (respectively, the class of jointly continuous functions $F : \mathbf{R} \times \mathbf{Y} \mapsto \mathbf{X}$).

Definition 2.3. A function $f \in C(\mathbf{R}, \mathbf{X})$ is said to be almost automorphic, respectively, compact almost automorphic, if, for every sequence of real numbers $(s'_n)_{n\in\mathbf{N}}$, there exists a subsequence $(s_n)_{n\in\mathbf{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

pointwise on $\mathbf{R}$ (respectively, uniformly on compacts of $\mathbf{R}$).

If the convergence above is uniform in $t \in \mathbf{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbf{X})$, respectively, $KAA(\mathbf{X})$, the collection of all almost automorphic functions $\mathbf{R} \mapsto \mathbf{X}$, respectively, compact almost automorphic functions $\mathbf{R} \to \mathbf{X}$. Note that $AA(\mathbf{X})$ equipped with the sup-norm $\|\cdot\|_\infty$ turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

Theorem 2.4 [**29, 30**]. *If $f, f_1, f_2 \in AA(\mathbf{X})$, then*:

(i) $f_1 + f_2 \in AA(\mathbf{X})$,

(ii) $\lambda f \in AA(\mathbf{X})$ *for any scalar* λ,

(iii) $f_\alpha \in AA(\mathbf{X})$ *where* $f_\alpha : \mathbf{R} \to \mathbf{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,

(iv) *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbf{R}\}$ *is relatively compact in* $\mathbf{X}$; *thus, f is bounded in norm,*

(v) *if $f_n \to f$ uniformly on $\mathbf{R}$ where each $f_n \in AA(\mathbf{X})$, then $f \in AA(\mathbf{X})$ too.*

In addition to the above-mentioned properties, we have the following property due to Bugajewski and Diagana [**9**]:

(vi) if $g \in L^1(\mathbf{R})$, then $f * g \in AA(\mathbf{R})$, where $f * g$ is the convolution of f with g on $\mathbf{R}$.

Let $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$ be another Banach space.

Definition 2.5. A jointly continuous function $F : \mathbf{R}\times\mathbf{Y} \mapsto \mathbf{X}$ is said to be almost automorphic, respectively, compact almost automorphic functions, in $t \in \mathbf{R}$ if $t \mapsto F(t,x)$ is almost automorphic, respectively, compact almost automorphic functions, for all $x \in K$ ($K \subset \mathbf{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s'_n)_{n\in\mathbf{N}}$, there exists a subsequence $(s_n)_{n\in\mathbf{N}}$ such that

$$G(t,x) := \lim_{n\to\infty} F(t+s_n, x)$$

and

$$\lim_{n\to\infty} G(t-s_n, x) = F(t,x)$$

pointwise on $\mathbf{R}$, respectively, uniformly on compacts of $\mathbf{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbf{Y},\mathbf{X})$, respectively, $KAA(\mathbf{Y},\mathbf{X})$.

For more on almost automorphic functions and related issues, we refer the reader to the excellent book by N'Guérékata [**29**].

Define

$$PAP_0(\mathbf{R},\mathbf{X}) := \left\{ f \in BC(\mathbf{R},\mathbf{X}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(s)\|\, ds = 0 \right\}.$$

Similarly, $PAP_0(\mathbf{Y},\mathbf{X})$ will denote the collection of all bounded continuous functions $F : \mathbf{R}\times\mathbf{Y} \mapsto \mathbf{X}$ such that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|F(s,x)\|\, ds = 0$$

uniformly in $x \in K$, where $K \subset \mathbf{Y}$ is any bounded subset.

Definition 2.6 (Liang et al. [**27, 37**]). A function $f \in BC(\mathbf{R},\mathbf{X})$ is called *pseudo almost automorphic* if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbf{X})$ and $\phi \in PAP_0(\mathbf{X})$. The collection of such functions will be denoted by $PAA(\mathbf{X})$.

The functions g and ϕ appearing in Definition 2.6 are respectively called the *almost automorphic* and the *ergodic perturbation* components of f.

Definition 2.7. A function $f \in BC(\mathbf{R}, \mathbf{X})$ is called compact *pseudo almost automorphic* if it can be expressed as $f = g + \varphi$, where $g \in KAA(\mathbf{X})$ and $\varphi \in PAP_0(\mathbf{X})$. The collection of such functions will be denoted by $KPAA(\mathbf{X})$.

Definition 2.8. A bounded continuous function $F : \mathbf{R} \times \mathbf{Y} \mapsto \mathbf{X}$ belongs to $AA(Y, \mathbf{X})$ whenever it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbf{Y}, \mathbf{X})$ and $\Phi \in PAP_0(\mathbf{Y}, \mathbf{X})$. The collection of such functions will be denoted by $PAA(\mathbf{Y}, \mathbf{X})$.

We now collect a few useful properties of pseudo almost automorphic functions.

Proposition 2.9. *If $g \in L^1(\mathbf{R})$, $f \in PAA(\mathbf{R})$, then $f * g \in PAA(\mathbf{R})$, where $f * g$ is the convolution of f with g on $\mathbf{R}$.*

The proof of Proposition 2.9 is based upon Bugajewski and Diagana [**9**] and Bugajewski, Diagana and Mahop [**10**].

A substantial result is the next theorem, which is due to Liang et al. [**37**].

Theorem 2.10 [**37**]. *The space $PAA(\mathbf{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.*

The next composition result, that is, Theorem 2.11, is a consequence of [**26**, Theorem 2.4].

Theorem 2.11. *Suppose $f : \mathbf{R} \times \mathbf{Y} \mapsto \mathbf{X}$ belongs to $PAA(\mathbf{Y}, \mathbf{X})$; $f = g + h$, with $x \mapsto g(t, x)$ being uniformly continuous on any bounded subset K of $\mathbf{Y}$ uniformly in $t \in \mathbf{R}$. Furthermore, we suppose that there exists an $L > 0$ such that*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|_{\mathbf{Y}}$$

for all $x, y \in \mathbf{Y}$ and $t \in \mathbf{R}$.

