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EQUAL SUMS OF LIKE POWERS AND EQUAL PRODUCTS OF INTEGERS

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ABSTRACT. Several mathematicians have studied the problem of finding two distinct sets of integers $x_1, \dots, x_s$ and $y_1, \dots, y_s$, such that $\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k$, $k = k_1, k_2, \dots, k_n$, where k_i are specified positive integers. The particular case when $k = 1, 2, \dots, n$ is the well-known Tarry-Escott problem. This paper is the first detailed study of the problem of finding two distinct sets of nonzero integers which, in addition to the conditions already mentioned, also satisfy the condition $x_1x_2\cdots x_s = y_1y_2\cdots y_s$. Parametric or numerical solutions are given in this paper for many diophantine systems of this type, two examples being the system of equations $\sum_{i=1}^{5} x_i^k = \sum_{i=1}^{5} y_i^k$, $k = 1, 2, 3, 5$, and $\prod_{i=1}^{5} x_i = \prod_{i=1}^{5} y_i$, and the system given by the equations $\sum_{i=1}^{8} x_i^k = \sum_{i=1}^{8} y_i^k$, $k = 1, 2, \dots, 6$, and $\prod_{i=1}^{8} x_i = \prod_{i=1}^{8} y_i$. It is also shown that certain diophantine systems with equal sums of powers and equal products do not have any nontrivial solutions. Some open problems are mentioned at the end of the paper.

1. Introduction. The general problem of equal sums of like powers consists in finding two distinct sets of integers x_i, y_i, $i = 1, 2, \dots, s$ such that the sums of the kth powers of the integers in both the sets are equal for several values of k. In other words, the problem is concerned with finding nontrivial solutions of the diophantine system

$$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k, \quad k = k_1, k_2, \dots, k_n, \tag{1.1}$$

where the exponents k_j, $j = 1, 2, \dots, n$ are specified positive integers. When the exponents k_j are taken as the consecutive integers $1, 2, \dots, n$, we get the well-known Tarry-Escott problem of degree n. This paper is concerned with obtaining solutions of diophantine systems of type (1.1)

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in nonzero integers x_i, y_i, $i = 1, 2, \ldots, s$ that satisfy the additional condition

$$\prod_{i=1}^{s} x_i = \prod_{i=1}^{s} y_i. \tag{1.2}$$

We note that various diophantine systems involving equal sums of like powers, with the equality holding for certain specified exponents, have been investigated by several mathematicians (see, for instance, [**2, 3, 10**], [**12**, Chapter XXIV, pages 705–713]). In particular, the Tarry-Escott problem has attracted considerable attention [**1, 5, 13, 14**]. However, until now very limited attention has been given to the problem of equal sums of like powers with the additional condition (1.2). Diophantine systems involving equal products which have been considered until now include the system of equations

$$\begin{aligned} x_1^k + x_2^k + x_3^k &= y_1^k + y_2^k + y_3^k, \\ x_1x_2x_3 &= y_1y_2y_3, \end{aligned} \tag{1.3}$$

for which solutions have been given in [**6**, page 301], [**14**, page 66] and [**16**] when $k = 1$; in [**7**], [**14**, pages 36–38] and [**15**] when $k = 2$; in [**8**] and [**14**, page 69] when $k = 3$; and in [**9**] when $k = 4$. In fact, the complete solution of the diophantine system (1.3) with the first equation being satisfied for both $k = 1$ and $k = 3$ is given in [**8**, pages 138–139]. It seems that the only other diophantine system involving equal products for which solutions have been published is the system of equations,

$$\begin{aligned} x_1^k + x_2^k + x_3^k + x_4^k &= y_1^k + y_2^k + y_3^k + y_4^k, \quad k = 2, 4, \\ x_1x_2x_3x_4 &= y_1y_2y_3y_4, \end{aligned} \tag{1.4}$$

for which partial solutions are given in [**14**, pages 67–68], [**17, 21**].

We will exclude from consideration any solutions of the simultaneous equations (1.1) and (1.2) in which any x_i, y_i is 0 as well as solutions in which s is even and y_i are a permutation of $-x_i$ since, in both these cases, (1.2) is trivially satisfied and the problem reduces to one involving only equal sums of like powers. We note that, since each of the equations (1.1) and (1.2) is homogeneous, if x_i, y_i, $i = 1, 2, \ldots, s$

is a primitive solution of these simultaneous equations, then ρx_i, ρy_i, $i = 1, 2, \ldots, s$, where ρ is any nonzero integer, is also a solution and all such proportional solutions will be considered equivalent. Further, we will write $\overline{x}$ to denote the s-tuple $(x_1, x_2, \ldots, x_s)$ and similarly $\overline{y}$, $\overline{X}$ and $\overline{Y}$ will denote the corresponding s-tuples with the value of s being evident in each case from the context.

2. Some general results. We now prove several lemmas giving general results about diophantine systems involving equal sums of like powers and equal products.

Lemma 1. *If there exist integers a_i, b_i, $i = 1, 2, \ldots, m$, satisfying the relations*

$$\sum_{i=1}^{m} a_i^k = \sum_{i=1}^{m} b_i^k, \quad k = k_1, k_2, \ldots, k_n, \tag{2.1}$$

then a solution of the simultaneous equations

$$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k, \quad k = k_1, k_2, \ldots, k_n, \tag{2.2}$$

and

$$\prod_{i=1}^{s} x_i = \prod_{i=1}^{s} y_i, \tag{2.3}$$

with $s = 2m$ is given in terms of arbitrary parameters p and q by

$$\begin{aligned} x_i &= pa_i, \quad x_{m+i} = qb_i, \quad i = 1, 2, \ldots, m, \\ y_i &= qa_i, \quad y_{m+i} = pb_i, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{2.4}$$

The straightforward proof is omitted. We note that, by a suitable choice of p and q, we can easily ensure that $pa_{i_1} = qa_{i_2}$, for some i_1, i_2 and, removing this common term from both sides, we can get a solution of the simultaneous equations (2.2) and (2.3) with $s = 2m - 1$.

Lemma 2. *If there exists a nontrivial solution of the diophantine system*

$$\begin{aligned} \sum_{i=1}^{s} x_i^k &= \sum_{i=1}^{s} y_i^k, \quad k = 1, 2, \ldots, n, \\ \prod_{i=1}^{s} x_i &= \prod_{i=1}^{s} y_i, \end{aligned} \tag{2.5}$$

then $s \geq n + 2$.

Proof. If $s \leq n+1$, it follows from (2.5) and the well-known relations between the elementary symmetric functions and sums of powers of the roots of an equation [**4**, pages 271–272], that all the elementary symmetric functions of x_i, $i = 1, 2, \ldots, s$, attain the same values as the corresponding elementary symmetric functions of y_i, $i = 1, 2, \ldots, s$; hence, x_i are the roots of the same equation of degree s as y_i, and thus must be a permutation of y_i. It follows that if there exists a nontrivial solution of the diophantine system (2.5), then $s \geq n + 2$. □

Lemma 3. *If there exist integers* a_i, b_i, $i = 1, 2, \ldots, n+2$, *satisfying the relations*

$$\sum_{i=1}^{n+2} a_i^k = \sum_{i=1}^{n+2} b_i^k, \quad k = 1, 2, \ldots, n, \tag{2.6}$$

then a solution of the diophantine system

$$\sum_{i=1}^{n+2} x_i^k = \sum_{i=1}^{n+2} y_i^k, \quad k = 1, 2, \ldots, n, \tag{2.7}$$

$$\prod_{i=1}^{n+2} x_i = \prod_{i=1}^{n+2} y_i, \tag{2.8}$$

is given by

$$x_i = ra_i + d, \qquad y_i = rb_i + d, \quad i = 1, 2, \ldots, n + 2, \tag{2.9}$$

with d and r being defined by

$$\begin{aligned} d &= -\rho\left\{\prod_{i=1}^{n+2} a_i - \prod_{i=1}^{n+2} b_i\right\}, \\ r &= \rho\left\{a_1a_2\cdots a_{n+2}\sum_{i=1}^{n+2} a_i^{-1} - b_1b_2\cdots b_{n+2}\sum_{i=1}^{n+2} b_i^{-1}\right\}, \end{aligned} \tag{2.10}$$

where ρ is a suitably chosen integer.

