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THE DECOMPOSITION OF TRIANGULAR NUMBERS

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ABSTRACT. In this paper, we prove that there are infinitely many triangular numbers which have two different ways to be decomposed as the product of two triangular numbers, each greater than 1.

1. Introduction. Among various kinds of figurate numbers, a famous one is the triangular number

$$\binom{n}{2} = 1 + 2 + 3 + \cdots + (n-1), \quad n \geqslant 2,\ n \in \mathbf{Z}.$$

Research on triangular numbers can be traced back to Pythagoras (570–501 B.C.). Many elegant properties of triangular numbers have been discovered by Fermat, Euler, Legendre, Gauss and other great mathematicians [**2**]: Legendre proved that no triangular number, except 1, is a cube or fourth power; Euler proved that there are infinitely many triangular square numbers (a positive integer which is simultaneously a triangular number and a perfect square); in 1796, Gauss showed that every natural number is a sum of at most three triangular numbers.

To find triangular numbers whose multiples are triangular, Euler proved the special case that there exist infinitely many pairs $\binom{r}{2}, \binom{s}{2}$, such that $\binom{r}{2} = 3\binom{s}{2}$. For related results, we refer to [**1**]. Notice $3 = \binom{3}{2}$, so that we deduce the following: there are infinitely many triangular numbers which can be decomposed as the product of two "non-1" triangular numbers, that is, each greater than 1. In view of

$$\binom{36}{2} = \binom{3}{2}\binom{21}{2} = \binom{4}{2}\binom{15}{2}$$

and

$$\binom{1225}{2} = \binom{15}{2}\binom{120}{2} = \binom{21}{2}\binom{85}{2},$$

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which imply that the decomposition is not always unique, a natural question may be asked: are there infinitely many triangular numbers which have two different ways to be decomposed as the product of two non-1 triangular numbers? Is it possible to have three different ways? In this paper, we completely solve the first question. As a matter of fact, we prove stronger results as follow:

Theorem 1. *For any triangular square number $r > 1$, $\binom{r}{2}$ has two different ways to be decomposed as the product of two non-*1 *triangular numbers.*

Theorem 2. *There are infinitely many positive integers which have three different ways to be decomposed as the product of two positive integers of the form $n^2 - 1$.*

For the second question, by numerical calculations, we find that:

*For a positive integer $n \leqslant 3\times 10^5$, $\binom{n}{2}$ has no more than two different ways to be decomposed as the product of two non-*1 *triangular numbers.*

Moreover, we have the following:

Conjecture 1. *There is no triangular number which has more than two different ways to be decomposed as the product of two non-*1 *triangular numbers.*

2. Preliminaries. Let $\mathbf{N}$ be the set of positive integers. Denote

$$\varepsilon = 3 + 2\sqrt{2},\ \overline{\varepsilon} = 3 - 2\sqrt{2},\ A_n = \varepsilon^n + \overline{\varepsilon}^n,\ B_n = \varepsilon^n - \overline{\varepsilon}^n.$$

We have

Lemma 1. *All the solutions of*

$$\binom{x}{2} = 2\binom{y}{2}, \quad (x, y \in \mathbf{N},\ x \geq 2,\ y \geq 2) \tag{1}$$

are

$$\begin{cases} x_n = \left(1 + \frac{1}{2}A_n + \frac{\sqrt{2}}{2}B_n\right) \Big/ 2 \\ y_n = \left(1 + \frac{1}{2}A_n + \frac{\sqrt{2}}{4}B_n\right) \Big/ 2, \end{cases} \tag{2}$$

i.e.,

$$\begin{cases} x_{n+2} = 6x_{n+1} - x_n - 2 & x_0 = 1,\ x_1 = 4 \\ y_{n+2} = 6y_{n+1} - y_n - 2 & y_0 = 1,\ y_1 = 3 \end{cases} \tag{3}$$

for $n \in \mathbf{N}$.