Then the function defined by $h(t) = f(t, \varphi(t))$ belongs to $PAA(\mathbf{X})$, provided $\varphi \in PAA(\mathbf{Y})$.

We also have:

Theorem 2.12 [**37**]. *If $f : \mathbf{R} \times \mathbf{Y} \mapsto \mathbf{X}$ belongs to $PAA(\mathbf{Y}, \mathbf{X})$ and if $x \mapsto f(t, x)$ is uniformly continuous on any bounded subset K of $\mathbf{Y}$ for each $t \in \mathbf{R}$, then the function defined by $h(t) = f(t, \varphi(t))$ belongs to $PAA(\mathbf{X})$ provided $\varphi \in PAA(\mathbf{Y})$.*

3. Main results. Throughout the rest of the paper we fix the real numbers α, β such that $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$.

Consider the nonautonomous differential equation

$$\frac{du}{dt} = A(t)u + F(t, u), \quad t \in \mathbf{R}, \tag{3.1}$$

where $F : \mathbf{R} \times \mathbf{X} \mapsto \mathbf{X}$ is jointly continuous.

Definition 3.1. A continuous function $u : \mathbf{R} \mapsto \mathbf{X}$ is said to be a mild solution to equation (3.1) provided that

$$u(t) = U(t, s)u(s) + \int_s^t U(t, r)F(r, u(r))\, dr.$$

for all $t \in \mathbf{R}$.

If F is a bounded jointly continuous function, it is not difficult to show that

$$u(t) = \int_{-\infty}^t U(t, s)F(s, u(s))\, ds, \quad t \in \tau \tag{3.2}$$

is a mild solution for (3.1).

To study the existence of pseudo almost automorphic solutions to equation (3.1), in addition to the previous assumptions, we suppose that the injection

$$\mathbf{X}_\beta \hookrightarrow \mathbf{X}$$

is compact and that the following additional assumptions hold:

(H.4) $R(\omega, A(\cdot)) \in KAA(B(\mathbf{X}))$.

(H.5) The function $F : \mathbf{R} \times \mathbf{X} \mapsto \mathbf{X}$ is compact pseudo almost automorphic in the first variable uniformly in the one. Furthermore, $u \mapsto F(t,u)$ is uniformly continuous on any bounded subset K of $\mathbf{X}$ for each $t \in \mathbf{R}$. Finally,

$$\|F(t,u)\|_\infty \le \mathcal{M}(\|u\|_\infty),$$

where $\mathcal{M} : \mathbf{R}^+ \mapsto \mathbf{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{M(r)}{r} = 0.$$

Throughout the rest of the paper, we set

$$Su(t) = \int_{-\infty}^{t} U(t,s)F(s,u(s))\,ds, \quad t \in \mathbf{R}.$$

Lemma 3.2 [**16**]. *Under assumptions* (H.1)–(H.5), *the mapping* $S : BC(\mathbf{R},\mathbf{X}) \mapsto BC(\mathbf{R},\mathbf{X}_\beta)$ *is well-defined and continuous.*

Lemma 3.3. *Under the assumptions* (H.1)–(H.5), *the integral operator* S *defined above maps* $KPAA(\mathbf{X})$ *into itself.*

Proof. Let $u \in KPAA(\mathbf{X})$. Setting $\phi(t) = F(t,u(t))$ and using Theorem 2.11, it follows that $\phi \in KPAA(\mathbf{X})$. Set $\phi = g + h$ where $g \in KAA(\mathbf{X})$ and $h \in PAP_0(\mathbf{X})$. Write $Su = S_1u + S_2u$, where

$$S_1u(t) = \int_{-\infty}^{t} U(t,s)g(s)\,ds \quad \text{and} \quad S_2u(t) = \int_{-\infty}^{t} U(t,s)h(s)\,ds.$$

We next show that $S_1u \in KAA(\mathbf{X})$ and $S_2u \in PAP_0(\mathbf{X})$. Indeed, since $g \in KAA(\mathbf{X})$, for every sequence of real numbers $(\tau_n')_{n\in\mathbf{N}}$, there exists a subsequence $(\tau_n)_{n\in\mathbf{N}}$ such that

$$\psi(t) := \lim_{n\to\infty} g(t+\tau_n)$$

and

$$\lim_{n\to\infty} \psi(t-\tau_n) = g(t)$$

uniformly on compacts of $\mathbf{R}$.

Set

$$M(t) = \int_{-\infty}^{t} U(t,s)g(s)\,ds$$

and

$$N(t) = \int_{-\infty}^{t} U(t,s)\psi(s)\,ds$$

for all $t \in \mathbf{R}$.

Now

$$\begin{aligned} M(t+\tau_n) - N(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n, s)g(s)\,ds \\ &\quad - \int_{-\infty}^{t} U(t,s)\psi(s)\,ds \\ &= \int_{-\infty}^{t} U(t+\tau, s+\tau_n)g(s+\tau_n)\,ds \\ &\quad - \int_{-\infty}^{t} U(t,s)\psi(s)\,ds \\ &= \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)(g(s+\tau_n) - \psi(s))\,ds \\ &\quad + \int_{-\infty}^{t} (U(t+\tau_n, s+\tau_n) - U(t,s))\psi(s)\,ds. \end{aligned}$$

Using the exponential stability of $U(t,s)$ and the Lebesgue dominated convergence theorem, one can easily see that

$$\left\| \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)(g(s+\tau_n) - \psi(s))\,ds \right\| \longrightarrow 0$$
$$\text{as } n \to \infty, \text{ uniformly on compacts of } \mathbf{R}.$$

Similarly, from [**6**], it follows that

$$\left\| \int_{-\infty}^{t} (U(t+\tau_n, s+\tau_n) - U(t,s))\psi(s)\,ds \right\| \longrightarrow 0$$
$$\text{as } n \to \infty, \text{ uniformly on compacts of } \mathbf{R}.$$

Therefore,

$$N(t) = \lim_{n\to\infty} M(t+\tau_n), \quad \text{uniformly on compacts of } \mathbf{R}.$$

Similarly,

$$M(t) = \lim_{n\to\infty} N(t-\tau_n), \quad \text{uniformly on compacts of } \mathbf{R}.$$

Therefore, $t \mapsto S_1 u(t)$ belongs to $KAA(\mathbf{X})$.