Proof. If d and r are arbitrary parameters, it follows from a well-known theorem [**13**, Theorem 1, page 614] on the Tarry-Escott problem that x_i, y_i defined by (2.9) satisfy the relations (2.7). Further,

$$\begin{aligned} \prod_{i=1}^{n+2} x_i - \prod_{i=1}^{n+2} y_i &= \prod_{i=1}^{n+2}(ra_i + d) - \prod_{i=1}^{n+2}(rb_i + d) \\ &= r^{n+1}d\left\{a_1a_2\cdots a_{n+2}\sum_{i=1}^{n+2} a_i^{-1} - b_1b_2\cdots b_{n+2}\sum_{i=1}^{n+2} b_i^{-1}\right\} \\ &\quad + r^{n+2}\left\{\prod_{i=1}^{n+2} a_i - \prod_{i=1}^{n+2} b_i\right\}, \end{aligned} \tag{2.11}$$

where in the right-hand side of the above equation, the coefficient of r^jd^{n+2-j} vanishes for $j = 0, 1, \ldots, n$ as it works out to be $\sum a_1a_2\cdots a_j - \sum b_1b_2\cdots b_j$ which is 0 in view of the relations (2.6). It immediately follows from (2.11) that, if we take d and r as defined by (2.10), then x_i, y_i satisfy condition (2.8) in addition to the relations (2.7). □

We note that, while applying Lemma 3 to symmetric solutions of (2.6) yields only trivial solutions of the equations (2.7) and (2.8), nonsymmetric solutions of (2.6) lead to nontrivial solutions of these equations.

Lemma 4. *The three diophantine systems*

$$\begin{aligned}\sum_{i=1}^{n+2} x_i^k &= \sum_{i=1}^{n+2} y_i^k, \quad k = 1, 2, \dots, n, n+2,\\ \prod_{i=1}^{n+2} x_i &= \prod_{i=1}^{n+2} y_i,\end{aligned} \tag{2.12}$$

and,

$$\begin{aligned}\sum_{i=1}^{n+2} x_i^k &= \sum_{i=1}^{n+2} y_i^k, \quad k = 1, 2, \dots, n,\\ \prod_{i=1}^{n+2} x_i &= \prod_{i=1}^{n+2} y_i,\\ \sum_{i=1}^{n+2} x_i &= 0,\end{aligned} \tag{2.13}$$

and,

$$\begin{aligned}\sum_{i=1}^{n+2} x_i^k &= \sum_{i=1}^{n+2} y_i^k, \quad k = 1, 2, \dots, n, n+2,\\ \sum_{i=1}^{n+2} x_i &= 0,\end{aligned} \tag{2.14}$$

are equivalent, that is, a nontrivial solution of any one of the above three diophantine systems is also a nontrivial solution of the remaining two.

Proof. We write $s_k = \sum_{i=1}^{n+2} x_i^k$ and note the well-known result [**4**, pages 271–272] that the product $\prod_{i=1}^{n+2} x_i$ can be expressed in terms of $s_1, s_2, \dots, s_{n+1}, s_{n+2}$. Thus, there must exist a relation of the type,

$$\prod_{i=1}^{n+2} x_i = f(s_1, s_2, \dots, s_n) + \lambda_1 s_1 s_{n+1} + \lambda_2 s_{n+2}, \tag{2.15}$$

where $f(s_1, s_2, \ldots, s_n)$ is some function of s_j, $j = 1, 2, \ldots, n$, and λ_1, λ_2 are necessarily nonzero constants. Again, writing $t_k = \sum_{i=1}^{n+2} y_i^k$, we get a similar relation,

$$\prod_{i=1}^{n+2} y_i = f(t_1, t_2, \ldots, t_n) + \lambda_1 t_1 t_{n+1} + \lambda_2 t_{n+2}. \tag{2.16}$$

For each of the three diophantine systems mentioned in the lemma, x_i, y_i, $i = 1, 2, \ldots, n+2$, satisfy the relations $s_k = t_k$, $k = 1, 2, \ldots, n$, and hence, in each case, we get on subtracting (2.16) from (2.15),

$$\prod_{i=1}^{n+2} x_i - \prod_{i=1}^{n+2} y_i = \lambda_1 s_1 (s_{n+1} - t_{n+1}) + \lambda_2 (s_{n+2} - t_{n+2}). \tag{2.17}$$

If x_i, y_i, $i = 1, 2, \ldots, n+2$ is a nontrivial solution of (2.12), it follows from a theorem of Bastien, quoted by Dickson [**12**, page 712], that $s_{n+1} \neq t_{n+1}$, and hence it follows from (2.17) that $s_1 = 0$, and hence x_i, y_i also satisfy (2.13) and (2.14). Similarly, it readily follows from (2.17) that a nontrivial solution of (2.13) or (2.14) also satisfies the remaining two diophantine systems stated in the lemma. □

Lemma 5. *If p, q, r, s, X, Y, U, V are nonzero integers satisfying the relations*

$$(X^k - Y^k)/(p^k - q^k) = (U^k - V^k)/(r^k - s^k), \quad k = k_1, k_2, \ldots, k_n, \tag{2.18}$$

then the nonzero integers x_i, y_i, $i = 1, 2, 3, 4$, defined by

$$\begin{aligned} x_1 &= rX, \quad x_2 = sY, \quad x_3 = qU, \quad x_4 = pV, \\ y_1 &= sX, \quad y_2 = rY, \quad y_3 = pU, \quad y_4 = qV, \end{aligned} \tag{2.19}$$

satisfy the simultaneous equations,

$$x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k, \quad k = k_1, \ldots, k_n, \tag{2.20}$$

$$x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4. \tag{2.21}$$

Proof. It follows from the relations (2.18) that

$$(rX)^k + (sY)^k + (qU)^k + (pV)^k = (sX)^k + (rY)^k + (pU)^k + (qV)^k,$$

and so x_i, y_i, as defined by (2.19), satisfy relations (2.20) and (2.21). □

3. Solutions of the Tarry-Escott problem with equal products of integers. We will now obtain solutions of the diophantine system,

$$\begin{aligned} \sum_{i=1}^{s} x_i^k &= \sum_{i=1}^{s} y_i^k, \quad k = 1, 2, \ldots, n, \\ \prod_{i=1}^{s} x_i &= \prod_{i=1}^{s} y_i, \end{aligned} \tag{3.1}$$

for all positive integer values of $n \leq 6$ and, with $s = n+2$, the minimum value of s for the existence of nontrivial solutions of (3.1).

3.1. The Tarry-Escott problem of degree 1 with equal products. While solutions of (3.1) with $n = 1$ and $s = 3$, that is, the equations,

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 \tag{3.2}$$

$$x_1x_2x_3 = y_1y_2y_3, \tag{3.3}$$

are given in [**6, 14, 16**], the solutions given below seem to be better. To obtain the complete solution of equations (3.2) and (3.3), we write

$$\overline{x} = (pX,\, qY,\, rZ), \quad \overline{y} = (qX,\, rY,\, pZ), \tag{3.4}$$

and note that, for any solution $\overline{x}$, $\overline{y}$ of (3.2) and (3.3), there exist rational numbers p, q, r, X, Y, Z such that (3.4) is satisfied. With $\overline{x}$, $\overline{y}$ defined by (3.4), equation (3.3) is identically satisfied while (3.2) gives

$$(p - q)X + (q - r)Y + (r - p)Z = 0. \tag{3.5}$$

Two complete solutions of this simple equation are given by

$$X = \alpha + r\beta, \quad Y = \alpha + p\beta, \quad Z = \alpha + q\beta, \tag{3.6}$$

and

$$X = (p+q)\alpha + r\beta, \quad Y = (q+r)\alpha + p\beta, \quad Z = (r+p)\alpha + q\beta, \tag{3.7}$$

where α, β, p, q, r are arbitrary parameters. Thus, for the system of equations (3.2) and (3.3), we get two complete solutions which are given by (3.4) where X, Y, Z are given either by (3.6) or by (3.7) and α, β, p, q, r are arbitrary parameters. Taking $(\alpha, \beta, p, q, r) = (1, 2, 1, 2, 3)$ in the first of these solutions, we get the numerical solution $\overline{x} = (7, 6, 15)$, $\overline{y} = (14, 9, 5)$ for the simultaneous equations (3.2) and (3.3).

3.2. The Tarry-Escott problem of degree 2 with equal products. We will now consider the simultaneous diophantine equations,

$$x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k, \quad k = 1, 2, \tag{3.8}$$

$$x_1x_2x_3x_4 = y_1y_2y_3y_4, \tag{3.9}$$

for which we give two parametric solutions, one of which is complete.

