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POSITIVE SOLUTIONS OF AN nTH ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. By using the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem, this paper investigates the multiplicity of positive solutions of an nth order three-point boundary value problem. In addition, we also give some examples to demonstrate our results.

1. Introduction. In this paper, we consider the nth order three-point boundary value problem (TPBVP)

$$\begin{cases} y^{(n)}(t) + Q(t, y, y', \dots, y^{(n-2)}) = P(t, y, y', \dots, y^{(n-1)}), & t \in [a, b], \\ y^{(i)}(a) = 0, & 0 \le i \le n-3, \\ \alpha y^{(n-2)}(\eta) + \beta y^{(n-1)}(a) = y^{(n-2)}(a), & \\ \quad \gamma y^{(n-2)}(\eta) - \delta y^{(n-1)}(b) = y^{(n-2)}(b), & \end{cases} \tag{1.1}$$

where $n \ge 2$, $a < \eta < b$, $\delta \ge 0$, $0 < \alpha < [b - \delta\eta + (\gamma - 1)(a - \beta)]/(b - \eta + \beta)$, $0 < \gamma < (b - a + \delta)/(\eta - a + \beta)$, $\beta - \alpha(b - \eta + \delta) > 0$.

The existence and multiplicity of positive solutions for three point boundary value problems for second-order differential equations have received a great deal of attention [**4–6, 8, 10**]. There are fewer results in the literature on three-point boundary value problems for higher-order differential equations [**2, 3, 9**].

We cite some appropriate references here [**2, 3**]. Eloe and Ahmad [**2**] considered higher-order TPBVP

$$\begin{cases} u^{(n)} + a(t) f(u) = 0, & t \in (0, 1), \\ u(0) = 0, \ u'(0) = 0, \dots, u^{(n-2)}(0) = 0, & \\ \alpha u(\eta) = u(1). & \end{cases} \tag{1.2}$$

The authors established the existence of at least one positive solution of the TPBVP (1.2) by using Krasnosel'skii fixed point theorem.

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Graef, Henderson, Wong and Yang [**3**] were interested in the following nth order TPBVP:

$$(1.3)\qquad \begin{cases} u^{(n)}(t) = F(t,u(t)), & t\in[0,1],\\ u^{(i)}(0)=0, & 0\le i\le n-3,\\ u^{(n-2)}(p)=0,\ u^{(n-1)}(1)=0. \end{cases}$$

They studied the existence of at least three positive solutions of the TPBVP (1.3). For this purpose, they used the Leggett-Williams fixed point theorem and the five-functional fixed point theorem.

In this paper, motivated by the above research efforts on multi-point boundary value problems, criteria for the existence of at least two or three positive solutions of the TPBVP (1.1) are established by using the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem. Moreover, examples are also included to illustrate our results. Thus, our results are new for differential equations.

2. Preliminary lemmas. We give the following conditions and lemmas which will be used later.

We assume that there exist continuous functions $f:[0,\infty)\to(0,\infty)$, and p, p_1, q, $q_1:[a,b]\to\mathbf{R}$ such that

(H1) The maps $u\in\mathbf{R}^n\to P(t,u)\in\mathbf{R}$ and $u\in\mathbf{R}^{n-1}\to Q(t,u)\in\mathbf{R}$ are continuous for all $t\in[a,b]$;

(H2) For $y\in[0,\infty)$,

$$q(t)\le \frac{Q(t,y,y_1,\dots,y_{n-2})}{f(y)}\le q_1(t),$$

$$p(t)\le \frac{P(t,y,y_1,\dots,y_{n-1})}{f(y)}\le p_1(t);$$

(H3) $q(t)-p_1(t)$ is nonnegative for $t\in[a,b]$.