To complete the proof, we have to show that $t \mapsto S_2 u(t) \in PAP_0(\mathbf{X})$. First, note that $t \mapsto S_2 u(t)$ is a bounded continuous function. It remains to show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|S_2 u(t)\| \, dt = 0.$$

Using the fact that the evolution family $U(t,s)$ is exponentially stable it follows that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|S_2 u(t)\| \, dt \\
&\qquad\qquad \le \lim_{T\to\infty} \frac{M}{2T} \int_{-T}^{T} \int_{0}^{+\infty} e^{-\delta s} \|h(t-s)\| \, ds \, dt \\
&\qquad\qquad \le \lim_{T\to\infty} M \int_{0}^{+\infty} e^{-\delta s} \frac{1}{2T} \int_{-T}^{T} \|h(t-s)\| \, dt \, ds.
\end{aligned}$$

Let $L_s(T) = (1/2T) \int_{-T}^{T} \|h(t-s)\| \, dt$. Since $PAP_0(\mathbf{X})$ is translation invariant, it follows that $t \mapsto h(t-s)$ belongs to $PAP_0(\mathbf{X})$ for each $s \in \mathbf{R}$, and hence

$$\lim_{T\mapsto\infty} \frac{1}{2T} \int_{-T}^{T} \|h(t-s)\| \, dt = 0$$

for each $s \in \mathbf{R}$.

One completes the proof by using the well-known Lebesgue dominated convergence theorem and the fact that $L_s(T) \mapsto 0$ as $T \to \infty$ for each $s \in \mathbf{R}$. $\square$

We need the following technical lemma:

Lemma 3.4. *For $x \in \mathbf{X}$, let α, β be real numbers such that $0 < \alpha < \beta < 1$ and $2\beta > \alpha + 1$. Then, for all $t > s$, there is a constant $r(\alpha, \beta)$, such that*

$$\|A(t)U(t,s)x\|_\beta \le r(\alpha,\beta)e^{-\delta(t-s)/4}(t-s)^{-\beta}\|x\|. \tag{3.3}$$

Proof. Let $x \in \mathbf{X}$. First of all, note that $\|A(t)U(t,s)\|_{B(\mathbf{X},\mathbf{X}_\beta)} \le K(t-s)^{-(1-\beta)}$ for all t, s such that $0 < t-s \le 1$ and $\beta \in [0,1]$.

Letting $t - s \ge 1$ and using (H.2) and the above-mentioned approximate, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\beta &= \|A(t)U(t,t-1)U(t-1,s)x\|_\beta \\
&\le \|A(t)U(t,t-1)\|_{B(\mathbf{X},\mathbf{X}_\beta)}\|U(t-1,s)x\| \\
&\le MKe^{\delta}e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-3\delta(t-s)/4}(t-s)^{\beta}(t-s)^{-\beta}e^{-\delta(t-s)/4}\|x\|.
\end{aligned}$$

Now since $e^{-3\delta(t-s)/4}(t-s)^{\beta} \to 0$ as $t \to \infty$, it follows that there exists $c_4(\beta) > 0$ such that

$$\|A(t)U(t,s)x\|_\beta \le c_4(\beta)(t-s)^{-\beta}e^{-\delta(t-s)/4}\|x\|.$$

Now, let $0 < t-s \le 1$. Using equation (2.2) and the fact that $2\beta > \alpha + 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\beta &= \|A(t)U\left(t, \frac{t+s}{2}\right)U\left(\frac{t+s}{2}, s\right)x\|_\beta \\
&\le \left\|A(t)U\left(t, \frac{t+s}{2}\right)\right\|_{B(\mathbf{X},\mathbf{X}_\beta)}\left\|U\left(\frac{t+s}{2}, s\right)x\right| \\
&\le k_1\left\|A(t)U\left(t, \frac{t+s}{2}\right)\right\|_{B(\mathbf{X},\mathbf{X}_\beta)}\left\|U\left(\frac{t+s}{2}, s\right)x\right\|_\alpha \\
&\le k_1K\left(\frac{t-s}{2}\right)^{\beta-1}c(\alpha)\left(\frac{t-s}{2}\right)^{-\alpha}e^{-\delta(t-s)/4}\|x\| \\
&= c_5(\alpha,\beta)(t-s)^{\beta-1-\alpha}e^{-\delta(t-s)/4}\|x\| \\
&\le c_5(\alpha,\beta)(t-s)^{-\beta}e^{-\delta(t-s)/4}\|x\|.
\end{aligned}$$

In summary, there exists an $r(\alpha, \beta) > 0$ such that

$$\|A(t)U(t,s)x\|_\beta \leq r(\alpha,\beta)(t-s)^{-\beta} e^{-\delta(t-s)/4}\|x\|$$

for all $t, s \in \mathbf{R}$ with $t \geq s$. □

Let $\gamma \in (0,1]$, and let $BC^\gamma(\mathbf{R}, \mathbf{X}_\alpha)$ be the collection of all bounded continuous functions from $\mathbf{R}$ into $\mathbf{X}_\alpha$ equipped with the following distance:

$$\Delta(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)},$$

where, for $h = f - g$,

$$\begin{aligned}\Delta_n(f,g) &= \Delta_n(h,0)\\ &= \|h\|_{C[-n,n]} + \gamma \cdot \sup\left\{\frac{\|h(t)-h(s)\|_\alpha}{|t-s|^\gamma} : t,s \in [-n,n],\ t \neq s\right\}.\end{aligned}$$

Let $\mathbf{H}_\gamma(\mathbf{R}, \mathbf{X}_\alpha)$ be the locally convex Fréchet space $(BC^\gamma(\mathbf{R}, \mathbf{X}_\alpha), \Delta)$.