3.2.1. It follows from Lemma 5 that a solution of the above simultaneous equations can be found quite simply by solving the equations

$$\begin{aligned} \frac{X-Y}{p-q} &= 2\alpha, & \frac{U-V}{r-s} &= 2\alpha, \\ \frac{X+Y}{p+q} &= 2\beta, & \frac{U+V}{r+s} &= 2\beta. \end{aligned} \tag{3.10}$$

The complete solution of these equations is readily found to be

$$\begin{aligned} X &= (p-q)\alpha + (p+q)\beta, & U &= (r-s)\alpha + (r+s)\beta, \\ Y &= (q-p)\alpha + (p+q)\beta, & V &= (s-r)\alpha + (r+s)\beta, \end{aligned} \tag{3.11}$$

and, with these values, (2.19) gives a parametric solution of the simultaneous equations (3.8) and (3.9) in terms of arbitrary parameters α, β, p, q, r, s. As an example, a numerical solution of these equations obtained by taking $(\alpha, \beta, p, q, r, s) = (1, 2, 1, 2, 3, 1)$ is given by $\overline{x} = (15, 7, 20, 6)$, $\overline{y} = (5, 21, 10, 12)$.

3.2.2. A second solution of the simultaneous equations (3.8) and (3.9) can be found by writing

$$\overline{x} = (X_1p,\ X_2q,\ X_3r,\ X_4s), \qquad \overline{y} = (X_1q,\ X_2r,\ X_3s,\ X_4p). \tag{3.12}$$

For any solution $\overline{x}$, $\overline{y}$ of equations (3.8) and (3.9) with x_i, y_i being nonzero for each i, there exist rational numbers p, q, r, s, X_1, X_2, X_3, X_4, given by

$$\begin{aligned} (p,\ q,\ r,\ s) &= (x_2^{-1}x_3^{-1}x_4^{-1}y_4,\ y_2^{-1}y_3^{-1}, \\ &\qquad\qquad x_2^{-1}y_3^{-1},\ x_2^{-1}x_3^{-1}), \\ (X_1,\ X_2,\ X_3,\ X_4) &= (y_1y_2y_3,\ x_2y_2y_3,\ x_2x_3y_3,\ x_2x_3x_4), \end{aligned} \tag{3.13}$$

such that the relations (3.12) are satisfied. The values of x_i, y_i given by (3.12) satisfy equation (3.9) while on substituting these values in equations (3.8), we get the conditions,

$$(p^k - q^k)X_1^k + (q^k - r^k)X_2^k + (r^k - s^k)X_3^k + (s^k - p^k)X_4^k = 0, \tag{3.14}$$

with $k = 1, 2$. The complete solution of these two equations in X_i, $i = 1, 2, 3, 4$, is readily obtained using the obvious solution $X_1 = X_2 = X_3 = X_4$, and may be written as follows:

$$\begin{aligned} X_1 &= (p-q)(-q+s)s\alpha^2 + (q-r)(pr + ps - 2qs \\ &\qquad - rs + s^2)\alpha\beta + (q-r)(-pr+qs)\beta^2, \\ X_2 &= (p-q)(-q+s)p\alpha^2 + (p-q)(2pq - pr - ps \\ &\qquad - rs + s^2)\alpha\beta + (q-r)(-p^2 + pq - rs + s^2)\beta^2, \\ X_3 &= (p-q)(-q+s)p\alpha^2 + (q-r)(p-q)(p+s)\alpha\beta \\ &\qquad + (q-r)(-pr+qs)\beta^2, \\ X_4 &= (p-q)(-q+s)q\alpha^2 + (q-r)(p-q)(2q+r-s)\alpha\beta \\ &\qquad + (q-r)(-pq+q^2-r^2+rs)\beta^2, \end{aligned} \tag{3.15}$$

where α, β, p, r, s, are arbitrary parameters. It now follows that the complete solution of equations (3.8) and (3.9) is given by (3.12) where X_i are defined by (3.15) in terms of the arbitrary parameters α, β, p, q, r, s. A numerical solution obtained by taking $(\alpha, \beta, p, q, r, s) = (1, 3, 1, 2, 3, 4)$ is given by $\overline{x} = (19, 50, 48, 80)$, $\overline{y} = (38, 75, 64, 20)$.

3.3. The Tarry-Escott problem of degree 3 with equal products. We will now obtain three parametric solutions, including a complete solution of the simultaneous diophantine equations,

$$\sum_{i=1}^{5} x_i^k = \sum_{i=1}^{5} y_i^k, \quad k = 1, 2, 3, \tag{3.16}$$

$$\prod_{i=1}^{5} x_i = \prod_{i=1}^{5} y_i. \tag{3.17}$$

3.3.1. If X_i, Y_i, $i = 1, 2, 3$, satisfy the relations,

$$X_1^k + X_2^k + X_3^k = Y_1^k + Y_2^k + Y_3^k, \quad k = 1, 2, \tag{3.18}$$

it follows immediately from a well-known theorem on the Tarry-Escott problem [**13**, Theorem 3, page 615] that

$$\begin{aligned} X_1^k + X_2^k + X_3^k &+ (Y_1 + h)^k + (Y_2 + h)^k + (Y_3 + h)^k \\ &= Y_1^k + Y_2^k + Y_3^k + (X_1 + h)^k + (X_2 + h)^k + (X_3 + h)^k, \end{aligned} \tag{3.19}$$

where $k = 1, 2, 3$, and h is arbitrary. We use the following, readily verifiable, solution of (3.18),

$$\begin{aligned} \overline{X} &= (\lambda_1 X - \lambda_3 Y + d,\ -\lambda_2 X - \lambda_1 Y + d,\ \lambda_3 X + \lambda_2 Y + d), \\ \overline{Y} &= (\lambda_3 X - \lambda_1 Y + d,\ -\lambda_2 X - \lambda_3 Y + d,\ \lambda_1 X + \lambda_2 Y + d), \end{aligned} \tag{3.20}$$

where $\lambda_1 = p + 2q$, $\lambda_2 = 2p + q$, $\lambda_3 = p - q$, and d, p, q, X and Y are arbitrary parameters. Taking $h = Y_2 - Y_3$, we get a solution of the simultaneous equations (3.16), and now applying Lemma 3 gives the following solution of equations (3.16) and (3.17):

$$\begin{aligned} \overline{x} = (&\lambda_1(2X + Y),\ \lambda_3(Y - X),\ \lambda_2(X + 2Y),\ -\lambda_1 X - \lambda_2 Y, \\ &- 2(\lambda_2 X + \lambda_3 Y)), \\ \overline{y} = (&\lambda_2 X + \lambda_3 Y,\ 2(\lambda_1 X + \lambda_2 Y),\ -\lambda_3(X + 2Y), \\ &- \lambda_2(2X + Y),\ \lambda_1(Y - X)), \end{aligned}$$

where λ_i are defined as before and p, q, X, Y are arbitrary parameters. A numerical solution obtained by taking $(p, q, X, Y) = (1, 5, 1, 2)$ is given by $\overline{x} = (44, -4, 35, -25, 2)$, $\overline{y} = (-1, 50, 20, -28, 11)$.

3.3.2. A second solution of the simultaneous equations (3.16) and (3.17) can be found by writing

$$\overline{x} = (X_1p,\, X_2q,\, X_3r,\, X_4s,\, X_5t), \qquad (3.21)$$
$$\overline{y} = (X_1q,\, X_2r,\, X_3s,\, X_4t,\, X_5p),$$

when equation (3.17) is identically satisfied while substituting these values in the three equations given by (3.16), we get the three equations,

$$(p^k-q^k)X_1^k+(q^k-r^k)X_2^k+(r^k-s^k)X_3^k+(s^k-t^k)X_4^k+(t^k-p^k)X_5^k=0. \qquad (3.22)$$

with $k = 1, 2, 3$. Three simple solutions of the simultaneous equations (3.22) are $\overline{X} = (1, 1, 1, 1, 1)$, $\overline{X} = (s, t, p, q, r)$, and $\overline{X} = (rst, stp, tpq, pqr, qrs)$. We will make use of the first two of these solutions to solve these three simultaneous equations. A similar procedure can be applied using any two of these three solutions to get more solutions of these equations.