We consider the TPBVP

$$(2.1)\qquad \begin{cases} y''(t)+h(t)=0, & t\in[a,b],\\ \alpha y(\eta)+\beta y'(a)=y(a),\ \gamma y(\eta)-\delta y'(b)=y(b). \end{cases}$$

Lemma 2.1. *Let*

$$d=(\gamma-1)(a-\beta)+(1-\alpha)(\delta+b)+\eta(\alpha-\gamma).$$

If $d \neq 0$, *then for* $h \in \mathcal{C}[a,b]$, *the* TPBVP (2.1) *has the unique solution*

$$\begin{aligned} y(t) = &- \int_a^t (t-s)h(s)\,ds \\ &+ \frac{t(\alpha-\gamma) - [\alpha(\delta+b)+\gamma(\beta-a)]}{d} \int_a^\eta (\eta-s)h(s)\,ds \\ &+ \frac{t(1-\alpha)+(\alpha\eta+\beta-a)}{d} \int_a^b (b-s+\delta)h(s)\,ds. \end{aligned}$$

Proof. From $y'' + h(t) = 0$, we have

$$y(t) = y(a) + y'(a)(t-a) - \int_a^t h(s)\,ds.$$

By using the boundary conditions

$$\alpha y(\eta) + \beta y'(a) = y(a), \qquad \gamma y(\eta) - \delta y'(b) = y(b),$$

we obtain

$$\begin{aligned} y(t) = &- \int_a^t (t-s)h(s)\,ds \\ &+ \frac{t(\alpha-\gamma) - [\alpha(\delta+b)+\gamma(\beta-a)]}{d} \int_a^\eta (\eta-s)h(s)\,ds \\ &+ \frac{t(1-\alpha)+(\alpha\eta+\beta-a)}{d} \int_a^t (b-s+\delta)h(s)\,ds. \qquad \square \end{aligned}$$

Lemma 2.2. *Let* $0 < \alpha < [b-\gamma\eta+\delta+(\gamma-1)(a-\beta)]/(b-\eta+\beta)$, $0 < \gamma < (b-a+\delta+\beta)/(\eta-a+\beta)$, $\beta-\alpha(b-\eta+\delta) > 0$. *If* $h \in \mathcal{C}[a,b]$ *with* $h \geq 0$, *then the unique solution* y *of the* TPBVP (2.1) *satisfies*:

$$y(t) \geq 0, \quad t \in [a,b].$$

Proof. From the fact that $y''(t) = -h(t) \leq 0$, we know that the graph of y is concave down on $[a,b]$. If $y(a) \geq 0$ and $y(b) \geq 0$, then the concavity of y implies that $y(t) \geq 0$ for $t \in [a,b]$. For $0 < \gamma < [\beta - \alpha(b-\eta+\delta)]/\beta$,

$$\begin{aligned}
y(a) = {} & \frac{1}{d}\int_a^b (b-s+\delta)[\alpha(\eta-a)+\beta]h(s)\,ds \\
& - \int_a^\eta (\eta-s)[\alpha(b-a+\delta)+\beta\gamma]h(s)\,ds \\
= {} & \frac{1}{d}[\beta(1-\gamma)-\alpha(b-\eta+\delta)]\int_a^\eta (\eta-s)h(s)\,ds \\
& + \frac{\alpha(\eta-a)+\beta}{d}(b-\eta+\delta)\int_a^\eta h(s)\,ds \\
& + \int_\eta^b (b-s+\delta)h(s)\,ds \\
\geq {} & 0.
\end{aligned}$$

