

## CONVEXITY AND OSCULATION IN NORMED SPACES

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**ABSTRACT.** Constructive properties of uniform convexity, strict convexity, near convexity, and metric convexity in real normed linear spaces are considered. Examples show that certain classical theorems, such as the existence of points of osculation, are constructively invalid. The methods used are in accord with principles introduced by Errett Bishop.

**Introduction.** Contributions to the constructive study of convexity are found in [1, 2, 9] and, most extensively, in a recent paper of Fred Richman [8]. The present paper consists mainly of comments related to the latter paper, and Brouwerian counterexamples concerning various properties of convexity and the existence of certain points.

**Constructive mathematics.** A characteristic feature of the constructivist program is meticulous use of the conjunction “or.” To prove “ $P$  or  $Q$ ” constructively, it is required that either we prove  $P$ , or we prove  $Q$ ; it is not sufficient to prove the contrapositive  $\neg(\neg P$  and  $\neg Q)$ .

To clarify the methods used here, we give examples of familiar properties of the reals which are constructively *invalid*, and also properties which are constructively *valid*. The following classical properties of a real number  $\alpha$  are constructively invalid: “Either  $\alpha < 0$  or  $\alpha = 0$  or  $\alpha > 0$ ,” and “If  $\neg(\alpha \leq 0)$ , then  $\alpha > 0$ .” The relation  $\alpha > 0$  is given a strict constructive definition, with far-reaching significance. Then, the relation  $\alpha \leq 0$  is defined as  $\neg(\alpha > 0)$ . A constructively valid property of the reals is the *Constructive Dichotomy lemma*: If  $\alpha < \beta$ , then for any real number  $\gamma$ , either  $\gamma > \alpha$ , or  $\gamma < \beta$ . This lemma is applied ubiquitously, as a constructive substitute for the constructively invalid *Trichotomy*. For more details, see [1].

Applying constructivist principles when reworking classical mathematics can have interesting and surprising results.<sup>1</sup>

**Brouwerian counterexamples.** To determine the specific nonconstructivities in a classical theory, and thereby indicating feasible direc-

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2010 AMS *Mathematics subject classification*. Primary 46B20.

Received by the editors on June 9, 2010, and in revised form on August 2, 2010.

DOI:10.1216/RMJ-2013-43-2-551 Copyright ©2013 Rocky Mountain Mathematics Consortium

tions for constructive work, *Brouwerian counterexamples* are used, in conjunction with *omniscience principles*. A Brouwerian counterexample is a proof that a given statement implies an omniscience principle. In turn, an omniscience principle would imply solutions or significant information for a large number of well-known unsolved problems.<sup>2</sup> Omniscience principles may be stated in terms of binary sequences or properties of real numbers; we will have need for two omniscience principles:

**Limited Principle of Omniscience (LPO).** *For any real number  $\alpha \geq 0$ , either  $\alpha = 0$  or  $\alpha > 0$ .*

**Lesser Limited Principle of Omniscience (LLPO).** *For any real number  $\alpha$ , either  $\alpha \leq 0$  or  $\alpha \geq 0$ .*

A statement is considered *constructively invalid* if it implies an omniscience principle. Following Bishop, we may at times use the italicized *not* to indicate a constructively invalid statement.<sup>3</sup>

**Normed spaces.** In a real normed linear space  $X$ , we denote by  $B_r(c)$  the closed ball with center  $c$  and radius  $r \geq 0$ . We write simply  $B$  for the unit ball  $B_1(0)$  and  $\partial B$  for its boundary. The convex hull of points  $u, v \in X$  will be denoted  $\mathcal{H}\{u, v\}$ .

**1. Uniform convexity.** A real normed linear space  $X$  is *uniformly convex* if, for every  $\varepsilon > 0$ , there exists a  $q < 1$  such that, if  $u, v \in \partial B$  with  $\|u - v\| \geq \varepsilon$ , then  $\|(u + v)/2\| \leq q$ .