Proof. Let $2x - 1 = X,\ 2y - 1 = Y$. Then (1) changes to

$$X^2 - 2Y^2 = -1. \tag{4}$$

All the positive integer solutions of (4) are given by

$$X_n + Y_n\sqrt{2} = (1 + \sqrt{2})\varepsilon^n, \quad (n \geq 0), \tag{5}$$

where

$$\begin{cases} X_n = \frac{A_n}{2} + \frac{B_n}{\sqrt{2}} \\ Y_n = \frac{A_n}{2} + \frac{B_n}{2\sqrt{2}}. \end{cases} \tag{6}$$

The recursive sequences derived from (5) are

$$\begin{cases} X_{n+2} = 6X_{n+1} - X_n & X_0 = 1,\ X_1 = 7 \\ Y_{n+2} = 6Y_{n+1} - Y_n & Y_0 = 1,\ Y_1 = 5. \end{cases} \tag{7}$$

Noting that $2x_n - 1 = X_n,\ 2y_n - 1 = Y_n$ and $x_n \geqslant 2$, we finally give all the solutions of (1) by (2) and (3) from (6) and (7) respectively. □

Lemma 2. *If* y_n *satisfies relation* (2), *then*

$$1 + 4y_n(y_n - 1)y_{n+1}(y_{n+1} - 1) = \left(\left(\frac{B_{n+1}}{4}\right)^2 - 1\right)^2. \tag{8}$$

Proof. Since

$$\begin{cases} \varepsilon^n = \overline{\varepsilon}\varepsilon^{n+1} = (3-2\sqrt{2})\varepsilon^{n+1} \\ \overline{\varepsilon}^n = \varepsilon\overline{\varepsilon}^{n+1} = (3+2\sqrt{2})\overline{\varepsilon}^{n+1}, \end{cases}$$

and

$$\begin{cases} A_n = \varepsilon^n + \overline{\varepsilon}^n \\ B_n = \varepsilon^n - \overline{\varepsilon}^n, \end{cases}$$

we have

$$\begin{cases} A_n = 3A_{n+1} - 2\sqrt{2}B_{n+1} \\ B_n = 3B_{n+1} - 2\sqrt{2}A_{n+1}. \end{cases} \tag{9}$$

Combining (2) with (9), we have

$$\begin{aligned} (10)\quad & 1 + 4y_n(y_n-1)y_{n+1}(y_{n+1}-1) \\ &= 1 + 4\left(\frac{1}{2}\left(1+\frac{1}{2}A_n+\frac{\sqrt{2}}{4}B_n\right)\right)\left(\frac{1}{2}\left(1+\frac{1}{2}A_n+\frac{\sqrt{2}}{4}B_n\right)-1\right) \\ &\times \left(\frac{1}{2}\left(1+\frac{1}{2}A_{n+1}+\frac{\sqrt{2}}{4}B_{n+1}\right)\right)\left(\frac{1}{2}\left(1+\frac{1}{2}A_{n+1}+\frac{\sqrt{2}}{4}B_{n+1}\right)-1\right) \\ &= -\frac{1}{64}A_{n+1}^4 + \frac{1}{256}B_{n+1}^4 - \frac{1}{64}A_{n+1}^2B_{n+1}^2 - \frac{1}{8}A_{n+1}^2 - \frac{1}{16}B_{n+1}^2 + \frac{5}{4}. \end{aligned}$$

Noting that

$$\begin{cases} A_{n+1} + B_{n+1} = 2\varepsilon^{n+1} \\ A_{n+1} - B_{n+1} = 2\overline{\varepsilon}^{n+1}, \end{cases}$$

we have

$$A_{n+1}^2 = B_{n+1}^2 + 4, \tag{11}$$

with which we finally reduce (10) to (8). □

Lemma 3. *All the solutions of*

$$\binom{s}{2} = t^2, \quad (s,t \in \mathbf{N},\ s \geq 2,\ t \geq 1) \tag{12}$$

are

(13)
$$\begin{cases} s_n = \frac{1}{2} + \frac{A_n}{4} \\ t_n = \frac{B_n}{4\sqrt{2}}, \end{cases}$$

i.e.,

(14)
$$\begin{cases} s_{n+2} = 6s_{n+1} - s_n - 2 & s_0 = 1,\ s_1 = 2 \\ t_{n+2} = 6t_{n+1} - t_n & t_0 = 0,\ t_1 = 1 \end{cases}$$

for $n \in \mathbf{N}$.