For $[\beta - \alpha(b-\eta+\delta)]/\beta < \gamma < (b-a+\delta+\beta)/\beta$,

$$\begin{aligned}
y(a) = {} & \frac{1}{d}\int_a^\eta \Big\{ s[\alpha(b-\eta+\delta)+\beta(\gamma-1)] \\
& \qquad - \alpha a(b-\eta+\delta) + \beta(b-\eta\gamma+\delta)\Big\} h(s)\,ds \\
& + \frac{\alpha(\eta-a)+\beta}{d}\int_\eta^b (b-s+\delta)h(s)\,ds \\
\geq {} & \frac{\beta}{d}\int_a^\eta [b-a+\delta-\gamma(\eta-a)]h(s)\,ds \\
& + \frac{\alpha(\eta-a)+\beta}{d}\int_\eta^b (b-s+\delta)h(s)\,ds \\
\geq {} & 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
y(b) = {} & \frac{1}{d}\Big\{[b-a+\beta-\alpha(b-\eta)]\int_a^b (b-s+\delta)h(s)\,ds \\
& - [\alpha\delta+\gamma(b-a+\beta)]\int_a^\eta (\eta-s)h(s)\,ds\Big\} - \frac{d}{d}\int_a^b (b-s)\,ds \\
= {} & \frac{1}{d}\Big\{[\gamma(b-\eta)+\delta]\int_a^\eta (s-a+\beta)h(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \gamma(\eta - a + \beta) \int_\eta^b (b-s)h(s)\,ds \\
&\qquad + \delta \int_\eta^b [\alpha(\eta - s) + \beta - a + s]h(s)\,ds \Big\} \\
&\geq 0,
\end{aligned}
$$

since $\beta - \alpha(b - \eta + \delta) > 0$. $\square$

Lemma 2.3. *Let* $0 \leq \alpha < [b - \gamma\eta + \delta + (\gamma - 1)(a - \beta)]/(b - \eta + \delta)$, $0 < \gamma < (b - a + \delta + \beta)/(\eta - a + \beta)$, $\beta - \alpha(b - \eta + \delta) > 0$. *If* $h \in \mathcal{C}[a, b]$ *and* $h \geq 0$, *the unique solution of* (2.1) *satisfies*

$$\min_{t\in[\eta,b]} y(t) \geq k\|y\|, \qquad \|y\| := \max_{t\in[a,b]} |y(t)|,$$

where

$$k := \min\left\{\frac{b-\eta}{b-a}\gamma,\ \frac{\eta-a}{b-a}\gamma,\ \frac{\eta-a}{b-a}\right\} \in (0,1). \tag{2.2}$$

Proof. Since $y''(t) = -h(t) \leq 0$, for all $t \in [a, b]$, we have

$$\min_{t\in[\eta,b]} y(t) = \min\{y(\eta), y(b)\}.$$

Now fix $\tau \in [a, b]$ such that $y(\tau) = \|y\|$. We divide the proof into two cases.

Case 1. $a \leq \tau \leq \eta$. Then we have $y(b) \leq y(\eta)$. Let

$$F(t) := y(t) - \frac{b-t}{b-\tau} y(\tau).$$

Therefore, $F(\tau) = 0$, $F(b) = u(b) \geq 0$ and $F''(t) = y''(t) \leq 0$ so that $F(t) \geq 0$ on $t \in [\tau, b)$. From

$$\frac{y(\eta)}{b-\eta} \geq \frac{y(\tau)}{b-\tau} \geq \frac{y(\tau)}{b-a},$$

together with $\gamma y(\eta) - \delta y'(b) = y(b)$, we have

$$y(b) + \delta y'(b) \geq \frac{b-\eta}{b-a}\,\gamma\|y\|.$$

So we have

$$\min_{t\in[\eta,b]} y(t) = y(b) \geq \frac{b-\eta}{b-a}\,\gamma\|y\|,$$

since $y'(b) \leq 0$ in this case.

Case 2. $\eta < \tau \leq b$. Let

$$H(t) := y(t) - \frac{t-a}{\tau-a}\,y(\tau).$$

Then $H(\tau) = 0$, $H(a) = y(a) \geq 0$ and $H''(t) = y''(t) \leq 0$ so that $H(t) \geq 0$ on $t \in (a,\tau]$. If $y(b) \leq y(\eta)$, from

$$\frac{y(\eta)}{\eta-a} \geq \frac{y(\tau)}{\tau-a} \geq \frac{y(\tau)}{b-a},$$

together with $\gamma y(\eta) - \delta y'(b) = y(b)$, we have

$$y(b) + \delta y'(b) \geq \frac{\eta-a}{b-a}\gamma\|y\|.$$

Therefore,

$$\min_{t\in[\eta,b]} y(t) = y(b) \geq \frac{\eta-a}{b-a}\gamma\|y\|,$$

since $y'(b) \leq 0$ in this case. If $y(\eta) \leq y(b)$, then

$$\min_{t\in[a,b]} y(t) = y(\eta) \geq \frac{\eta-a}{b-a}\|y\|. \qquad \square$$

Let B be the Banach space defined by

$$B = \{y \in \mathcal{C}^n[a,b] : y^{(i)}(a) = 0,\ 0 \leq i \leq n-3\} \tag{2.3}$$