Lemma 1 in [8] is stated below as Lemma 1.1. This lemma extends the defining property of uniform convexity, stated for unit vectors, to arbitrary vectors; it is required for Richman's characterization of nearly convex subsets of uniformly convex spaces.<sup>4</sup>

The proof here fills a minor gap in the proof found in [8], where the unit vectors at the end need not attain the required distance.

**Lemma 1.1 [8].** *Let  $X$  be a real normed linear space. If  $X$  is uniformly convex, then for any  $\varepsilon > 0$  there exists  $q < 1$  with the following property: For any point  $c \in X$  and any real number  $r \geq 0$ , if  $u, v \in B_r(c)$  with  $\|u - v\| \geq r\varepsilon$ , then  $\|(u + v)/2 - c\| \leq qr$ .*

*Proof.*<sup>5</sup> Given  $\varepsilon > 0$ , choose  $q' < 1$  using  $\varepsilon' = \varepsilon/2$  in the definition of uniform convexity. Set  $\delta = \min\{\varepsilon/12, (1 - q')/4, 1/3\}$ , and take  $q = 1 - \delta$ .

(a) First consider the situation in which  $r > 0$ ; we may assume that  $c = 0$  and  $r = 1$ . Let  $u, v \in B$  with  $\|u - v\| \geq \varepsilon$ . Either  $\|u\| < 1 - 2\delta$  or  $\|u\| > 1 - 3\delta$ . In the first case,  $\|(u + v)/2\| < ((1 - 2\delta) + 1)/2 = q$ . Similarly, there are two cases for  $v$ .

In the fourth case, both norms are  $> 1 - 3\delta$ . Set  $\bar{u} = u/\|u\|$ ,  $\bar{v} = v/\|v\|$ ,  $m = (u + v)/2$  and  $m' = (\bar{u} + \bar{v})/2$ . Thus,  $\|u - \bar{u}\| = 1 - \|u\| < 3\delta$ , and similarly for  $v$ . Now

$$\begin{aligned} \|\bar{u} - \bar{v}\| &\geq \|u - v\| - \|u - \bar{u}\| - \|\bar{v} - v\| \\ &> \varepsilon - 3\delta - 3\delta \geq \varepsilon/2 = \varepsilon', \end{aligned}$$

so  $\|m'\| \leq q'$ , and it follows that

$$\begin{aligned} \|m\| &\leq \|m'\| + \|m - m'\| \\ &\leq \|m'\| + \frac{1}{2}(\|u - \bar{u}\| + \|v - \bar{v}\|) \\ &< q' + 3\delta \leq 1 - \delta = q. \end{aligned}$$

This verifies the required property when  $r > 0$ .

(b) Now consider the general situation, where  $r \geq 0$ ; we may assume that  $c = 0$ . Let  $u, v \in B_r(0)$  with  $\|u - v\| \geq r\varepsilon$ , and let  $e > 0$ . Either  $r > 0$ , or  $r < e$ . In the first case, (a) applies, so  $\|(u + v)/2\| \leq qr < qr + e$ . In the second case,  $\|(u + v)/2\| \leq r < e \leq qr + e$ . Thus,  $\|(u + v)/2\| \leq qr$ .  $\square$

Also required for [8, Theorem 4], is [8, Lemma 3], stated below as Lemma 1.2. The proof here avoids an incorrect assumption at the beginning of the proof found in [8].

**Lemma 1.2** [8]. *Let  $X$  be a uniformly convex real normed linear space, and let  $c$  and  $d$  be points of  $X$ . Then  $\text{diam}(B_r(c) \cap B_s(d)) \rightarrow 0$  as  $r + s \rightarrow \|c - d\|$ .*<sup>6</sup>

*Proof.*<sup>7</sup> Set  $\rho = \|c - d\|$ . Given  $\varepsilon > 0$ , either  $\rho < \varepsilon/4$  or  $\rho > 0$ . In the first case, set  $\delta = \varepsilon/4$ . If  $|r + s - \rho| < \delta$ , then  $r \leq r + s \leq \rho + \delta < \varepsilon/2$ , so  $\text{diam } B_r(c) \leq 2r < \varepsilon$ .