Lemma 3.5 [**16**]. *Under assumptions* (H.1)–(H.5)*, the mapping S defined previously maps bounded sets of $BC(\mathbf{R}, \mathbf{X})$ into bounded sets of $BC^\gamma(\mathbf{R}, \mathbf{X}_\beta)$ for some* $0 < \gamma < 1$.

Proof. Follows the same lines as in Diagana [**16**]. □

The proof of the next lemma follows along the same lines as that of Lemma 3.3 and hence is omitted.

Lemma 3.6. *The integral operator S maps bounded sets of $KPAA(\mathbf{X})$ into bounded sets of $BC^{1-\beta}(\mathbf{R}, \mathbf{X}_\beta) \cap KPAA(\mathbf{X})$.*

Similarly, the next lemma is a consequence of [**21**, Proposition 3.3].

Lemma 3.7. *The set $BC^{1-\beta}(\mathbf{R}, \mathbf{X}_\beta)$ is compactly contained in $BC(\mathbf{R}, \mathbf{X})$, that is, the canonical injection $id : BC^{1-\beta}(\mathbf{R}, \mathbf{X}_\beta) \mapsto BC(\mathbf{R}, \mathbf{X})$ is compact, which yields that*

$$id : BC^{1-\beta}(\mathbf{R}, \mathbf{X}_\beta) \cap KPAA(\mathbf{X}) \mapsto KPAA(\mathbf{X})$$

is compact, too.

Theorem 3.8. *Suppose assumptions* (H.1)–(H.5) *hold. Then the nonautonmous differential equation* (3.1) *has at least one pseudo almost automorphic solution.*

Proof. The proof follows along the same lines as that of [**21**, Proposition 3.4]. Let us recall that, in view of Lemma 3.5, we have

$$\|Su\|_{\beta,\infty} \leq d(\beta,\delta)\mathcal{M}(\|u\|_{\infty}$$

and

$$\|Su(t_2) - Su(t_1)\|_{\beta} \leq s(\alpha,\beta,\delta)\mathcal{M}(\|u\|_{\infty})|t_2 - t_1|$$

for all $u \in BC(\mathbf{R}, \mathbf{X}_{\beta})$, $t_1, t_2 \in \mathbf{R}$ with $t_1 \neq t_2$, where $d(\beta,\delta)$ and $s(\alpha,\beta,\delta)$ are positive constants. Consequently, $u \in BC(\mathbf{R}, \mathbf{X})$ and $\|u\|_{\infty} < R$ yield $Su \in BC^{1-\beta}(\mathbf{R}, \mathbf{X}_{\beta})$ and $\Delta(Su, 0) < R_1$ where $R_1 = c(\alpha,\beta,\delta)\mathcal{M}(R)$. Using the fact that $\|x\| \leq c\|x\|_{\beta}$ for all $x \in \mathbf{X}_{\beta}$, it follows that there exists an $r > 0$ such that, for all $R \geq r$, the following hold

$$S\Big(B_{KPAA(\mathbf{X})}(0,R)\Big) \subset B_{BC^{1-\beta}(\mathbf{R},\mathbf{X}_{\beta})}(0,R) \cap B_{KPAA(\mathbf{X})}(0,R).$$

In view of the above, it follows that $S : D \mapsto D$ is continuous and compact, where D is the ball in $KPAA(\mathbf{X})$ of radius R with $R \geq r$. Using the Schauder fixed point theorem it follows that S has a fixed-point, which obviously is a pseudo almost automorphic mild solution to (3.1). ◻

4. Pseudo almost automorphic solutions to some second-order differential equations. In this section, we study the existence of pseudo almost automorphic solutions to some classes of nonautonmous second-order abstract differential equations based on the theory of operator matrices and the previous existence results for equation (3.1). For more on the basic theory of operator matrices, we refer the reader to [**5, 7, 18, 24, 25, 32–36**].

In this section, we take $\mathbf{X} = D(A) \times \mathbf{H}$. We have previously seen that each $u \in \mathbf{H}$ can be written in terms of the sequence of orthogonal projections E_n as follows:

$$u = \sum_{n=1}^{\infty} \sum_{k=1}^{\gamma_n} \langle u, e_n^k \rangle e_n^k = \sum_{n=1}^{\infty} E_n u.$$

Moreover, for each $u \in D(A)$,

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u.$$

Theorem 4.1. *Under the previous assumptions and if $\mathcal{G}$ satisfies* (H.5)*, then the nonautonomous differential equation* (1.4) *has at least one bounded solution*

$$\begin{pmatrix} u \\ v \end{pmatrix} \in D(A) \times \mathbf{H},$$

which in addition is pseudo almost automorphic.

Proof. The proof is slightly similar to the higher-order case (Diagana [**15, 16**]). Indeed, for all $z := \binom{u}{v} \in D = D(A(t)) = D(A) \times \mathbf{H}$, we obtain the following

$$\begin{aligned} A(t)z &= \begin{pmatrix} 0 & I_{\mathbf{H}} \\ -b(t)A & -a(t)I_{\mathbf{H}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} v \\ -b(t)Au - a(t)v \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} E_n v \\ -b(t) \sum_{n=1}^{\infty} \lambda_n E_n u - a(t) \sum_{n=1}^{\infty} E_n v \end{pmatrix} \\ &= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ -b(t)\lambda_n & -a(t) \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \sum_{n=1}^{\infty} A_n(t) P_n z, \end{aligned}$$

where

$$P_n := \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \quad n \geq 1,$$

and

$$A_n(t) := \begin{pmatrix} 0 & 1 \\ -b(t)\lambda_n & -a(t) \end{pmatrix}, \quad n \geq 1.$$