We write

$$\overline{X} = (su + v,\, tu + v,\, pu + v,\, qu + v,\, ru + v), \qquad (3.23)$$

when (3.22) is satisfied identically for $k = 1$, while it is expressible as

$$(p - q)(p - s)(q - s) = (p - t)(q - r)(p - q - r + t) \qquad (3.24)$$

for $k = 2$, and it reduces to the following linear equation in u, v for $k = 3$:

$$\begin{aligned}&(p^3r^2-p^3s^2-p^2r^3+p^2s^3+q^3s^2-q^3t^2-q^2s^3+q^2t^3+r^3t^2-r^2t^3)u\\ &\quad+(p^3r-p^3s-pr^3+ps^3+q^3s-q^3t-qs^3+qt^3+r^3t-rt^3)v=0.\end{aligned} \qquad (3.25)$$

Now (3.24) is equivalent to the three linear equations,

$$p-s=\alpha(p-t), \quad q-s=\beta(q-r), \quad \alpha\beta(p-q)=p-q-r+t, \qquad (3.26)$$

and hence is readily solved for r, s, t in terms of arbitrary rational parameters α, β, p, q, and then (3.25) immediately gives a solution for u, v. These values of r, s, t, u, v substituted in (3.23) give the values of X_i and, finally, substituting all these values in (3.21) gives a solution of the simultaneous equations (3.16) and (3.17) in terms of arbitrary rational parameters α, β, p, q. While we omit this cumbersome solution in terms of polynomials of degree 10, we note that a special case of this solution when $\alpha = 2$, $\beta = 3$, is given in terms of arbitrary parameters p and q by (3.21) where

$$\overline{X} = (774p-793q,\ 527p-546q,\ 280p-299q,\ 261p-280q,\ 432p-451q),$$

and $(r, s, t) = (9p-8q, 27p-26q, 14p-13q)$. A numerical solution obtained by taking $p = 2$, $q = 1$ is given by $\overline{x} = (1510, 508, 2610, 6776, 6195)$, $\overline{y} = (755, 5080, 7308, 3630, 826)$.

3.3.3. To obtain the complete solution of (3.16) and (3.17), we write

$$\begin{aligned}
x_1 &= d,\\
x_2 &= a_1\alpha + (b_1 - b_2)\beta + d,\\
x_3 &= \left(\sum_{i=1}^{2} a_i\right)\alpha + (b_1 - b_3)\beta + d,\\
x_4 &= \left(\sum_{i=1}^{3} a_i\right)\alpha + (b_1 - b_4)\beta + d,\\
x_5 &= \left(\sum_{i=1}^{4} a_i\right)\alpha + b_1\beta + d,\\
y_1 &= b_1\beta + d,\\
y_2 &= a_1\alpha + d,\\
y_3 &= \left(\sum_{i=1}^{2} a_i\right)\alpha + (b_1 - b_2)\beta + d,\\
y_4 &= \left(\sum_{i=1}^{3} a_i\right)\alpha + (b_1 - b_3)\beta + d,\\
y_5 &= \left(\sum_{i=1}^{4} a_i\right)\alpha + (b_1 - b_4)\beta + d,
\end{aligned} \tag{3.27}$$

where a_i, b_i, d, α, β are arbitrary parameters with $a_i \neq 0$. With these values, equation (3.16) is satisfied identically for $k = 1$, and conversely, given any known solution X_i, Y_i, $i = 1, 2, 3, 4, 5$, of the equations (3.16), it is easily seen that there exist values of the parameters a_i, b_i, d, α, β which, substituted in (3.27), give $x_i = X_i$, $y_i = Y_i$, $i = 1, \ldots, 5$. Substituting the values of x_i, y_i, given by (3.27) in (3.16) with $k = 2$, we get, on removing the factor $2\alpha\beta$, the condition

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = 0, \tag{3.28}$$

and, accordingly, we take, without loss of generality, the following values of b_i in terms of three arbitrary linear parameters p, q and r:

$$b_1 = a_1^{-1}p, \quad b_2 = a_2^{-1}q, \quad b_3 = a_3^{-1}r, \quad b_4 = -a_4^{-1}(p + q + r). \tag{3.29}$$

Substituting the values of x_i, y_i given by (3.27) and the above values of b_i in (3.16) with $k = 3$, we get

$$\phi_1(p, q, r)\alpha + \phi_2(p, q, r)\beta = 0, \tag{3.30}$$

where

$$\begin{aligned}
\phi_1(p, q, r) &= (a_1 + 2a_2 + 2a_3 + a_4)p + (a_2 + 2a_3 + a_4)q + (a_3 + a_4)r,\\
\phi_2(p, q, r) &= (a_1 + a_4)a_2a_3p^2 + (a_2 + a_4)a_1a_3q^2 + (a_3 + a_4)a_1a_2r^2\\
&\quad + 2a_1a_2a_3(pq + qr + rp).
\end{aligned}$$

Thus, on taking $\alpha = -\rho\phi_2(p, q, r)$, $\beta = \rho\phi_1(p, q, r)$, where ρ is an arbitrary rational number, equation (3.16) is satisfied with $k = 3$, and accordingly (3.27) gives a solution of the simultaneous equations (3.16) where b_i, $i = 1, 2, 3, 4$ are defined by (3.29) and α, β have the values given above while a_i, $i = 1, 2, 3, 4$ and p, q, r, s are arbitrary integer parameters with $a_i \neq 0$. This solution is complete since we have taken the complete solution of all the intermediate equations, and we can accordingly work backwards to determine the values of the arbitrary parameters that will yield any given solution of the three simultaneous diophantine equations given by (3.16).

Since we have obtained the complete solution of equations (3.16), on applying Lemma 3, we obtain the complete solution of the simultaneous equations (3.16) and (3.17). This solution, as also the complete solution of the three equations (3.16), is too cumbersome to write, and is hence omitted.

3.4. The Tarry-Escott problem of degree 4 with equal products. We will now obtain a parametric solution of the simultaneous diophantine equations

$$\sum_{i=1}^{6} x_i^k = \sum_{i=1}^{6} y_i^k, \quad k = 1,\, 2,\, 3,\, 4, \tag{3.31}$$

$$\prod_{i=1}^{6} x_i = \prod_{i=1}^{6} y_i. \tag{3.32}$$

We note that a well-known, readily verifiable, solution of the diophantine system

$$\sum_{i=1}^{3} x_i^k = \sum_{i=1}^{3} y_i^k, \quad k = 1,\, 2,\, 4, \tag{3.33}$$

is given by

$$\begin{aligned} x_1 &= (p+2q)X - (p-q)Y, & y_1 &= (p-q)X - (p+2q)Y, \\ x_2 &= -(2p+q)X - (p+2q)Y, & y_2 &= -(2p+q)X - (p-q)Y, \\ x_3 &= (p-q)X + (2p+q)Y, & y_3 &= (p+2q)X + (2p+q)Y. \end{aligned} \tag{3.34}$$

We use the above values of $x_1,\, x_2,\, x_3,\, y_1,\, y_2,\, y_3$ and, similarly, we write

$$\begin{aligned} x_4 &= (r-s)U - (r+2s)V, & y_4 &= (r+2s)U - (r-s)V, \\ x_5 &= -(2r+s)U - (r-s)V, & y_5 &= -(2r+s)U - (r+2s)V, \\ x_6 &= (r+2s)U + (2r+s)V, & y_6 &= (r-s)U + (2r+s)V. \end{aligned} \tag{3.35}$$

With these values of $x_i,\, y_i$, equation (3.31) is identically satisfied for $k = 1,\, 2$ and 4, while for $k = 3$, equation (3.31) reduces, by suitable transpositions, to

$$pq(p+q)XY(X+Y) = rs(r+s)UV(U+V). \tag{3.36}$$

Writing $X = \alpha U$, $Y = \beta V$ and removing the factor UV, equation (3.36) reduces to a linear equation in U and V which gives the following solution of (3.36):

$$\begin{aligned} U &= p^2q\alpha\beta^2 + pq^2\alpha\beta^2 - r^2s - rs^2, \\ V &= -p^2q\alpha^2\beta - pq^2\alpha^2\beta + r^2s + rs^2, \\ X &= \alpha(p^2q\alpha\beta^2 + pq^2\alpha\beta^2 - r^2s - rs^2), \\ Y &= \beta(-p^2q\alpha^2\beta - pq^2\alpha^2\beta + r^2s + rs^2). \end{aligned} \tag{3.37}$$

With these values of U, V X, Y, a nonsymmetric solution of (3.31) is given by (3.34) and (3.35) where α, β, p, q, r, s are rational parameters. On applying Lemma 3, we now obtain a multi-parameter solution of the equations (3.31) and (3.32). As this solution is too cumbersome to write, we give below a one-parameter solution, obtained by taking $p = 1$, $q = 2$, $r = -1$, $s = 3$ and $\beta = 1$, and denoting the polynomial $c_0\alpha^4 + c_1\alpha^3 + \cdots + c_4$ by $(c_0, c_1, c_2, c_3, c_4)$:

$$\begin{aligned}
x_1 &= (20, -84, -85, 21, 20), & y_1 &= (20, -84, 41, -105, 20),\\
x_2 &= (20, -21, 104, -105, 20), & y_2 &= (20, 105, 104, 21, 20),\\
x_3 &= (20, 105, 41, 84, 20), & y_3 &= (20, -21, -85, 84, 20),\\
x_4 &= (20, -105, 104, -21, 20), & y_4 &= (20, 84, -85, -21, 20),\\
x_5 &= (20, 84, 41, 105, 20), & y_5 &= (20, -105, 41, -84, 20),\\
x_6 &= (20, 21, -85, -84, 20), & y_6 &= (20, 21, 104, 105, 20).
\end{aligned}$$

Taking $\alpha = 3$, we get the numerical solution $\overline{x} = (-95, 121, 364, -23, 328, 85)$, $\overline{y} = (-41, 391, 40, 220, -77, 247)$.