In the second case, choose  $q < 1$  in Lemma 1.1 using  $\varepsilon' = \varepsilon/\rho$ , and set  $\delta = \min\{\varepsilon/4, \rho(1-q)\}$ . Let  $|r+s-\rho| < \delta$ , and let  $u, v \in B_r(c) \cap B_s(d)$ . Either  $r > \rho - \delta$  or  $r < \rho$ . In the first of these subcases,  $s < \rho - r + \delta < 2\delta$ , so  $\|u-v\| \leq 2s < 4\delta \leq \varepsilon$ . Similarly, there are two subcases for  $s$ .

There remains only the fourth subcase, where both  $r < \rho$  and  $s < \rho$ . Suppose  $\|u-v\| > \varepsilon$ . Then  $\|u-v\| > \varepsilon'\rho > \varepsilon'r$ , with  $r > 0$ , and similarly for  $s$ . Thus  $\|(u+v)/2-c\| \leq qr$  and  $\|(u+v)/2-d\| \leq qs$ , so

$$\rho = \|c-d\| \leq q(r+s) < r+s$$

and

$$r+s-\rho \geq r+s-q(r+s) = (1-q)(r+s) > (1-q)\rho \geq \delta,$$

which contradicts the choice of  $r$  and  $s$ ; thus,  $\|u-v\| \leq \varepsilon$ .  $\square$

**2. Near and metric convexity.** A subset  $S$  of a metric space  $(M, \rho)$  is *nearly convex* if for any  $x, y \in S$ ,  $\lambda > 0$ , and  $\mu > 0$ , with  $\rho(x, y) < \lambda + \mu$ , there exists a  $z \in S$  such that  $\rho(x, z) < \lambda$  and  $\rho(z, y) < \mu$ . This version of convexity was introduced in [2] in connection with continuity problems.

Richman [8] has defined a subset  $S$  of a metric space to be *metrically convex* if, for any  $x, y \in S$ ,  $\lambda \geq 0$ , and  $\mu \geq 0$ , with  $\rho(x, y) = \lambda + \mu$ , there exists a  $z \in S$  such that  $\rho(x, z) = \lambda$  and  $\rho(z, y) = \mu$ . It will follow from Example 3.1 and Theorem 3.3 that the statement “Every real normed linear space is metrically convex” is constructively invalid.

The following is a comment in [8]; a proof is included here.

**Proposition 2.1** [8]. *A metrically convex subset of a metric space is nearly convex.*

*Proof.* Let  $x, y \in S$ ,  $\lambda > 0$ , and  $\mu > 0$ , with  $\rho(x, y) < \lambda + \mu$ . Set  $\gamma = \lambda + \mu - \rho(x, y)$ . Either  $\lambda < \gamma$  or  $\lambda > \gamma/2$ . In the first case, set  $\lambda' = 0$  and  $\mu' = \mu + \lambda - \gamma$ . Then  $0 \leq \lambda' < \lambda$ ,  $0 \leq \mu' < \mu$ , and  $\rho(x, y) = \lambda' + \mu'$ , so  $z$  may be selected in  $S$  using metric convexity. Similarly, there are two cases for  $\mu$ .

In the fourth case, we have  $\lambda > \gamma/2$  and  $\mu > \gamma/2$ . Set  $\lambda' = \lambda - \gamma/2$  and  $\mu' = \mu - \gamma/2$ . Then  $0 \leq \lambda' < \lambda$ ,  $0 \leq \mu' < \mu$ , and  $\rho(x, y) = \lambda' + \mu'$ , so again metric convexity provides a suitable element  $z \in S$ .  $\square$

The statement in [8], “Convex subsets of a normed space are metrically convex,” is incorrect. Convex subsets that are *not* metrically convex are found nearly everywhere, as is shown by the following Brouwerian counterexample.

**Example 2.2.** Every nontrivial convex subset  $S$  of a real normed linear space  $X$  contains a convex subset  $T$  such that the statement “ $T$  is metrically convex” is constructively invalid; the statement implies LLPO.