Now, the characteristic equation for $A_n(t)$ is given by

$$\lambda^2 + a(t)\lambda + \lambda_n b(t) = 0. \tag{4.1}$$

Throughout the rest of the paper, we suppose

$$0 < \widetilde{a_0} := \inf_{t\in\mathbf{R}} a(t) \leq \sup_{t\in\mathbf{R}} a(t) := a_0, \quad \inf_{t\in\mathbf{R}} b(t) := b_0 > 0,$$

and

$$a_0^2 < 4\lambda_1 b_0. \tag{4.2}$$

From equation (4.2) it easily follows that the discriminant of equation (4.1) defined by $L_n(t) = a^2(t) - 4\lambda_n b(t) < 0$ for all $t \in \mathbf{R}$, $n \geq 1$, and hence all roots of (4.1) are nonzero (with nonzero real and imaginary parts) complex roots given by

$$\lambda_1^n(t) = \frac{-a(t) + i\sqrt{-L_n(t)}}{2}$$

and

$$\lambda_2^n(t) = \overline{\lambda_1^n(t)} = \frac{-a(t) - i\sqrt{-L_n(t)}}{2},$$

that is,

$$\sigma(A_n(t)) = \{\lambda_1^n(t), \lambda_2^n(t)\}.$$

In addition to the above, suppose

$$a_0 \in \left(0, 2\pi\sqrt{\frac{\lambda_1 b_0}{1+\pi^2}}\right).$$

Setting

$$\frac{1}{d} := \sup_{\substack{t\in\mathbf{R}\\ n\geq 1}} \left[\frac{1}{\sqrt{-L_n(t)}}\right],$$

one can easily see that $1/d \leq 1/\sqrt{4\lambda_1 b_0 - a_0^2}$; and hence, if we set

$$\widetilde{\omega} := \tan^{-1}\left(\frac{a_0}{2\sqrt{4\lambda_1 b_0 - a_0^2}}\right),$$

then

$$0 < \widetilde{\omega} < \frac{\pi}{2}.$$

Define

$$S_\omega = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| \leq \omega\},$$

where $\omega = (\pi/2) + \widetilde{\omega} \in ((\pi/2), \pi)$.

On the other hand, one can show without difficulty that $A_n(t) = K_n^{-1}(t) J_n(t) K_n(t)$, where $J_n(t), K_n(t)$ and $K_n^{-1}(t)$ are respectively given by

$$J_n(t) = \begin{pmatrix} \lambda_1^n(t) & 0 \\ 0 & \lambda_2^n(t) \end{pmatrix}, \qquad K_n(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1^n(t) & \lambda_2^n(t) \end{pmatrix},$$

and

$$K_n^{-1}(t) = \frac{1}{\lambda_1^n(t) - \lambda_2^n(t)} \begin{pmatrix} -\lambda_2^n(t) & 1 \\ \lambda_1^n(t) & -1 \end{pmatrix}.$$

For $\lambda \in S_\omega$ and $z \in \mathbf{X}$, one has

$$\begin{aligned} R(\lambda, A(t))z &= \sum_{n=1}^{\infty} (\lambda - A_n(t))^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n(t)(\lambda - J_n(t)P_n)^{-1} K_n^{-1}(t) P_n z. \end{aligned}$$

Hence,

$$\begin{aligned}
&\|R(\lambda, A(t))z\|^2 \\
&\qquad \leq \sum_{n=1}^{\infty} \|K_n(t)P_n(\lambda - J_n(t)P_n)^{-1}K_n^{-1}(t)P_n\|_{B(\mathbf{X})}^2 \|P_n z\|^2 \\
&\qquad \leq \sum_{n=1}^{\infty} \|K_n(t)P_n\|_{B(\mathbf{X})}^2 \|(\lambda - J_n(t)P_n)^{-1}\|_{B(\mathbf{X})}^2 \\
&\qquad\quad \times \|K_n^{-1}(t)P_n\|_{B(\mathbf{X})}^2 \|P_n z\|^2.
\end{aligned}$$

Moreover, for $z := \binom{z_1}{z_2} \in \mathbf{X}$, we obtain

$$\begin{aligned}
\|K_n(t)P_n z\|^2 &= \|E_n z_1 + E_n z_2\|^2 \\
&\quad + \|\lambda_1^n(t)E_n z_1 + \lambda_2^n(t)E_n z_2\|^2 \\
&\leq 3(1 + \lambda_1^n(t)^2)\|z\|^2.
\end{aligned}$$

Thus, there exists a $C_1 > 0$ such that

$$\|K_n(t)P_n z\| \leq C_1 \lambda_1^n(t)\|z\| \quad \text{for all } n \geq 1 \text{ and } t \in \mathbf{R}.$$

Similarly, for $z := \binom{z_1}{z_2} \in \mathbf{X}$, one can show that there is a $C_2 > 0$ such that

$$\|K_n^{-1}(t)P_n z\| \leq \frac{C_2}{\lambda_1^n}\|z\| \quad \text{for all } n \geq 1 \text{ and } t \in \mathbf{R}.$$

Now, for $z \in \mathbf{X}$, we have

$$\begin{aligned}
\|(\lambda - J_n P_n)^{-1} z\|^2 &= \left\| \begin{pmatrix} 1/(\lambda - \lambda_1^n) & 0 \\ 0 & 1/(\lambda - \lambda_2^n) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 \\
&\leq \frac{1}{|\lambda - \lambda_1^n|^2}\|z_1\|^2 + \frac{1}{|\lambda - \lambda_2^n|^2}\|z_2\|^2.
\end{aligned}$$

Let $\lambda_0 > 0$. Define the function

$$\eta(\lambda) := \frac{1 + |\lambda|}{|\lambda - \lambda_2^n|}.$$

It is clear that η is continuous and bounded on the closed set

$$\Sigma := \{\lambda \in \mathbf{C} : \ |\lambda| \leq \lambda_0, \ |\arg \lambda| \leq \omega\}.$$

