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ON LIMITS OF SEQUENCES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We study functions which are the pointwise limit of a sequence of holomorphic functions. In one complex variable this is a classical topic, though we offer some new points of view and new results. Some novel results for solutions of elliptic equations will be treated. In several complex variables the question seems to be new, and we explore some new avenues.

1. Introduction. It is a standard and well-known fact from complex function theory (which appears to be due to Stieltjes [**23**], although see also Vitali's theorem in [**24**] and Weierstrass's complete works [**25**]) that, if $\{f_j\}$ is a sequence of holomorphic functions on a planar domain Ω and if the sequence converges *uniformly on compact subsets of* Ω, then the limit function is holomorphic on Ω. Certainly this result is one of several justifications for equipping the space of holomorphic functions on Ω with the compact-open topology (see also [**15**], where this point of view is developed in detail from the perspective of functional analysis).

Considerably less well known is the following result of Osgood [**19**]:

Theorem 1. *Let* $\{f_j\}$ *be a sequence of holomorphic functions on a planar domain* Ω. *Assume that the* f_j *converge* pointwise *to a limit function* f *on* Ω. *Then* f *is holomorphic on a dense, open subset* V *of* Ω. *The convergence is uniform on compact subsets of the dense, open set.*

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This result is not entirely obvious; it is certainly surprising and interesting. For completeness, we now offer a proof of the theorem:

Proof of Theorem 1. Let U be a nonempty open subset of Ω with compact closure in Ω. Define, for $k = 1, 2, \ldots$,

$$S_k = \{z \in \overline{U} : |f_j(z)| \leq k \quad \text{for all } j \in \mathbf{N}\}.$$

Since the f_j converge at each $z \in \overline{U}$, certainly the set $\{f_j(z) : j \in \mathbf{N}\}$ is bounded for each fixed z. So each $z \in \overline{U}$ lies in some S_k. In other words,

$$\overline{U} = \bigcup_k S_k.$$

Now of course $\overline{U}$ is a complete metric space (in the ordinary Euclidean metric), so the Baire category theorem tells us that some S_k must be "somewhere dense" in $\overline{U}$. This means that $\overline{S_k}$ will contain a nontrivial Euclidean metric ball (or disc) in $\overline{U}$. Call the ball $\mathcal{B}$. Now it is a simple matter to apply Montel's theorem on $\mathcal{B}$ to find a subsequence f_{j_k} that converges uniformly on compact sets to a limit function g. But of course g must coincide with f, and g (hence f) must be holomorphic on $\mathcal{B}$.

Since the choice of U in the above arguments was arbitrary, the conclusion of the theorem follows. □

Remark. An alternative approach, which avoids the explicit use of Montel's theorem, is as follows. Once one has identified an S_k whose closure contains a ball or disc $D(P, r)$, let $\gamma : [0, 1] \to \Omega$ be a simple, closed, rectifiable curve in $D(P, r)$. Then of course the image $\widetilde{\gamma} \equiv \{\gamma(t) : t \in [0, 1]\}$ of γ is a compact set. Let $\varepsilon > 0$. By Lusin's theorem, the sequence f_j converges uniformly on some subset $E \subseteq \widetilde{\gamma}$ with the property that the linear measure of $\widetilde{\gamma} \setminus E$ is less than ε. Let K be a compact subset of the open region surrounded by γ, and let $\delta > 0$ be the Euclidean distance of K to $\widetilde{\gamma}$. Let $\varepsilon^* > 0$ and choose $J > 0$ so large that, when $\ell, m > J$, then

$$|f_\ell(z) - f_m(z)| < \varepsilon^*$$

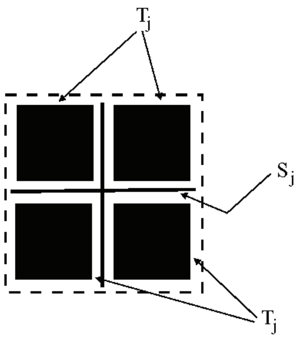

FIGURE 1. The sets S_j and T_j.

for all $z \in E$. Then, for $w \in K$,

$$\begin{aligned}|f_\ell(w) - f_m(w)| &= \left| \frac{1}{2\pi i} \oint_\gamma \frac{f_\ell(\zeta) - f_m(\zeta)}{\zeta - w} \, d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_E \frac{\varepsilon^*}{\delta} \, ds + \frac{1}{2\pi} \int_{\widetilde{\gamma} \setminus E} \frac{2k}{\delta} \, ds \\ &\leq \frac{\varepsilon^* \cdot |E|}{2\pi \cdot \delta} + \frac{\varepsilon \cdot 2k}{2\pi \cdot \delta}.\end{aligned}$$

Thus, we see that we have uniform convergence on K. And the holomorphicity follows as usual. □

The book [**20**] contains a nice treatment of some of the one-complex-variable theory related to Osgood's theorem.