*Proof.* By *nontrivial* we mean that  $S$  contains at least two distinct points  $u$  and  $v$ . Given any real number  $\alpha$ , let  $T$  be the convex hull of the points  $(u + v)/2 \pm |\alpha|(u - v)/\|u - v\|$ . Then  $T \subset S$  (since we may assume that  $|\alpha|$  is small), and  $T$  is isometric with the subset  $V = \mathcal{H}\{-|\alpha|, |\alpha|\}$  of  $\mathbf{R}$ .

Set  $x = -|\alpha|$ ,  $y = |\alpha|$ ,  $\lambda = |\alpha| + \alpha$ , and  $\mu = |\alpha| - \alpha$ . Then  $x, y \in V$ ,  $\lambda \geq 0$ ,  $\mu \geq 0$ , and  $\lambda + \mu = \rho(x, y)$ . Under the hypothesis “ $T$  is metrically convex,” so also is  $V$ ; thus, there exists a  $z \in V$  such that  $\rho(x, z) = \lambda$  and  $\rho(z, y) = \mu$ . Since  $-|\alpha| \leq z \leq |\alpha|$ , we have  $z + |\alpha| = \rho(x, z) = \lambda = |\alpha| + \alpha$ , so  $z = \alpha$ .

This shows that  $\alpha \in \mathcal{H}\{-|\alpha|, |\alpha|\}$ , so there exists a  $t$  with  $0 \leq t \leq 1$  and  $\alpha = (1 - t)(-|\alpha|) + t|\alpha| = (2t - 1)|\alpha|$ . Note that, if  $\alpha > 0$ , then  $t = 1$ , while if  $\alpha < 0$ , then  $t = 0$ . Either  $t < 1$  or  $t > 0$ , and it follows that either  $\alpha > 0$  is contradictory or  $\alpha < 0$  is contradictory. Thus, either  $\alpha \leq 0$  or  $\alpha \geq 0$ , and LLPO results.  $\square$

**Convex hulls and intervals.** It is easily seen that the convex hull  $\mathcal{H}\{a, b\}$  is a dense subset of the closed interval  $[a, b]$  whenever  $a \leq b$  and coincides with the interval when  $a < b$ . Peter Schuster has raised the question of whether these sets are always equal.<sup>8</sup> A negative answer was given by [2, Example 10.11]. However, a simpler example is included in the proof of Example 2.2 above. Although the real number  $\alpha$  lies in the closed interval  $[-|\alpha|, |\alpha|]$ , the statement “ $\alpha \in \mathcal{H}\{-|\alpha|, |\alpha|\}$  for all  $\alpha \in \mathbf{R}$ ” implies LLPO.

Example 2.2 notwithstanding, we have the following result for *complete* convex subsets:

**Theorem 2.3.** *A complete convex subset  $S$  of a real normed linear space  $X$  is metrically convex.*

*Proof.*<sup>9</sup> Let  $x, y \in S$ ,  $\lambda \geq 0$ ,  $\mu \geq 0$ , with  $\|x - y\| = \lambda + \mu$ ; set  $\rho = \lambda + \mu$ . When  $\rho > 0$ , we write  $w = (\mu x + \lambda y)/\rho$ ; in this situation, we have  $\|w - x\| = \lambda$  and  $\|w - y\| = \mu$ . For all positive integers  $n$ , define

$$S_n = \{x : \rho < 1/n\} \cup \{w : \rho > 0\}.$$

Let  $m < m'$ , and  $y \in S_{m'}$ . If  $y = x$ , then  $\rho < 1/m' < 1/m$ , so  $y \in S_m$ , while if  $y = w$ , with  $\rho > 0$ , then again  $y \in S_m$ ; thus  $S_{m'} \subset S_m$ . Also, if  $x, w \in S_n$ , with  $\rho > 0$ , then  $\|w - x\| = \lambda \leq \rho < 1/n$ ; thus  $\text{diam } S_n < 1/n$ .

Thus the sequence  $\{\overline{S_n}\}_n$  of closures is a nested sequence of nonvoid closed subsets of  $S$  with diameters tending to zero. Since  $S$  is complete, there is a unique point  $z$  in the intersection of this sequence.