3.5. The Tarry-Escott problem of degree 5 with equal products. We will now obtain a parametric solution of the simultaneous equations,

$$\sum_{i=1}^{7} x_i^k = \sum_{i=1}^{7} y_i^k, \quad k = 1, 2, \ldots, 5, \tag{3.38}$$

$$\prod_{i=1}^{7} x_i = \prod_{i=1}^{7} y_i. \tag{3.39}$$

We take x_i, y_i, $i = 1, 2, 3$, as defined by (3.34) and write

$$\begin{aligned}
&x_4 = Z_1 + Z_2 + Z_3, && x_5 = Z_1 - Z_2 - Z_3,\\
&x_6 = -Z_1 + Z_2 - Z_3, && x_7 = -Z_1 - Z_2 + Z_3,\\
&y_4 = -x_4, \quad y_5 = -x_5, && y_6 = -x_6, \quad y_7 = -x_7,
\end{aligned} \tag{3.40}$$

when (3.38) is satisfied identically for $k = 1, 2, 4$, while for $k = 3$, equation (3.38) reduces to the equation,

$$16Z_1Z_2Z_3 = 27pq(p+q)XY(X+Y), \tag{3.41}$$

and, for $k=5$, equation (3.38) reduces, on using (3.41), to the equation

$$2(Z_1^2+Z_2^2+Z_3^2)=3(p^2+pq+q^2)(X^2+XY+Y^2). \tag{3.42}$$

To solve equations (3.41) and (3.42), we find a trivial solution of equations (3.38) by taking $Y=X$, $Z_3=Z_1+Z_2$, and use it to obtain the following trivial solution of equations (3.41) and (3.42):

$$Y=X,\quad Z_1=3pX/2,\quad Z_2=3qX/2,\quad Z_3=3(p+q)X/2. \tag{3.43}$$

We now take $Z_1=3pX/2$ in the two equations (3.41) and (3.42) so that they may be treated as two quadratic equations in the four variables X, Y, Z_2, Z_3, and one solution of these equations is given by (3.43). We now substitute

$$X=at+1,\quad Y=bt+1,\quad Z_2=ct+3q/2,\quad Z_3=3(p+q)/2, \tag{3.44}$$

in (3.41) and (3.42), both of which yield one nonzero solution for t. We readily find suitable values of a, b, c such that these two nonzero values of t are equal, and this leads to a solution of (3.41) and (3.42), and hence to a solution of the simultaneous equations (3.38) in terms of the parameters p and q. Writing the polynomial $c_0p^n+c_1p^{n-1}q+c_2p^{n-2}q^2+\cdots+c_nq^n$ as $(c_0, c_1, c_2, \ldots, c_n)$, this parametric solution may be written as follows:

$$\begin{aligned}
x_1&=(-24,\,-174,\,-507,\,-735,\,-612,\,-432,\,-324,\,-108),\\
x_2&=(6,\,33,\,42,\,27,\,180,\,468,\,432,\,108),\\
x_3&=(18,\,141,\,465,\,708,\,432,\,-36,\,-108,\,0),\\
x_4&=(-12,\,-60,\,-105,\,-129,\,-270,\,-504,\,-432,\,-108),\\
x_5&=(-18,\,-159,\,-510,\,-771,\,-450,\,180,\,324,\,108),\\
x_6&=(12,\,102,\,264,\,216,\,-162,\,-324,\,-108,\,0),\\
x_7&=(18,\,117,\,351,\,684,\,882,\,648,\,216,\,0),\\
y_1&=(-24,\,-186,\,-573,\,-873,\,-540,\,144,\,324,\,108),\\
y_2&=(6,\,75,\,327,\,780,\,1008,\,612,\,108,\,0),\\
y_3&=(18,\,111,\,246,\,93,\,-468,\,-756,\,-432,\,-108),\\
y_4&=(12,\,60,\,105,\,129,\,270,\,504,\,432,\,108),\\
y_5&=(18,\,159,\,510,\,771,\,450,\,-180,\,-324,\,-108),
\end{aligned}$$

$$y_6 = (-12, -102, -264, -216, 162, 324, 108, 0),$$
$$y_7 = (-18, -117, -351, -684, -882, -648, -216, 0).$$

Now, on using Lemma 3, we obtain a solution of the diophantine system given by (3.38) and (3.39) in terms of polynomials of degree 14 which are not being given explicitly. As a numerical example, when $p = 1$, $q = 2$, we get the following solution of (3.38) and (3.39):

$$\overline{x} = (19893, -6407, 4658, 19278, 928, 9563, -5577),$$
$$\overline{y} = (1703, -4437, 20878, -7182, 11168, 2533, 17673).$$

3.6. The Tarry-Escott problem of degree 6 with equal products. Shuwen [**19**] found one solution for the diophantine system (3.1) with $n = 6$ and $s = 8$, viz., $\overline{x} = (1899, 1953, 1957, 2079, 2117, 2231, 2241, 2323)$, $\overline{y} = (1909, 1919, 2001, 2037, 2163, 2187, 2263, 2321)$. I obtained by trial a numerical solution of the simultaneous equations $\sum_{i=1}^{8} x_i^k = y_i^k$, $k = 1, 2, \ldots, 6$ and, applying Lemma 3, obtained a second numerical solution of (3.1) with $n = 6$, $s = 8$, namely, $\overline{x} = (-175, -88, -85, 5, 119, 134, 296, 326)$, $\overline{y} = (-163, -136, -34, 14, 56, 185, 275, 335)$.

4. An extension of the Tarry-Escott problem with equal products. In this section we consider the simultaneous diophantine equations,

$$\begin{aligned} &\sum_{i=1}^{n+2} x_i^k = \sum_{i=1}^{n+2} y_i^k, \quad k = 1, 2, \ldots, n, n+2 \\ &\prod_{i=1}^{n+2} x_i = \prod_{i=1}^{n+2} y_i, \end{aligned} \tag{4.1}$$

for positive integer values of $n \le 4$. We obtain parametric solutions of this system when $n = 1$, 2 or 3, and infinitely many solutions when $n = 4$. We will use the results of the first four subsections of Section 3 together with Lemma 4 to obtain these solutions.

4.1. When $n = 1$, the diophantine system (4.1) is given by (1.3) with $k = 1$ and 3. Two complete parametric solutions of (1.3) with

$k = 1$ are given in subsection 3.1. It follows from Lemma 4 that, if we choose the parameters such that $x_1 + x_2 + x_3 = 0$, we will obtain a complete solution of (4.1) with $n = 1$. Both the parametric solutions given in subsection 3.1 lead to the following complete solution of the diophantine system (4.1) with $n = 1$:

$$\begin{aligned} \overline{x} &= (p(pq - r^2),\ q(qr - p^2),\ r(pr - q^2)), \\ \overline{y} &= (q(pq - r^2),\ r(qr - p^2),\ p(pr - q^2)). \end{aligned} \tag{4.2}$$

A numerical solution obtained by taking $(p, q, r) = (1, 2, 3)$ is given by $\overline{x} = (7, -10, 3)$, $\overline{y} = (14, -15, 1)$.

We note that the above complete solution of the diophantine system (4.1) with $n = 1$ is much simpler and neater than the complete solution given in [**8**].