On the other hand, it is clear that η is bounded for $|\lambda| > \lambda_0$. Thus, η is bounded on S_ω. If we take

$$N = \sup\left\{\frac{1+|\lambda|}{|\lambda - \lambda_j^n|} : \lambda \in S_\omega, \quad n \geq 1;\ j = 1, 2,\ t \in \mathbf{R}\right\}.$$

Therefore,

$$\|(\lambda - J_n P_n)^{-1} z\| \leq \frac{N}{1+|\lambda|}\|z\|, \quad \lambda \in S_\omega.$$

Consequently,

$$\|R(\lambda, A(t))\| \leq \frac{K}{1+|\lambda|}$$

for all $\lambda \in S_\omega$ and $t \in \mathbf{R}$.

First of all, note that the domain $D = D(A(t))$ is independent of t. Now note that the operator $A(t)$ is invertible with

$$A(t)^{-1} = \begin{pmatrix} -a(t)b(t)^{-1}A^{-1} & -b(t)^{-1}A^{-1} \\ I_{\mathbf{H}} & 0 \end{pmatrix}, \quad t \in \mathbf{R}.$$

Hence, for $t, s, r \in \mathbf{R}$, one has

$$\begin{aligned}&(A(t) - A(s))A(r)^{-1} \\ &\quad = \begin{pmatrix} 0 & 0 \\ [-a(r)b(r)^{-1}(b(s) - b(t)) + (a(s) - a(t))]I_{\mathbf{H}} & -b(r)^{-1}(b(s) - b(t)) \end{pmatrix},\end{aligned}$$

and hence, assuming that there exist $L_0, L_1 \geq 0$ and $\mu \in (0, 1]$ such that

$$|a(t) - a(s)| \leq L_0|t - s|^\mu, \qquad |b(t) - b(s)| \leq L_1|t - s|^\mu,$$

it easily follows that there exists a $C > 0$ such that

$$\|(A(t) - A(s))A(r)^{-1} z\| \leq C|t - s|^\mu \|z\|.$$

In summary, the family of operators $\{A(t)\}_{t\in\mathbf{R}}$ satisfy Acquistpace-Terreni conditions. Consequently, there exists an evolution family $U(t, s)$ associated with it. Let us now check that $U(t, s)$ has exponential dichotomy. First of all, note that, for every $t \in \mathbf{R}$, the family of linear operators $A(t)$ generate an analytic semigroup $(e^{\tau A(t)})_{\tau\geq 0}$ on $\mathbf{X}$ given by

$$e^{\tau A(t)}z = \sum_{n=0}^{\infty} K_n(t)^{-1}P_n e^{\tau J_n} P_n K_n(t) P_n z, \quad z \in \mathbf{X}.$$

On the other hand, we have

$$\|e^{\tau A(t)}z\| = \sum_{n=0}^{\infty} \|K_n(t)^{-1}P_n\|_{B(\mathbf{X})}\|e^{\tau J_n}P_n\|_{B(\mathbf{X})}\|K_n(t)P_n\|_{B(\mathbf{X})}\|P_n z\|,$$

with, for each $z = \binom{z_1}{z_2}$,

$$\begin{aligned}\|e^{\tau J_n}P_n z\|^2 &= \left\| \begin{pmatrix} e^{\lambda_1^n \tau}E_n & 0 \\ 0 & e^{\lambda_2^n \tau}E_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 \\ &\leq \|e^{\lambda_1^n \tau}E_n z_1\|^2 + \|e^{\lambda_2^n \tau}E_n z_2\|^2 \\ &\leq e^{-2\delta\tau}\|z\|^2,\end{aligned}$$

where $\delta = \widetilde{a_0}/4$. Therefore,

$$\|e^{\tau A(t)}\| \leq Ce^{-\delta\tau}, \quad \tau \geq 0. \tag{4.3}$$

Using the continuity of a, b and the equality

$$R(\lambda, A(t)) - R(\lambda, A(s)) = R(\lambda, A(t))(A(t) - A(s))R(\lambda, A(s)),$$

it follows that the mapping $J \ni t \mapsto R(\lambda, A(t))$ is strongly continuous for $\lambda \in S_\omega$ where $J \subset \mathbf{R}$ is an arbitrary compact interval. Therefore, $A(t)$ satisfies the assumptions of [**31**, Corollary 2.3], and thus the evolution family $(U(t,s))_{t\geq s}$ is exponentially stable.

It remains to check assumption (H.4). For that, we need to show that $A^{-1}(\cdot) \in KAA(B(\mathbf{X}))$. Since $t \mapsto a(t)$, $t \mapsto b(t)$ and $t \mapsto b(t)^{-1}$ are compact almost automorphic for every sequence of real numbers $(s_n')_{n\in\mathbf{N}}$, there exists a subsequence $(s_n)_{n\in\mathbf{N}}$ such that

$$\widetilde{a}(t) := \lim_{n\to\infty} a(t+s_n),$$

$$\lim_{n\to\infty} \widetilde{a}(t-s_n) = a(t),$$

$$\widetilde{b}(t) := \lim_{n\to\infty} b(t+s_n)$$

and

$$\lim_{n\to\infty} \widetilde{b}(t-s_n) = b(t)$$

uniformly on compacts of $\mathbf{R}$.