To show that the point  $z$  satisfies the required conditions, let  $\varepsilon > 0$ . If  $\rho > 0$ , then  $S_n = \{w\}$  eventually, so  $z = w$  and  $\|z - x\| = \lambda$ . On the other hand, if  $\rho < \varepsilon$ , then  $\lambda < \varepsilon$ , and  $\|s - x\| < \varepsilon$  for all  $s \in S_1$ , so  $\|z - x\| \leq \varepsilon$ . It follows that  $|\|z - x\| - \lambda| \leq \varepsilon$ . Thus,  $\|z - x\| = \lambda$ . Similarly,  $\|z - y\| = \mu$ . Thus  $S$  is metrically convex.  $\square$

**Corollary 2.4.** *Every complete real normed linear space  $X$  is metrically convex.*

In the proof of Theorem 2.3, the passage from the sequence  $\{S_n\}_n$  to the sequence  $\{\overline{S_n}\}_n$  of closures is necessary. Although each set  $S_n$  contains at most two elements, these sets are *not* closed. To demonstrate this, consider any real number  $\alpha \geq 0$ , and take  $X = \mathbf{R}$ ,  $x = 0$ ,  $y = \alpha$ ,  $\lambda = \alpha$ , and  $\mu = 0$ . Then each set  $S_n$  is included in the following Brouwerian counterexample.

**Example 2.5.** The statement “In a complete real normed linear space, any nonvoid subset containing at most two elements is closed” is constructively invalid; the statement implies LPO.

*Proof.* Given  $\alpha \geq 0$ , choose any real number  $c > 0$ , and define

$$S = \{0 : \alpha < c\} \cup \{\alpha : \alpha > 0\}.$$

To show that  $\alpha$  lies in the closure of  $S$ , let  $\varepsilon > 0$ . We may assume that  $\varepsilon < c$ . Either  $\alpha < \varepsilon$  or  $\alpha > 0$ . In the first case,  $0$  is a point of  $S$  with distance to  $\alpha$  less than  $\varepsilon$ . In the second case,  $\alpha \in S$ , and this suffices. Thus,  $\alpha \in \overline{S}$ . By hypothesis,  $\alpha \in S$ , so  $\alpha$  must lie in one of the sets forming the union. Hence LPO results.  $\square$

**3. Osculation.** Two closed balls,  $B_{r_1}(c_1)$  and  $B_{r_2}(c_2)$ , in a real normed linear space  $X$ , are said to be *osculating* if  $\|c_1 - c_2\| = r_1 + r_2$ . When  $r_1 + r_2 > 0$ , the balls are *nondegenerate*.

Theorem 5 in [8] considers various conditions on  $X$  which ensure that a point common to two osculating balls is *unique*. On the other hand, while the *existence* of a common point is evident in the case of nondegenerate osculating balls, a common point does *not* always exist, as the Brouwerian counterexample below will show. Thus, the implication “(2) implies (1)” in [8, Theorem 5], must be understood to involve only uniqueness.

**Example 3.1.** The statement “Osculating balls in a real normed linear space  $X$  always have at least one common point” is constructively invalid; it implies LLPO.

*Proof.* Given any real number  $\alpha$ , set  $X = \mathbf{R}\alpha$ , the linear subspace of  $\mathbf{R}$  generated by the single point  $\alpha$ , with the induced norm.<sup>10</sup> Set  $\alpha^+ = \max\{\alpha, 0\}$  and  $\alpha^- = \max\{-\alpha, 0\}$ . Since  $\alpha^+ + \alpha^- = |\alpha|$ , the balls  $B_{\alpha^-}(0)$  and  $B_{\alpha^+}(\alpha)$  in  $X$  are osculating.

By hypothesis, there exists a point  $x$  of  $X$  in the intersection  $I$  of these balls, with  $x = \lambda\alpha$  for some  $\lambda \in \mathbf{R}$ . Note that, if  $\alpha > 0$ , then  $I = B_0(0) \cap B_{|\alpha|}(\alpha)$ , so  $x = 0$  and  $\lambda = 0$ ; while, if  $\alpha < 0$ , then  $I = B_{|\alpha|}(0) \cap B_0(\alpha)$ , so  $x = \alpha$  and  $\lambda = 1$ . Either  $\lambda > 0$  or  $\lambda < 1$ , and it follows that either  $\alpha > 0$  is contradictory or  $\alpha < 0$  is contradictory. Thus, either  $\alpha \leq 0$  or  $\alpha \geq 0$ , and LLPO results.  $\square$

It will be convenient to state for reference a few basic properties of osculating balls, all of which are included in [8].