4.2. We will now obtain a parametric solution of the diophantine system

$$\begin{aligned} x_1^k + x_2^k + x_3^k + x_4^k &= y_1^k + y_2^k + y_3^k + y_4^k, \quad k = 1, 2, 4, \\ x_1x_2x_3x_4 &= y_1y_2y_3y_4. \end{aligned} \tag{4.3}$$

It follows from Lemma 4 that any solution of the simultaneous equations (3.8) and (3.9) that also satisfies the condition $x_1 + x_2 + x_3 + x_4 = 0$, will be a solution of the diophantine system (4.3). While imposing this condition on the solution of (3.8) and (3.9) given in subsection 3.2.1 leads to trivial solutions of system (4.3), when we impose this condition on the solution given by (3.12) and (3.15), we get

$$\begin{aligned} &(p-q)(-q+s)(pq+pr+ps+qs)\alpha^2 + (2p^2q^2 + p^2qr - 2p^2r^2 \\ &\quad - p^2rs - 2pq^3 + pq^2s + pqr^2 + pqs^2 - pr^2s - 2q^3s + q^2rs \\ &\quad + 2qr^2s - qrs^2)\alpha\beta - (q-r)(q+r)(p+s)(p-q+r-s)\beta^2 = 0. \end{aligned} \tag{4.4}$$

Taking $s = p-q+r$, we easily find nonzero values of α, β satisfying condition (4.4). We thus obtain the following solution of the diophantine system (4.3):

$$x_1 = (p^3r - 2p^2q^2 - 3p^2qr + 3p^2r^2 + 3pq^3 - 3pq^2r$$

$$
\begin{aligned}
&\quad - 3pqr^2 + 4pr^3 - q^4 + 3q^3r - 3q^2r^2 + r^4)p,\\
x_2 &= (p^4 + 4p^3r - 3p^2q^2 - 3p^2qr + 3p^2r^2 + 3pq^3\\
&\quad - 3pq^2r - 3pqr^2 + pr^3 - q^4 + 3q^3r - 2q^2r^2)q,\\
x_3 &= (p^3q - 2p^2r^2 + pq^3 + 3pq^2r - 2pqr^2 - pr^3\\
&\quad - q^4 + q^3r + q^2r^2 - qr^3)r,\\
x_4 &= -(p^3q + p^3r - p^2q^2 + 2p^2qr + 2p^2r^2 - pq^3\\
&\quad - 3pq^2r + q^4 - q^3r - qr^3)(p - q + r),\\
y_1 &= (p^3r - 2p^2q^2 - 3p^2qr + 3p^2r^2 + 3pq^3 - 3pq^2r\\
&\quad - 3pqr^2 + 4pr^3 - q^4 + 3q^3r - 3q^2r^2 + r^4)q,\\
y_2 &= (p^4 + 4p^3r - 3p^2q^2 - 3p^2qr + 3p^2r^2 + 3pq^3\\
&\quad - 3pq^2r - 3pqr^2 + pr^3 - q^4 + 3q^3r - 2q^2r^2)r,
\end{aligned}
$$

$$
\begin{aligned}
y_3 &= (p^3q - 2p^2r^2 + pq^3 + 3pq^2r - 2pqr^2 - pr^3\\
&\quad - q^4 + q^3r + q^2r^2 - qr^3)(p - q + r),\\
y_4 &= -(p^3q + p^3r - p^2q^2 + 2p^2qr + 2p^2r^2 - pq^3\\
&\quad - 3pq^2r + q^4 - q^3r - qr^3)p,
\end{aligned}
$$

where p, q, r are arbitrary parameters. A numerical solution obtained by taking $(p, q, r) = (1, -2, 3)$ is given by $\overline{x} = (9, 62, 67, -138)$ and $\overline{y} = (-18, -93, 134, -23)$. Another parametric solution of the diophantine system (4.3) may be obtained by taking s such that $pq + pr + ps + qs = 0$ when again we can find suitable α, β such that condition (4.4) is satisfied. This solution of degree 8 in arbitrary parameters p, q and r is omitted.

4.3. We will now obtain parametric solutions of the simultaneous diophantine equations,

$$\sum_{i=1}^{5} x_i^k = \sum_{i=1}^{5} y_i^k, \quad k = 1, 2, 3, 5, \tag{4.5}$$

$$\prod_{i=1}^{5} x_i = \prod_{i=1}^{5} y_i. \tag{4.6}$$

4.3.1. In view of Lemma 4, to solve the above diophantine system, it suffices to impose the condition

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0, \tag{4.7}$$

on the parametric solution given in subsection 3.3.1 for the simultaneous equations (3.16) and (3.17). This immediately yields the following solution of the diophantine system given by (4.5) and (4.6):

$$\begin{aligned}\overline{x} &= (-p^2-3pq-2q^2,\ pq-q^2,\ -2p^2-pq,\ p^2+pq+q^2,\ 2(p^2+pq+q^2)),\\ \overline{y} &= (-p^2-pq-q^2,\ -2(p^2+pq+q^2),\ p^2-pq,\ 2p^2+3pq+q^2,\ pq+2q^2).\end{aligned}$$

A numerical solution obtained by taking $p = 3$ and $q = 1$ is given by $\overline{x} = (-20, 2, -21, 13, 26)$, $\overline{y} = (-13, -26, 6, 28, 5)$.

4.3.2. As the solution obtained in the previous subsection is somewhat special since it necessarily has $y_1 = -x_4$ and $y_2 = -x_5$, we will obtain another parametric solution of the system of equations (4.5) and (4.6). We first obtain a simple, though not complete, solution of the equations (4.5) with $k = 1, 2, 3$, by choosing x_i, y_i as in (3.27) and following the same method as in subsection 3.3.3. After obtaining equation (3.30), we choose the arbitrary parameters a_i, $i = 1, 2, 3, 4$ and p, q, r such that (3.30) is satisfied identically for all values of α and β. To do this, we first choose p, q, r, as given below,

$$\begin{aligned}p &= (a_2+2a_3+a_4)(a_3+a_4)g,\\ q &= (a_1+2a_2+2a_3+a_4)(a_3+a_4)h,\\ r &= -(a_1+2a_2+2a_3+a_4)(a_2+2a_3+a_4)(g+h)\end{aligned} \tag{4.8}$$

when the coefficient of α in (3.30) vanishes, and further taking $a_4 = -a_2$, equation (3.30) reduces to

$$(a_1^2+2a_1a_2+a_1a_3+a_2^2-a_2a_3)g+a_1(a_1+a_2+2a_3)h = 0, \tag{4.9}$$

which is readily solved for g and h leading to the following solution which satisfies equation (4.5) simultaneously for $k = 1, 2, 3$:

$$
\begin{aligned}
x_1 &= d,\\
x_2 &= a_1\alpha - (a_1 + a_2)(a_1 + a_2 + a_3)\beta + d,\\
x_3 &= (a_1 + a_2)\alpha - 2(a_1 + a_2 + a_3)a_2\beta + d,\\
x_4 &= (a_1 + a_2 + a_3)\alpha - (a_1^2 + 2a_1a_2 + a_1a_3\\
&\qquad\qquad + a_2^2 + 3a_2a_3)\beta + d,\\
x_5 &= (a_1 + a_3)\alpha - 2a_2a_3\beta + d,\\
y_1 &= -2a_2a_3\beta + d,\\
y_2 &= a_1\alpha + d,\\
y_3 &= (a_1 + a_2)\alpha - (a_1 + a_2)(a_1 + a_2 + a_3)\beta + d,\\
y_4 &= (a_1 + a_2 + a_3)\alpha - 2(a_1 + a_2 + a_3)a_2\beta + d,\\
y_5 &= (a_1 + a_3)\alpha - (a_1^2 + 2a_1a_2 + a_1a_3\\
&\qquad\qquad + a_2^2 + 3a_2a_3)\beta + d.
\end{aligned}
\tag{4.10}
$$

We now choose d such that (4.7) is satisfied, and substituting the resulting values of x_i, y_i in equation (4.5) with $k = 5$, and removing the factor $-2a_1a_2a_3\alpha\beta(a_1+a_2)(a_1+a_2+a_3)(\alpha-2a_2\beta)\{\alpha-(a_1+a_2+a_3)\beta\}$, we get the condition,

$$
\begin{aligned}
&(a_1^2 + a_1a_3 + a_2^2 - a_3^2)\alpha^2 - (2a_1^3 + 3a_1^2a_2 + 3a_1^2a_3 + 4a_1a_2^2\\
&\quad + 3a_1a_2a_3 + a_1a_3^2 + 3a_2^3 + 2a_2^2a_3 - 5a_2a_3^2)\alpha\beta\\
&\quad + (2a_1^4 + 4a_1^3a_2 + 4a_1^3a_3 + 4a_1^2a_2^2 + 6a_1^2a_2a_3 + 2a_1^2a_3^2\\
&+ 4a_1a_2^3 + 4a_1a_2^2a_3 + 2a_1a_2a_3^2 + 2a_2^4 + 2a_2^3a_3 - 4a_2^2a_3^2)\beta^2 = 0.
\end{aligned}
\tag{4.11}
$$