Consider

$$B(t) = \begin{pmatrix} 0 & I_{\mathbf{H}} \\ -\widetilde{b}(t)A & -\widetilde{a}(t)I_{\mathbf{H}} \end{pmatrix}, \quad t \in \mathbf{R}$$

and

$$\widetilde{B}(t) = \begin{pmatrix} -\widetilde{a}(t)\widetilde{b}(t)^{-1}A^{-1} & -\widetilde{b}(t)^{-1}A^{-1} \\ I_{\mathbf{H}} & 0 \end{pmatrix}, \quad t \in \mathbf{R}.$$

We have the following identity:

$$A(t+s_n)^{-1} - \widetilde{B}(t) = A^{-1}(t+s_n)(A(t+s_n) - B(t))\widetilde{B}(t). \tag{4.4}$$

Now

$$A(t+s_n) - B(t) = \begin{pmatrix} 0 & 0 \\ -(b(t+s_n) - \widetilde{b}(t))A & -(a(t+s_n) - \widetilde{a}(t))I_{\mathbf{H}} \end{pmatrix}.$$

Therefore, for $z := \binom{z_1}{z_2} \in D$, one has

$$\begin{aligned} &\|(A(t+s_n) - B(t))z\| \\ &\qquad \leq \|(b(t+s_n) - \widetilde{b}(t))Az_1\| + \|(a(t+s_n) - \widetilde{a}(t))z_2\| \\ &\qquad \leq \varepsilon(\|Az_1\| + \|z_2\|) \\ &\qquad \leq \varepsilon\|z\|_D, \end{aligned}$$

and, using equation (4.4), we obtain

$$\begin{aligned} &\|A(t+s_n)^{-1}y - \widetilde{B}(t)y\| \\ &\qquad \leq \|A(t+s_n)^{-1}(A(t+s_n) - B(t))\widetilde{B}(t)y\| \\ &\qquad \leq \|A(t+s_n)^{-1}\|_{B(\mathbf{X})} \\ &\qquad\quad + \|(A(t+s_n) - B(t))\|_{B(D,\mathbf{X})}\|\widetilde{A}(t)^{-1}y\|_D, \quad y \in \mathbf{X}. \end{aligned}$$

Since $\|\widetilde{B}(t)y\|_D \le c\|y\|$, then

$$\|A(t+s_n)^{-1}y - \widetilde{B}(t)y\| \le c'\varepsilon\|y\|.$$

Similarly, one can show that

$$\|\widetilde{B}(t-s_n)y - A(t)^{-1}y\| \le c''\varepsilon\|y\|.$$

Therefore, $t \mapsto A(t)^{-1}$ is compact almost automorphic with respect to operator topology.

Therefore, if $\mathcal{G}$ satisfies (H.5), then then the nonautonomous differential equation (1.4) has at least one bounded solution, which in addition is pseudo almost automorphic. □

Remark 4.2. In view of the previous proof, it follows that equation (1.3) has at least one solution, which in addition is pseudo almost automorphic.

5. Existence of pseudo almost automorphic solutions to some second-order boundary value problems. Let $\Omega \subset \mathbf{R}^N$ be an open bounded subset. In this section, we study the existence of pseudo almost automorphic mild solutions to modified versions of the so-called (nonautonomous) Sine-Gordon equations. For that, we suppose that $a_0, a_1 : \mathbf{R} \times \Omega \mapsto (0, \infty)$ are compact almost automorphic functions and satisfy the previous assumptions satisfied by a and b. (Here, we take $a_1 = a$ and $a_0 = b$.) Moreover, we suppose that there exists a $\delta_0 > 0$ such that

$$\inf_{\substack{t\in\mathbf{R}\\ x\in\Omega}} a_1(t,x) \ge \delta_0.$$

In both examples, we make extensive use of the fact that the injection

$$D((-\Delta_D)^{1/2}) = W_0^{1,2}(\Omega) = H_0^1(\Omega) \hookrightarrow L^2(\Omega) \tag{5.1}$$

is compact, where Δ_D is the Laplace operator on $L^2(\Omega)$ equipped with Dirichlet boundary conditions.

We have

Proposition 5.1. *Let* $(\mathbf{X}, \|\cdot\|)$ *and* $(\mathbf{Y}, \|\cdot\|_0)$ *be Banach spaces such that* $\mathbf{X}$ *is compactly embedded into* $\mathbf{Y}$: $\mathbf{X} \subset\subset \mathbf{Y}$. *Then*

$$(\mathbf{Y}, \mathbf{X})_{\theta,q_0} \subset\subset \mathbf{Y}$$

whenever $0 < q_0 \leq \infty$ *and* $0 < \theta < 1$.

Proof. Suppose that we are given a bounded subset $\{x_l\}_{l=1}^{\infty} \subset (\mathbf{Y}, \mathbf{X})_{\theta,1}$. We shall show that $\{x_l\}_{l=1}^{\infty}$ is convergent in $\mathbf{Y}$ if we pass to a subsequence.

We choose $x_{l,j,1} \in \mathbf{X}$ and $x_{l,j,2} \in \mathbf{Y}$ so that

$$x_l = x_{l,j,1} + x_{l,j,2}, \ \|x_{l,j,1}\| + 2^j \|x_{l,j,2}\|_0 \leq 2K(x_l, 2^j).$$

Passage to a subsequence allows us to assume that $\{x_{l,j,1}\}_{l=1}^{\infty}$ is convergent in $\mathbf{Y}$ for each $j \in \mathbf{Z}$. We claim that $\{x_l\}_{l=1}^{\infty}$ is convergent in $\mathbf{Y}$. To prove this, we let $l_1, l_2 \in \mathbf{N}$. Then we have

$$\begin{aligned}\|x_{l_1} - x_{l_2}\|_0 &\leq \|x_{l_1,j,1} - x_{l_2,j,1}\|_0 + \|x_{l_1,j,2} - x_{l_2,j,2}\|_0 \\ &\leq \|x_{l_1,j,1} - x_{l_2,j,1}\|_0 + \|x_{l_1,j,2}\|_0 + \|x_{l_2,j,2}\|_0 \\ &\leq C(\|x_{l_1,j,1} - x_{l_2,j,1}\|_0 + 2^{-j\theta}\|x\|_{(\mathbf{Y},\mathbf{X})_{\theta,1}}).\end{aligned}$$

If we let $l_1, l_2 \to \infty$, then we have

$$\limsup_{l_1,l_2\to\infty} \|x_{l_1} - x_{l_2}\|_0 \leq C 2^{-j\theta}\|x\|_{(\mathbf{Y},\mathbf{X})_{\theta,1}}$$

for all $j \in \mathbf{Z}$. If we let $j \to \infty$, then we have

$$\limsup_{l_1,l_2\to\infty} \|x_{l_1} - x_{l_2}\|_0 \leq 0.$$

Consequently, the compactness of $(\mathbf{Y}, \mathbf{X})_{\theta,q_0} \subset\subset \mathbf{Y}$ was established. □

Corollary 5.2. *If* $\beta \in (0, 1)$ *and* $q \in (0, \infty]$, *then the injection*

$$(L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega))_{\beta,q} \hookrightarrow L^2(\Omega)$$

is compact.