**Lemma 3.2.** *Let  $B_{r_1}(c_1)$  and  $B_{r_2}(c_2)$  be osculating balls in a real normed linear space.*

- (a) *If  $v$  is a common point, then  $v$  lies on the boundary of each ball.*
- (b) *If  $u$  and  $v$  are common points, then all points in the convex hull  $\mathcal{H}\{u, v\}$  are also common points.*
- (c) *When  $r_1 + r_2 > 0$ , the point  $z = (r_2c_1 + r_1c_2)/(r_1 + r_2)$  is a common point.*
- (d) *Let  $r_1 + r_2 > 0$ . If a common point  $v$  lies on the line through the centers, then  $v$  is the point  $z$  specified in (c).*

*Proof.* In (a), we have  $\|v - c_i\| \leq r_i$ . Also,

$$\|v - c_1\| \geq \|c_1 - c_2\| - \|v - c_2\| \geq (r_1 + r_2) - r_2 = r_1.$$

Thus,  $\|v - c_1\| = r_1$ , and  $v \in \partial B_{r_1}(c_1)$ . Similarly,  $v \in \partial B_{r_2}(c_2)$ .

The remaining properties follow from simple calculations.  $\square$

From the definition of metric convexity and Lemma 3.2 (a) we have the following.

**Theorem 3.3.** *Let  $X$  be a real normed linear space. Osculating balls in  $X$  always have at least one common point if and only if  $X$  is metrically convex.*

From this theorem and Corollary 2.4, we obtain the following.

**Corollary 3.4.** *In a complete real normed linear space, osculating balls always have at least one common point.*

**4. Strict convexity.** A real normed linear space  $X$  is *strictly convex* if the convex hull  $\mathcal{H}\{u, v\}$  of two points is contained in the boundary  $\partial B$  of the unit ball  $B$  only when  $u = v$ . It follows that, for any ball  $B_r(c)$ , if  $\mathcal{H}\{u, v\} \subset \partial B_r(c)$ , then  $u = v$ .

The following is Richman's theorem relating strict convexity to the uniqueness of points of osculation.



**Theorem 4.1**<sup>11</sup> [8]. *For any real normed linear space  $X$ , the following conditions are equivalent.*

- (1) *Any two osculating balls have at most one common point.*
- (2) *Any two nondegenerate osculating balls have at most one common point.*
- (3) *Any two osculating unit balls have at most one common point.*
- (4)  *$X$  is strictly convex.*
- (5) *If  $x$  and  $y$  are points of  $X$  with  $\|x + y\| = \|x\| + \|y\| \neq 0$ , then  $x$  and  $y$  are linearly dependent.*

*Proof.*<sup>12</sup> That (1) implies (2) implies (3) is self-evident.

(3) implies (4). Let  $\mathcal{H}\{u, v\} \subset \partial B$ . Then  $\|(u + v)/2\| = 1$ , so  $B_1(0)$  and  $B_1(u + v)$  are osculating unit balls. Since  $u$  and  $v$  both lie in each ball,  $u = v$ .

(4) implies (1). Let  $B_{r_1}(c_1)$  and  $B_{r_2}(c_2)$  be osculating balls, and let  $u$  and  $v$  be common points. By Lemma 3.2 (a), (b),  $\mathcal{H}\{u, v\} \subset \partial B_{r_1}(c_1)$ , so  $u = v$ .

(1) implies (5). Given  $\|x + y\| = \|x\| + \|y\| > 0$ , the balls  $B_{\|x\|}(x)$  and  $B_{\|y\|}(-y)$  are osculating and nondegenerate, so by Lemma 3.2 (c) they have a common point of the form  $z = \lambda x + (1 - \lambda)(-y)$ . Since 0 is clearly a common point,  $z = 0$ , and this results in a linear dependence relation for  $x$  and  $y$ .