The next example is inspired by ideas in [**26**, pages 131–133]. It demonstrates that Osgood's theorem has substance and describes a situation that actually occurs. A thorough discussion of many of the ideas treated here, from a somewhat different point of view, appears in [**2**]. In fact, [**2**] presents quite a different construction of an example that illustrates Theorem 1.

Example 1. Let

$$U = \{z \in \mathbf{C} : |\mathrm{Re}\, z| < 1, |\mathrm{Im}\, z| < 1\}.$$

For $j = 1, 2, \ldots$, define

$$S_j = \{z \in U : \operatorname{Re} z = 0 \text{ or } \operatorname{Im} z = 0,\\ |\operatorname{Re} z| \leq 1 - 1/[j+2],\ |\operatorname{Im} z| \leq 1 - 1/[j+2]\}.$$

Also define

$$T_j = \{z \in U : 1/[j+2] \leq |\operatorname{Re} z| \leq 1 - 1/[j+2],\\ 1/[j+2] \leq |\operatorname{Im} z| \leq 1 - 1/[j+2]\}.$$

We invite the reader to examine Figure 1 to appreciate these sets.

Now, for each j, we apply Runge's theorem on $S_j \cup T_j$. Notice that the complement of $S_j \cup T_j$ is connected, so that we can push the poles of the approximating functions to the complement of U. We are then able to produce for each j a holomorphic function f_j on U such that

$$|f_j(z) - 0| < \frac{1}{j} \quad \text{for } z \in T_j,$$
$$|f_j(z) - 1| < \frac{1}{j} \quad \text{for } z \in S_j.$$

Then it is easy to see that the sequence $\{f_j\}$ converges pointwise to the function f given by

$$f(z) = \begin{cases} 0 & \text{if } z \in U \setminus \{z \in U : \operatorname{Re} z = 0 \text{ or } \operatorname{Im} z = 0\} \\ 1 & \text{if } z \in U \cap \{z \in U : \operatorname{Re} z = 0 \text{ or } \operatorname{Im} z = 0\}. \end{cases}$$

Thus, the limit function f is holomorphic on a dense open subset of U, and the exceptional set is the two axes in U. □

One might ask what more can be said about the open, dense set V on which the limit function f is holomorphic. Put in other words, what can one say about $\Omega \setminus V$? Lavrentiev [**14**] was the first to give a characterization of those open sets on which a pointwise convergent sequence of holomorphic functions can converge uniformly. Siciak [**21**] has given a rather different answer in the language of capacity theory.

In fact, a suitable version of the theorem is true for harmonic functions, or more generally for solutions of a uniformly elliptic partial differential equation of second order. We shall prove such results later in the present paper. See also [**5, 17, 18, 23**] for related results.

1. More results in the classical setting. Our first new result for planar domains concerns harmonic functions:

Theorem 2. *Let $\{f_j\}$ be a sequence of harmonic functions on a planar domain Ω. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is harmonic on a dense open subset of Ω.*

Sketch of the proof of the theorem. Proceed as in the proof of the result for holomorphic functions. It is certainly true that a collection of harmonic functions on a planar domain that is uniformly bounded on compacta will have a subsequence that converges uniformly on compact sets. This follows from easy estimates on the Poisson kernel. The rest of the argument is the same as before. □

Remark. A result of this nature is already contained in [**9**, Theorem 1].