with the norm $\|y\| = \max_{t\in[a,b]} |y^{(n-2)}(t)|$, and let

(2.4)
$$\mathcal{P} = \Big\{ y \in B : y^{(n-2)}(t) \geq 0 \text{ for } t \in [a,b], \ \min_{t\in[a,b]} y^{n-2}(t) \geq k\|y\| \Big\},$$

where k is as in (2.2).

Let $g(t,s)$ be the Green's function for the TPBVP:

$$\begin{cases} y^{(n)}(t) = 0, & t \in [a,b], \\ y^{(i)}(a) = 0, & 0 \leq i \leq n-3, \\ \alpha y(\eta) + \beta y'(a) = y(a), \ \gamma y(\eta) - \delta y'(b) = y(b). & \end{cases}$$

The solutions of the TPBVP (1.1) are the fixed points of the operator $A : \mathcal{B} \to \mathcal{B}$ defined by
(2.5)
$$Ay(t) = \int_a^b g(t,s)[Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds, \quad t \in [a,b].$$

It can be verified that

$$G(t,s) = g^{(n-2)}(t,s) \tag{2.6}$$

is the Green's function of the TPBVP:

$$\begin{cases} y''(t) = 0, & t \in [a,b], \\ \alpha y(\eta) + \beta y'(a) = y(a), \ \gamma y(\eta) - \delta y'(b) = y(b). & \end{cases}$$

A direct calculation gives the following:

$$G(t,s) = \begin{cases} G_1(t,s), & a \leq s \leq \eta, \\ G_2(t,s), & \eta < s \leq b, \end{cases}$$

where

$$G_1(t,s) = \frac{1}{d}\begin{cases} [\gamma(t-\eta) + \sigma(b) - t + \delta](s+\beta-a), & s \leq t, \\ [\gamma(s-\eta)+b-s](t+\beta-a)+\alpha(\eta-\sigma(b))(t-s), & t \leq s, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{d}\begin{cases} [s(1-\alpha) + \alpha\eta + \beta - a](b-t+\delta) \\ \quad +\gamma(\eta - a + \beta)(t-s), & s \leq t, \\ [t(1-\alpha) + \alpha\eta + \beta - a](b+\delta-s), & t \leq s. \end{cases}$$

From (2.6), it follows that

(2.7)

$$\begin{aligned}(Ay)^{(n-2)}(t) = \int_a^b G(t,s)[Q(s,y,y',\dots,y^{(n-2)})\\ - P(s,y,y',\dots,y^{(n-1)})]\,ds,\\ = -\int_a^t (t-s)[Q(s,y,y',\dots,y^{(n-2)})\\ - P(s,y,y',\dots,y^{(n-1)})]\,ds\\ + \frac{t(\alpha-\gamma)-[\alpha(\delta+b)+\gamma(\beta-a)]}{d}\\ \times \int_a^\eta (\eta-s)[Q(s,y,y',\dots,y^{(n-2)})\\ - P(s,y,y',\dots,y^{(n-1)})]\,ds\\ + \frac{t(1-\alpha)+(\alpha\eta+\beta-a)}{d}\\ \times \int_a^b (b+\delta-s)[Q(s,y,y',\dots,y^{(n-2)})\\ - P(s,y,y',\dots,y^{(n-1)})]\,ds, \quad t\in[a,b].\end{aligned}$$

Solving TPBVP (1.1) in B is equivalent to finding fixed points of the operator $A^{(n-2)}$ defined by (2.7).