(5) implies (3). Let  $B_1(c_1)$  and  $B_1(c_2)$  be osculating unit balls, and let  $v$  be any common point. Then

$$\|(c_1 - v) + (v - c_2)\| = \|c_1 - c_2\| = 2 = \|c_1 - v\| + \|v - c_2\|,$$

so  $c_1 - v$  and  $v - c_2$  are linearly dependent. We then have  $\lambda(v - c_1) + \mu(v - c_2) = 0$ , where either  $\lambda \neq 0$  or  $\mu \neq 0$ . We may assume that  $\mu \neq 0$  and, furthermore, that  $\mu = 1$ ; thus,  $(\lambda + 1)v = \lambda c_1 + c_2$ . It follows that  $(\lambda + 1)(v - c_1) = c_2 - c_1$ ; taking norms here, we find that  $|\lambda + 1| = 2$ , so  $\lambda$  is either -3 or 1. Also,  $(\lambda + 1)(v - c_2) = \lambda(c_1 - c_2)$ ; taking norms here, we have  $|\lambda| = 1$ , so -3 is ruled out, and it follows that  $v = (c_1 + c_2)/2$ .  $\square$

## ENDNOTES

1. For an exposition of the constructivist program, see Bishop's [1, Chapter 1], "A Constructivist Manifesto," see also [3, 6].

2. This method was introduced by L.E.J. Brouwer in 1908 to demonstrate that use of the *Law of excluded middle* inhibits mathematics from attaining its full significance.

3. For more information concerning Brouwerian counterexamples, and other omniscience principles, see [2, 4].

4. [8, Theorem 4]: *A subset of a uniformly convex space is nearly convex if and only if its closure is convex.*

5. The main ideas in this proof are drawn from [8].

6. The diameter of the intersection may not be amenable to constructive calculation; the expression *diameter tends to zero* means only that one may find arbitrarily small upper bounds.

7. The main ideas in this proof are drawn from [8]. The proof here avoids the incorrect assumption  $r + s \leq \|x - y\| + \inf\{r, s\}$  found in [8]. The proof here also removes the restriction to distinct centers; this would allow a slight simplification in the proof of [8, Theorem 4].

8. [9, page 446].

9. The original proof of Theorem 2.3 constructed a Cauchy sequence of points converging to the required point  $z$ ; this method is more typical of constructive practice. However, constructing such a sequence requires an appeal to the *axiom of countable choice*. The nested-sequence method of proof used here, suggested by the referee, avoids this axiom. Avoiding use of the *axiom of countable choice* is considered a desirable goal by many constructivists. For more details concerning this issue, see [7, 9].

10. For an extensive discussion of this space, see [5].

11. [8, Theorem 5]. Only the first five conditions are listed here; the sixth condition concerns a problem that remains open.

12. The proof here is essentially the same as in [8], with several simplifications.

## REFERENCES

1. Errett Bishop, *Foundations of constructive analysis*, McGraw-Hill Book Co., New York-Toronto-London, 1967.
2. Mark Mandelkern, *Constructive continuity*, Mem. Amer. Math. Soc. **42** (1983), number 277.
3. ———, *Constructive mathematics*, Math. Mag. **58** (1985), 272–280.
4. ———, *Brouwerian counterexamples*, Math. Mag. **62** (1989), 3–27.
5. Fred Richman, *Meaning and information in constructive mathematics*, Amer. Math. Month. **89** (1982), 385–388.
6. ———, *Existence proofs*, Amer. Math. Month. **106** (1999), 303–308.
7. ———, *Constructive mathematics without choice*, Synthese Lib. **306**, 199–205, Kluwer Academic Publishers, Dordrecht, 2001.
8. ———, *Near convexity, metric convexity and convexity*, Rocky Mountain J. Math. **37** (2007), 1305–1314.
9. Peter M. Schuster, *Unique existence, approximate solutions, and countable choice*, Theoret. Comp. Sci. **305** (2003), 433–455.

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