Theorem 3. *Let $\mathcal{L}$ be a uniformly elliptic operator, having locally bounded, C^2 coefficients, of order 2 on a planar domain Ω. Let $\{f_j\}$ be a sequence of functions that are annihilated by $\mathcal{L}$ on Ω. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is annihilated by $\mathcal{L}$ on a dense open subset of Ω.*

Proof. The proof is the same as the last result. The only thing to check is that a collection of functions annihilated by $\mathcal{L}$ that is bounded on compact sets will have a subsequence that converges uniformly on compact sets. This will follow, as in the harmonic case, from the Poisson formula for $\mathcal{L}$ (see [**8**]). The rest of the argument is the same. □

Theorem 4. *Let $\{f_j\}$ be a sequence of holomorphic functions on a planar domain Ω. Suppose that there is a constant $M > 0$ such that $|f_j(z)| \leq M$ for all j and for all $z \in \Omega$. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is holomorphic on all of Ω.*

Remark. Of course the new feature in this last theorem of Stieltjes [**22**] is that we are assuming that the family $\{f_j\}$ is uniformly bounded. This Tauberian hypothesis gives a stronger conclusion. The proof will now be a bit different.

Proof. Let U be an open subset of Ω. Then the argument from the proof of Theorem 1 applies immediately on U. Thus, the limit function is holomorphic on U. Since the choice of U was arbitrary, we are finished. □

In fact, there is a much weaker condition (than in the last theorem) that will give the same result:

Theorem 5. *Let $\{f_j\}$ be a sequence of holomorphic functions on a planar domain Ω. Suppose that there is a nonnegative, integrable function g on Ω such that $|f_j(z)| \leq g(z)$ for all j and for all $z \in \Omega$. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is holomorphic on all of Ω.*

Proof. The proof is simplicity itself. Suppose that K is a compact subset of Ω. Let φ be a C_c^∞ function on Ω that is identically equal to 1 on K. Then

$$f_j(z) = \frac{1}{\pi} \iint \frac{f_j(\zeta)\overline{\partial}\varphi(\zeta)}{z - \zeta}\, dA(\zeta)$$

for each j. Here dA is Lebesgue area measure on $\mathbf{C}$. As a consequence,

$$|f_j(z)| \leq \frac{1}{\pi} \iint \frac{g(\zeta)|\overline{\partial}\varphi(\zeta)|}{|z - \zeta|}\, dA(\zeta).$$

Since K has positive distance from the support of $\overline{\partial}\varphi$, we may now conclude that the $\{f_j\}$ are uniformly bounded on K. Theorem 4 therefore tells us that f is holomorphic on all of Ω. □

The following result is discussed but not proved in [**4**]:

Theorem 6. *Let $\{f_j\}$ be a sequence of Schlicht functions on the unit disc D that converges pointwise. Then the limit function is holomorphic on all of D.*

Proof. By the "growth theorem" (see [**7**]), any Schlicht function f satisfies

$$|f(z)| \leq |z| \cdot (1 - |z|)^{-2}$$

for all z in the disc D. It follows that the f_j are uniformly bounded on compact subsets of D. So they form a normal family. Thus, there is a subsequence $\{f_{j_k}\}$ that converges uniformly on compact sets to some limit function g. That function g is of course holomorphic everywhere. But it must coincide with the pointwise limit function. □

It is easy to see that results of the kind we are discussing here cannot be true in the category of real analytic functions. Indeed, the Weierstrass approximation theorem tells us that *any* continuous function is the uniform limit on compact sets of polynomials, hence of real analytic functions. But we do have the following modified result:

Proposition 7. *Let $\{f_j\}$ be a sequence of real analytic functions on the bounded interval (a, b). For each $\ell = 0, 1, 2, \ldots$, let $g^{(\ell)}$ denote the ℓth derivative of function g. Assume that, for each ℓ, there are positive constants K and R so that, for all j and ℓ,*

$$|f_j^{(\ell)}| \leq K \cdot \frac{\ell!}{R^\ell}.$$

Further assume that the sequence $\{f_j\}$ converges pointwise to some function f on (a, b). Then f is real analytic on (a, b).

The proposition bears out the heuristic that a sequence of real analytic functions converging to a non-analytic function like $f(x) = |x|$ on the interval $(-1, 1)$ must have derivatives blowing up. Note that, for simplicity, we have stated the result in one dimension. But it clearly holds in any dimension.

Proof of Proposition 7. Without loss of generality, assume that $a = -b$, so that point 0 lies in the center of the domain interval. Now call the interval $(-a, a)$.