Lemma 2.4. *Under the hypotheses* (H1)–(H3)*, the operator A is a completely continuous operator such that $A(\mathcal{P})\subset\mathcal{P}$.*

Proof. From (H1) and the continuity of $G(t,s)$, it follows that the operator A defined by (2.5) is completely continuous in $\mathcal{B}$. By Lemma 2.2, Lemma 2.3 and the definition of $\mathcal{P}$, we get $A\mathcal{P}\subset\mathcal{P}$.

Lemma 2.5 [**3**]. (a) *Let $y\in B$. Then,*

$$|y^{(i)}(t)| \le \frac{(t-a)^{n-2-i}}{(n-2-i)!}\|y\|, \quad t\in[a,b],\ 0\le i\le n-3.$$

In particular,

$$|y(t)| \le \frac{(b-a)^{n-2}}{(n-2)!}\|y\| \quad t\in[a,b]. \tag{2.8}$$

(b) *Let $y \in P$. Then*:

$$y^{(i)}(t) \geq 0, \quad t \in [a,b],\ 0 \leq i \leq n-3,$$

and

$$y^{(i)}(t) \geq \frac{(t-a)^{n-2-i}}{(n-2-i)!} k\|y\|, \quad t \in [\eta, b],\ 0 \leq i \leq n-3.$$

In particular,

$$y(t) \geq \frac{(\eta-a)^{n-2}}{(n-2)!} k\|y\|, \quad t \in [\eta, b]. \tag{2.9}$$

3. Existence of two positive solutions. In this section, using Theorem 3.1, the Avery-Henderson fixed-point theorem, we prove the existence of at least two positive solutions of the TPBVP (1.1).

Theorem 3.1 [**1**]. *Let $\mathcal{P}$ be a cone in a real Banach space S. If η and ψ are increasing, nonnegative continuous functionals on $\mathcal{P}$, let θ be a nonnegative, continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\psi(u) \leq \theta(u) \leq \eta(u) \quad \textit{and} \quad \|u\| \leq M\psi(u)$$

for all $u \in \overline{\mathcal{P}(\psi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \quad \textit{for all } 0 \leq \lambda \leq 1 \textit{ and } u \in \partial P(\theta, q).$$

If $A : \overline{\mathcal{P}(\psi, r)} \to \mathcal{P}$ is a completely continuous operator satisfying:

(i) *$\psi(Au) > r$ for all $u \in \partial\mathcal{P}(\psi, r)$,*

(ii) *$\theta(Au) < q$ for all $u \in \partial\mathcal{P}(\theta, q)$,*

(iii) *$\mathcal{P}(\eta, p) \neq \{\ \}$ and $\eta(Au) > p$ for all $u \in \partial\mathcal{P}(\eta, p)$, then A has at least two fixed points u_1 and u_2 such that*

$$p < \eta(u_1) \quad \textit{with } \theta(u_1) < q \textit{ and } q < \theta(u_2) \textit{ with } \psi(u_2) < r.$$

Let the Banach space $\mathcal{B} = \mathcal{C}[a,b]$ with the maximum norm. Again, define the cone $\mathcal{P} \subset \mathcal{B}$ by (2.4) and the operator $A : \mathcal{P} \to \mathcal{P}$ by (2.5).

Let the nonnegative, increasing, continuous functionals ψ, θ and η be defined on the cone $\mathcal{P}$ by

$$\begin{aligned} \psi(y) &:= \min_{t\in[\eta,b]} y^{(n-2)}(t), \\ \theta(y) &:= \max_{t\in[\eta,b]} y^{(n-2)}(t), \\ \eta(y) &:= \max_{t\in[a,b]} y^{(n-2)}(t), \end{aligned} \tag{3.1}$$

and let $\mathcal{P}(\psi, r) := \{y \in \mathcal{P} : \psi(y) < r\}$.