Proof. Since the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows that $H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, too. One obtains the result by taking $\mathbf{Y} = L^2(\Omega)$ and $\mathbf{X} = H_0^1(\Omega) \cap H^2(\Omega)$ and using Proposition 5.1. □

Clearly, letting $q = \infty$, it follows that the injection

$$\mathbf{H}_\beta = (L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega))_{\beta,\infty} \hookrightarrow L^2(\Omega) \tag{5.2}$$

is compact.

5.1. One-dimensional nonautonomous sine-Gordon equations. Let $l > 0$, and let $\mathbf{H} = L^2(0, l)$ when it is equipped with its natural topology. Our main objective in this subsection is to study the existence of pseudo almost automorphic solutions to a slightly modified version of the so called Sine-Gordon equation with Dirichlet boundary conditions, which had been studied in the literature especially by Leiva [**23**] in the following form:

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} + k \sin u = p(t, x), \quad t \in \mathbf{R},\ x \in (0, l) \tag{5.3}$$

$$u(t, 0) = u(t, l) = 0, \quad t \in \mathbf{R} \tag{5.4}$$

where c, d, k are positive constants, $p : \mathbf{R} \times (0, l) \mapsto \mathbf{R}$ is continuous and bounded.

Namely, we are interested in the system of second-order partial differential equations given by:

$$\frac{\partial^2 u}{\partial t^2} + a_1(t, x)\frac{\partial u}{\partial t} - a_0(t, x)\frac{\partial^2 u}{\partial x^2} = Q(t, x, u), \quad t \in \mathbf{R},\ x \in (0, l) \tag{5.5}$$

$$u(t, 0) = u(t, l) = 0, \quad t \in \mathbf{R} \tag{5.6}$$

where $a_1, a_0 : \mathbf{R} \times (0, l) \mapsto (0, \infty)$ are compact almost automorphic functions and $Q : \mathbf{R} \times (0, l) \times L^2(0, l) \mapsto L^2(0, l)$ is compact pseudo almost automorphic.

Let us take

$$Av = -\Delta v = -v'' \quad \text{for all } v \in D(A) = H_0^1(0, l) \cap H^2(0, l).$$

Suppose $Q : \mathbf{R} \times (0,l) \times L^2(0,l) \mapsto L^2(0,l)$ is compact pseudo almost automorphic in $t \in \mathbf{R}$ uniformly in $x \in (0,l)$ and $u \in L^2(0,l)$. Furthermore, $u \mapsto Q(t,x,u)$ is uniformly continuous on any bounded subset K of $L^2(0,l)$ for all $t \in \mathbf{R}$ and $x \in (0,l)$. Finally,

$$\|Q(t,x,u)\|_\infty \leq \mathcal{M}(\|u\|_\infty),$$

where $\mathcal{M} : \mathbf{R}^+ \mapsto \mathbf{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathcal{M}(r)}{r} = 0.$$

Consequently, taking into account the previous facts including equation (5.2), then system (5.5)–(5.6) has at least one pseudo almost automorphic mild solution.

5.2. N-dimensional nonautonomous sine-Gordon equations. Let $\Omega \subset \mathbf{R}^N$ ($N \geq 1$) be an open bounded subset with C^2 boundary $\partial\Omega$, and let $\mathbf{H} = L^2(\Omega)$ be equipped with its natural topology. In this subsection, we are interested in the so called N-dimensional nonautonomous sine-Gordon equation, which generalizes the previous example, that is, the system of second-order partial differential equations given by

(5.7)
$$\frac{\partial^2 u}{\partial t^2} + a_1(t,x)\frac{\partial u}{\partial t} - a_0(t,x)\Delta u = R(t,x,u), \quad t \in \mathbf{R},\ x \in \Omega$$

(5.8)
$$u(t,x) = 0, \quad t \in \mathbf{R},\ x \in \partial\Omega$$

where $a_1, a_0 : \mathbf{R} \times \Omega \mapsto (0,\infty)$ are compact almost automorphic functions, and $R : \mathbf{R} \times \Omega \times L^2(\Omega) \mapsto L^2(\Omega)$ is compact pseudo almost automorphic.

Define the linear operator A as follows:

$$Au = -\mathbf{\Delta} u \quad \text{for all } u \in D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Suppose that $R : \mathbf{R} \times \Omega \times L^2(\Omega) \mapsto L^2(\Omega)$ is compact pseudo almost automorphic in $t \in \mathbf{R}$ uniformly in $x \in \Omega$ and $u \in L^2(\Omega)$. Furthermore, $u \mapsto R(t,x,u)$ is uniformly continuous on any bounded subset K of $L^2(\Omega)$ for all $t \in \mathbf{R}$ and $x \in \Omega$. Finally,

$$\|R(t,x,u)\|_\infty \leq \mathbf{N}(\|u\|_\infty),$$

where $\mathbf{N} : \mathbf{R}^+ \mapsto \mathbf{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathbf{N}(r)}{r} = 0.$$

Consequently, taking into account the previous facts including (5.2), then the system 5.7)–(5.8) has at least one pseudo almost automorphic mild solution.

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