Fix a positive integer N. With a simple diagonalization argument, we may choose a subsequence f_{j_k} so that $f_{j_k}^{(\ell)}(0)$ converges to some number α_ℓ as $k \to \infty$, each $\ell = 0, 1, 2, \ldots, N$. For each k, we may write

$$f_{j_k}(x) = \sum_{\ell=0}^{N} \frac{f_{j_k}^{(\ell)}}{\ell!} \cdot x^\ell + \mathcal{O}(x^{N+1}).$$

Letting $k \to +\infty$, we find that

$$f(x) = \sum_{\ell=0}^{N} \frac{\alpha_\ell}{\ell!} \cdot x^\ell + \mathcal{O}(x^{N+1}).$$

Of course, a similar identity holds at each point of $(-a, a)$ (not just at 0). It follows then, from the converse of the Taylor theorem (see [**1, 12**]), that f is C^∞ and $f^{(\ell)}$ at each point x is given as the limit of a subsequence of the $f_j^{(\ell)}$ at that point. Since the argument applies to show that every subsequence of $\{f_j\}$ has a subsequence with this property, we may conclude that

$$\lim_{j\to\infty} f_j^{(\ell)}(x) = f^{(\ell)}(x)$$

for each x in the $(-a, a)$ and each $\ell = 0, 1, 2, \ldots$.

Now we have assumed that

$$|f_j^{(\ell)}(x)| \le K \cdot \frac{\ell!}{R^\ell}$$

for x in a compact subset of $(-a, a)$. Letting $j \to \infty$ gives

$$|f^{(\ell)}(x)| \le K \cdot \frac{\ell!}{R^\ell}.$$

We conclude then that f is real analytic, as desired. □

2. Results in several complex variables. The first result in $\mathbf{C}^n$ is as follows.

Theorem 8. *Let $\{f_j\}$ be a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbf{C}^n$. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is holomorphic on a dense open subset of Ω.*

Also the convergence is uniform on compact subsets of the dense open set.

Proof. The argument is the same as that for Theorem 1. We need only note that Montel's theorem is still valid. The rest of the argument is unchanged. □

Remark. Just as in the remark following the proof of Theorem 1, we could use the Henkin-Ramirez integral formula on small balls (see [**11**, Chapter 8]) to give an alternative proof of this result.

Theorem 9. *Let $\{f_j\}$ be a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbf{C}^n$. Assume that the f_j converge pointwise to a limit function f on Ω. Let ℓ be any complex line in $\mathbf{C}^n$. Then the limit function f is holomorphic on a dense open subset of $\ell \cap \Omega$.*

Proof. Of course, we simply apply the argument from the proof of Theorem 1 on $\ell \cap \Omega$. □

Remark. This is a stronger result than Theorem 8. One may note that something similar could be proved with "'complex line" replaced by "complex analytic variety." It is not clear what the optimal result might be.

We note that [**5**] contains some results which characterize those sets whose characteristic function is the pointwise limit of a sequence of holomorphic functions. The main result of [**5**] was anticipated in the paper [**3**]. See also [**6**].

With a little effort, one may produce results in several complex variables that introduce a new way to think about these theorems. An example is this:

Theorem 10. *Let $\{f_j\}$ be a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbf{C}^n$, $n \geq 2$. Suppose that there is a function f on Ω such that, for each analytic disc $\varphi : D \to \Omega$, the sequence $\{f_j \circ \varphi\}$ converges uniformly on $\overline{D}$ to $f \circ \varphi$. Then f is holomorphic on Ω.*

Proof. Fix $z \in \Omega$ and $c > 0$ small. Fix an index j, and let $\varphi(\zeta) = (0, 0, \ldots, c \cdot \zeta, 0, \ldots 0)$, where the ζ appears in the jth position.

Then our hypothesis says that $f_j \circ \varphi(\zeta) = f_j((0, 0, \ldots, \zeta, 0, \ldots 0)$ converges uniformly, for $\zeta \in D$ to f. But the limit function will of course be holomorphic. So f is holomorphic in each variable separately. It follows then from Hartogs's theorem (see [**11**]) that f is genuinely holomorphic as a function of n variables. □

3. Concluding remarks. It is clear that there is more to learn in the several complex variable setting. We would like a result that has a chance of being sharp, so that the exceptional set for convergence can be characterized (as in [**21**] for one complex variable). This matter will be explored in future papers.

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