Proof. Let $2s - 1 = X, 2t = Y$. Then (12) changes to

(15)
$$X^2 - 2Y^2 = 1.$$

All the positive integer solutions of (15) are given by

(16)
$$X_n + Y_n\sqrt{2} = \varepsilon^n, \quad (n \geq 0),$$

where

(17)
$$\begin{cases} X_n = \frac{A_n}{2} \\ Y_n = \frac{B_n}{2\sqrt{2}}. \end{cases}$$

The recursive sequences derived from (16) are

(18)
$$\begin{cases} X_{n+2} = 6X_{n+1} - X_n & X_0 = 1,\ X_1 = 3 \\ Y_{n+2} = 6Y_{n+1} - Y_n & Y_0 = 0,\ Y_1 = 2. \end{cases}$$

Noting that $2s_n - 1 = X_n$, $2t_n = Y_n$ and $s_n \geqslant 2$, we finally give all the solutions of (12) by (13) and (14) from (17) and (18), respectively. □

3. Proofs of the theorems.

Proof of Theorem 1. By Lemma 1,

(19)
$$2\binom{y_n}{2}\binom{y_{n+1}}{2} = \binom{x_n}{2}\binom{y_{n+1}}{2} = \binom{x_{n+1}}{2}\binom{y_n}{2}, \quad (n \geq 1).$$

By Lemma 2,

$$1+8\left(2\binom{y_n}{2}\binom{y_{n+1}}{2}\right)=1+4y_n(y_n-1)y_{n+1}(y_{n+1}-1) \tag{20}$$
$$=\left(\left(\frac{B_{n+1}}{4}\right)^2-1\right)^2.$$

By Lemma 3, t_{n+1}^2 is a triangular square number and

$$1+8\binom{t_{n+1}^2}{2}=(2t_{n+1}^2-1)^2=\left(\left(\frac{B_{n+1}}{4}\right)^2-1\right)^2,\quad (n\in\mathbf{N}). \tag{21}$$

Combining (19), (20) and (21), we have

$$\binom{t_{n+1}^2}{2}=\binom{x_n}{2}\binom{y_{n+1}}{2}=\binom{x_{n+1}}{2}\binom{y_n}{2},\quad (n\in\mathbf{N}). \tag{22}$$

We now prove that $y_n\neq x_n$ and $y_n\neq y_{n+1}$. In view of $\binom{x_n}{2}=2\binom{y_n}{2}$, we deduce $y_n\neq x_n$. Now we show $y_{n+1}>y_n$ by induction. First of all, $y_1>y_0$. Suppose $y_{n+1}>y_n$; then by (3), we have

$$y_{n+2}-y_{n+1}=(6y_{n+1}-y_n-2)-y_{n+1}$$
$$=5y_{n+1}-y_n-2>4y_n-2>0.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. By (22),

$$8\left(8\binom{t_{n+1}^2}{2}+1-1\right)=\left(8\binom{x_n}{2}+1-1\right)\left(8\binom{y_{n+1}}{2}+1-1\right)$$
$$=\left(8\binom{x_{n+1}}{2}+1-1\right)\left(8\binom{y_n}{2}+1-1\right).$$

Simplifying, we have

$$(3^2-1)\left((2t_{n+1}^2-1)^2-1\right)=\left((2x_n-1)^2-1\right)\left((2y_{n+1}-1)^2-1\right)$$
$$=\left((2x_{n+1}-1)^2-1\right)\left((2y_n-1)^2-1\right).$$

This completes the proof of Theorem 2. □

As examples, take $n = 1,\ 2.$ We have

$$\begin{aligned} 40320 &= \left(3^2-1\right)\left(71^2-1\right) = \left(5^2-1\right)\left(41^2-1\right) = \left(7^2-1\right)\left(29^2-1\right), \\ 47980800 &= \left(3^2-1\right)\left(2449^2-1\right) = \left(29^2-1\right)\left(239^2-1\right) \\ &= \left(41^2-1\right)\left(169^2-1\right). \qquad \square \end{aligned}$$

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