We now choose a_1, a_2, a_3 as follows,

$$
a_1 = 2uv + v^2, \qquad a_2 = u^2 - v^2 + uv, \qquad a_3 = -u^2 - v^2,
\tag{4.12}
$$

when the coefficient of α^2 in equation (4.11) vanishes, and hence (4.11) is readily solved for α and β, and thereby we obtain a solution of the simultaneous equations (4.5) with $k = 1, 2, 3, 5$ in terms of arbitrary parameters u and v. Since this solution satisfies the condition (4.7), it follows from Lemma 4 that this solution must satisfy equation (4.6) as well. Thus, we obtain the following solution of the simultaneous equations (4.5) and (4.6):

$$
x_1 = -2(2u^4+6u^3v+5u^2v^2-uv^3+2v^4)(2u^4-4u^3v+5u^2v^2+4uv^3-3v^4),
$$

$$\begin{aligned}
x_2 &= -(4u^4+7u^3v+5u^2v^2-7uv^3-v^4)(2u^4+u^3v+5u^2v^2-6uv^3+2v^4),\\
x_3 &= 2(u^4-2u^3v-5u^2v^2-3uv^3+v^4)(u^4-7u^3v-5u^2v^2+7uv^3-4v^4),\\
x_4 &= (4u^4+2u^3v-5u^2v^2+8uv^3-v^4)(3u^4+4u^3v-5u^2v^2-4uv^3-2v^4),\\
x_5 &= 2(u^4+3u^3v-5u^2v^2+2uv^3+v^4)(u^4+8u^3v+5u^2v^2+2uv^3-4v^4),\\
y_1 &= 2(3u^4+4u^3v-5u^2v^2-4uv^3-2v^4)(2u^4+u^3v+5u^2v^2-6uv^3+2v^4),\\
y_2 &= -2(u^4-2u^3v-5u^2v^2-3uv^3+v^4)(4u^4+2u^3v-5u^2v^2+8uv^3-v^4),\\
y_3 &= (u^4+8u^3v+5u^2v^2+2uv^3-4v^4)(2u^4-4u^3v+5u^2v^2+4uv^3-3v^4),\\
y_4 &= -2(u^4+3u^3v-5u^2v^2+2uv^3+v^4)(4u^4+7u^3v+5u^2v^2-7uv^3-v^4),\\
y_5 &= (2u^4+6u^3v+5u^2v^2-uv^3+2v^4)(u^4-7u^3v-5u^2v^2+7uv^3-4v^4),
\end{aligned}$$

where u and v are arbitrary parameters. A numerical solution obtained by taking $u = 2$, $v = -1$, is given by $\overline{x} = (-584, -1189, 646, 11, 1116)$, $\overline{y} = (116, -209, -1314, 1271, 136)$.

4.4. We will now obtain numerical solutions of the simultaneous diophantine equations,

$$\sum_{i=1}^{6} x_i^k = \sum_{i=1}^{6} y_i^k, \quad k = 1, 2, 3, 4, 6, \tag{4.13}$$

$$\prod_{i=1}^{6} x_i = \prod_{i=1}^{6} y_i. \tag{4.14}$$

To solve this diophantine system, we choose x_i, y_i as defined by (3.34) and (3.35) when (4.13) is satisfied identically for $k = 1, 2, 4$, while for $k = 3$, equation (4.13) reduces, as before, to (3.36). When $k = 6$, equation (4.13) reduces, on using (3.36), to

$$\begin{aligned}(p-q)(p+2q)(2p+q)&(X-Y)(X+2Y)(2X+Y)\\ &= (r-s)(r+2s)(2r+s)(U-V)(U+2V)(2U+V).\end{aligned} \tag{4.15}$$

To solve (3.36) and (4.15), we write $X = -2U$, $Y = V$, when these two equations reduce to the following two equations respectively:

$$(4pq^2 + 4p^2q - r^2s - rs^2)U = (2pq^2 + 2p^2q + r^2s + rs^2)V, \tag{4.16}$$

$$\begin{aligned}(16p^3 + 24p^2q - 24pq^2 - 16q^3 &+ 2r^3 + 3r^2s - 3rs^2 - 2s^3)U\\ &= (4p^3 + 6p^2q - 6pq^2 - 4q^3 + 4r^3 + 6r^2s - 6rs^2 - 4s^3)V.\end{aligned} \tag{4.17}$$

We eliminate U and V from these two equations, then substitute $r = p$, $s = 1$ in the eliminant and, omitting the factor $p(2q+1)(3p+2q+1)$ which leads to trivial solutions, we get the condition

$$6p^3 + 6p^2q - 4pq^2 - 4q^3 + 3p^2 - 4pq + 4q^2 - p - 8q + 2 = 0 \tag{4.18}$$

which represents an elliptic curve with one rational point on the curve being $(p, q) = (-1, 0)$. Using the birational transformation defined by the relations

$$\begin{aligned} p &= \frac{-9\xi^3 + 80\xi^2 - 6\xi\eta - 20\xi + 180\eta + 1400}{9\xi^3 - 184\xi^2 + 1000\xi - 1600}, \\ q &= \frac{-58\xi^2 + 33\xi\eta + 550\xi - 300\eta - 3400}{9\xi^3 - 184\xi^2 + 1000\xi - 1600}, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \xi &= \frac{100p + 60q - 40}{11p + 2q + 11}, \\ \eta &= \frac{-450p^2 - 3840pq - 2280q^2 - 1920p + 1320q - 1470}{121p^2 + 44pq + 4q^2 + 242p + 44q + 121}, \end{aligned} \tag{4.20}$$

equation (4.18) reduces to the Weierstrass minimal form of the elliptic curve that is given by

$$\eta^2 = \xi^3 + 100. \tag{4.21}$$

One rational point P on this elliptic curve is easily seen to be $(\xi, \eta) = (-4, 6)$. Doubling this point yields a second rational point $2P$ given by $(\xi, \eta) = (24, -118)$ while $3P$ is given by $(\xi, \eta) = (-19/49, 3429/343)$. As the rational point $3P$ on the elliptic curve (4.21) does not have integer co-ordinates, it follows from the Nagell-Lutz theorem on elliptic curves [**20**, page 56] that this is not a point of finite order. Thus, there exist infinitely many rational points on the elliptic curve (4.21), and these can be obtained by the group law. Further, a reference to Cremona's well-known tables [**11**] on elliptic curves shows that the rank of this elliptic curve is 1, which reconfirms the existence of infinitely many rational points on this curve. These infinitely many rational points on the curve (4.21) yield infinitely many values of p, q satisfying equation (4.18). With these values of p, q and $r = p$, $s = 1$, equations

(4.16) and (4.17) can be solved for U, V and, working backwards, we can obtain infinitely many solutions of the simultaneous equations (4.13).

We also note that the values of x_i thus obtained from (3.34) and (3.35) necessarily satisfy the relation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$, and hence it follows from Lemma 4 that the solutions obtained above for equations (4.13) also simultaneously satisfy equation (4.14). We can thus get infinitely many integer solutions of the diophantine system given by (4.13) and (4.14).

While the rational points P and $2P$ on the elliptic curve (4.21) lead to trivial solutions of the simultaneous equations (4.13) and (4.14), the point $3P$ leads to the following nontrivial solution:

$$\begin{aligned}\overline{x} &= (10541,\ 1175,\ -11716,\ -1460,\ 7897,\ -6437),\\ \overline{y} &= (11461,\ -10865,\ -596,\ 4700,\ 3683,\ -8383).\end{aligned}$$

5. Other diophantine systems with equal sums of like powers and equal products of integers.

5.1. We will now obtain a parametric solution of the diophantine system

$$\begin{aligned}x_1^k + x_2^k + x_3^k + x_4^k &= y_1^k + y_2^k + y_3^k + y_4^k, \quad k = 1, 3\\ x_1x_2x_3x_4 &= y_1y_2y_3y_4,\end{aligned} \tag{5.1}$$

It follows from Lemma 5 that a solution of this diophantine system can be found by solving the following equations:

$$\frac{X - Y}{p - q} = \frac{U - V}{r - s}, \tag{5.2}$$

$$\frac{X^2 + XY + Y^2}{p^2 + pq + q^2} = \frac{U^2 + UV + V^2}{r^2 + rs + s^2}. \tag{5.3}$$

Equations (5.2) and (5.3) may be considered as a linear and a quadratic equation in the variables X, Y, U, V with a known solution $X = p$, $Y = q$, $U = r$, $V = s$. Eliminating V from these two equations, we get a quadratic equation in X, Y, U which is readily solved using the known solution. We thus obtain the following solution of equations (5.2) and (5.3):

$$\begin{aligned}
X &= (pr-qs)(p^2r-q^2s)m^2+(ps-qr)(-pr+qs)n^2p \\
&\quad -(p^3r^2-p^3rs-2p^2qrs+2pq^2r^2+2pq^2s^2-q^3r^2-q^3rs)mn, \\
Y &= (ps-qr)(-pr+qs)m^2q+(ps-qr)(p^2s-q^2r)n^2+ \\
&\quad (p^3r^2+p^3rs-2p^2qr^2-2p^2qs^2+2pq^2rs-q^3r^2+q^3rs)mn, \\
U &= (ps-qr)(-pr+qs)m^2r+(ps-qr)(-pr+qs)n^2r \\
&\quad -(p^2r^2-p^2rs+p^2s^2-2pqrs+q^2r^2-q^2rs+q^2s^2)mnr, \\
V &= -(pr-qs)(pr^2-qs^2)m^2-(ps-qr)(ps^2-qr^2)n^2 \\
&\quad -(p^2r^2s+p^2rs^2-2pqs^3+rq^2s^2+sq^2r^2-q^2r^3-p^2r^3)mn,
\end{aligned}$$

where m, n, p, q, r, s, are arbitrary parameters. With these values of X, Y, U, V, a solution of the diophantine system (5.1) is given by (2.19). A numerical solution obtained by taking $(m, n, p, q, r, s) = (2, 9, 1, 2, 1, 3)$ is given by $\overline{x} = (13, 93, 47, 22)$ and $\overline{y} = (39, 31, 11, 94)$.

5.2. As has been mentioned in the introduction, a large number of results on equal sums of like powers are already known. These can be used together with Lemma 1 to obtain solutions of diophantine systems with equal sums of powers and equal products. As an example, we will show how infinitely many solutions can be obtained for the diophantine system,

$$\begin{aligned}
\sum_{i=1}^{5} x_i^k &= \sum_{i=1}^{5} y_i^k, \quad k = 2, 3, 4, \\
x_1x_2x_3x_4x_5 &= y_1y_2y_3y_4y_5,
\end{aligned} \tag{5.4}$$

We will use the known solutions of the diophantine system

$$\sum_{i=1}^{3} a_i^k = \sum_{i=1}^{3} b_i^k, \quad k = 2, 3, 4, \tag{5.5}$$

to obtain solutions of the diophantine system (5.4). As has been shown in [**10**], infinitely many integer solutions of (5.5) can be obtained, a numerical example being given by $(a_1, a_2, a_3) = (-815, 358, 1224)$ and

$(b_1, b_2, b_3) = (-776, -410, 1233)$. Taking $p = 388$, $q = 205$, and applying Lemma 1 to this solution, we get a numerical solution of the diophantine system (5.4) which is given by $\overline{x} = (-316220, 138904, 474912, -84050, 252765)$ and $\overline{y} = (-167075, 73390, 250920, -301088, 478404)$. As we know infinitely many solutions of (5.5), we can obtain infinitely many solutions of (5.4).

6. Some diophantine systems with no nontrivial solutions. In this section we prove that certain diophantine systems involving equal sums of powers and equal products have no nontrivial solutions.

Theorem 1. *The simultaneous diophantine equations,*

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3, \tag{6.1}$$

$$x_1^4 + x_2^4 + x_3^4 = y_1^4 + y_2^4 + y_3^4, \tag{6.2}$$

$$x_1 x_2 x_3 = y_1 y_2 y_3, \tag{6.3}$$

have no nontrivial solutions in integers.

Proof. If there exists a nonzero solution of the simultaneous diophantine equations (6.1), (6.2) and (6.3), we may write $p = x_1 + x_2 + x_3 = y_1 + y_2 + y_3$, $r = x_1x_2x_3 = y_1y_2y_3$, $q_1 = x_1x_2 + x_2x_3 + x_3x_1$, $q_2 = y_1y_2 + y_2y_3 + y_3y_1$. The following identity in symmetric functions of x_1, x_2, x_3 is readily verified:

$$x_1^4 + x_2^4 + x_3^4 = p^4 - 4p^2q_1 + 4pr + 2q_1^2. \tag{6.4}$$

Similarly, we have the identity,

$$y_1^4 + y_2^4 + y_3^4 = p^4 - 4p^2q_2 + 4pr + 2q_2^2. \tag{6.5}$$

On subtracting (6.5) from (6.4), we get in view of (6.2),

$$2(q_1 - q_2)(q_1 + q_2 - 2p^2) = 0. \tag{6.6}$$

Since $p^2 - 2q_1 = x_1^2 + x_2^2 + x_3^2 > 0$, it follows that $q_1 < p^2/2$, and similarly, $q_2 < p^2/2$. Thus, $q_1 + q_2 - 2p^2 < -p^2$ and hence cannot be 0. It now follows from (6.6) that $q_1 = q_2$, that is, $x_1x_2 + x_2x_3 + x_3x_1 =$

$y_1y_2 + y_2y_3 + y_3y_1$. This, together with the relations (6.1) and (6.3) implies that x_1, x_2, x_3 must be a permutation of y_1, y_2, y_3. This proves the theorem. □

It is interesting to note here that if, in the above diophantine system, we replace equation (6.3) by the cubic equation $x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3$, the resulting diophantine system, namely,

$$x_1^k + x_2^k + x_3^k = y_1^k + y_2^k + y_3^k, \quad k = 1,\, 3,\, 4, \tag{6.7}$$

has nontrivial solutions in integers [**6**, pages 305–306].

Theorem 2. *The diophantine system*

$$\begin{aligned} \sum_{i=1}^{n} x_i^{2k} &= \sum_{i=1}^{n} y_i^{2k}, \quad k = 1,\, 2, \dots, n-1, \\ x_1x_2\cdots x_n &= y_1y_2\cdots y_n, \end{aligned} \tag{6.8}$$

has no nontrivial solutions in integers.

Proof. Writing $X_i = x_i^2$, $Y_i = y_i^2$, $i = 1, 2, \dots, n$, a nontrivial solution of system (6.8) implies the existence of a nontrivial solution of the diophantine system,

$$\begin{aligned} \sum_{i=1}^{n} X_i^{k} &= \sum_{i=1}^{n} Y_i^{k}, \quad k = 1,\, 2, \dots, n-1, \\ X_1X_2\cdots X_n &= Y_1Y_2\cdots Y_n. \end{aligned} \tag{6.9}$$

It follows from Lemma 2 that the system (6.9) has no nontrivial solutions and the theorem follows. □

Corollary. *The diophantine system*

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2, \\ x_1^4 + x_2^4 + x_3^4 &= y_1^4 + y_2^4 + y_3^4, \\ x_1x_2x_3 &= y_1y_2y_3, \end{aligned} \tag{6.10}$$

has no nontrivial solutions in integers.

The corollary is simply a special case of Theorem 2 with $n = 3$. It is mentioned here since, just as in the case of the diophantine system of Theorem 1, we do have nontrivial solutions of the diophantine system (5.5) obtained by replacing the condition of equal products in the system (6.10) by the condition $x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3$.

We also note that Piezas, in his online book on algebraic identities [**18**, Chapter 028, subsection 8.4], has given a theorem on equal sums of like powers based on the existence of nontrivial integer solutions of the diophantine system (6.8) with $n = 4$. Since the system (6.8) has no nontrivial solutions, the theorem of Piezas is of no help in obtaining new results on equal sums of like powers.

7. Some open problems. There remain several open problems concerning equal sums of like powers and equal products of integers. Some of these are as follows:

(i) Find parametric/numerical solutions of the diophantine system (3.1) when $n > 6$, and with $s = n + 2$.

(ii) Do there exist nontrivial solutions of the diophantine system (3.1) with $s = n + 2$ for any arbitrary n? If not, for given n, find the least value of s for which the diophantine system (3.1) has nontrivial solutions.

(iii) Find parametric/numerical solutions of the diophantine system (4.1) when $n > 4$. Do there exist nontrivial solutions of the diophantine system (4.1) for any arbitrary n?

It would indeed be very interesting to find solutions of the diophantine system (4.1) for any arbitrary value of n.

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