Finally, define constants

$$m := \left(\frac{\alpha(b+\eta) + \gamma a + b + \beta}{d} \int_a^b (b+\delta-s)(q_1(s) - p(s))\, ds \right)^{-1}, \tag{3.2}$$

$$\begin{aligned} M := \min\Bigg\{ &\frac{\eta - a + \beta}{d} \int_\eta^b (b+\delta-s)(q(s) - p_1(s))\, ds, \\ &\frac{\gamma(\eta - a + \beta)}{d} \int_\eta^b (b-s)(q(s) - p_1(s))\, ds \Bigg\}. \end{aligned} \tag{3.3}$$

Theorem 3.2. *Assume* (H1)–(H3) *hold, and* $0 \le \alpha < [b - \gamma\eta + \delta + (\gamma-1)(a-\beta)]/(b-\eta+\beta)$, $0 < \gamma < (b-a+\delta+\beta)/(\eta-a+\beta$, $\beta - \alpha(b-\eta+\delta) > 0$. *Suppose there exist positive numbers* $0 < p < q < r$ *such that the function* f *satisfies the following conditions*:

(i) $f(y) > p/M$ *for* $t \in [\eta, b]$ *and* $y \in [(kp(\eta-a)^{n-2})/(n-2)!, (p(b-a)^{n-2})/(n-2)!]$,

(ii) $f(y) < qm$ *for* $t \in [a,b]$ *and* $y \in [0, (q(b-a)^{n-2})/k(n-2)!]$,

(iii) $f(y) > r/M$ *for* $t \in [\eta, b]$ *and* $y \in [r, (r(b-a)^{n-2})/k(n-2)!]$,

where k, m *and* M *are as defined in* (2.2), (3.2) *and* (3.3), *respectively. Then the* TPBVP (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$p < \max_{t\in[a,b]} y_1(t) \quad \text{with} \quad \max_{t\in[\eta,b]} y_1(t) < q,$$

$$q < \max_{t\in[\eta,b]} y_2(t) \quad \text{with} \quad \min_{t\in[\eta,b]} y_2(t) < r.$$

Proof. From (3.1), for each $y \in \mathcal{P}$, we have

$$\psi(y) \leq \theta(y) \leq \eta(y), \tag{3.4}$$

$$\|y\| \leq \frac{1}{k} \min_{t \in [\eta, b]} y^{(n-2)}(t) \tag{3.5}$$
$$= \frac{1}{k}\psi(y) \leq \frac{1}{k}\theta(y) \leq \frac{1}{k}\eta(y).$$

For any $y \in \mathcal{P}$, (3.4) and (3.5) imply

$$\psi(y) \leq \theta(y) \leq \eta(y), \qquad \|y\| \leq \frac{1}{k}\psi(y).$$

For all $y \in \mathcal{P}$, $\lambda \in [0, 1]$, we have

$$\theta(\lambda y) = \max_{t \in [\eta, b]} (\lambda y)^{(n-2)}(t) = \lambda \max_{t \in [\eta, b]} y^{(n-2)}(t) = \lambda\theta(y).$$

It is clear that $\theta(0) = 0$.

We now show that the remaining conditions of Theorem 3.1 are satisfied.

Firstly, we shall verify that condition (iii) of Theorem 3.1 is satisfied. Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\eta, p) \neq \{\ \}$. Since $y \in \partial\mathcal{P}(\eta, p)$, $kp \leq y^{(n-2)}(t) \leq \|y\| = p$ for $t \in [\eta, b]$. In view of (2.8) and (2.9), it follows that $y(t) \in [(kp(\eta - a)^{n-2})/(n-2)!, (p(b-a)^{n-2})/(n-2)!]$, for $t \in [\eta, b]$. Therefore,

$$\begin{aligned}
\eta(Ay) &= \max_{t \in [a, b]} (Ay)^{(n-2)}(t) \\
&\geq (Ay)^{(n-2)}(b) \\
&= \frac{1}{d}\Big\{[b - a + \beta - \alpha(b - \eta)] \int_a^b (b - s + \delta) h(s)\, ds \\
&\qquad - [\alpha\delta + \gamma(b - a + \beta)] \int_a^\eta (\eta - s) h(s)\, ds\Big\} - \frac{d}{d}\int_a^b (b - s)\, ds \\
&= \frac{1}{d}\Big\{[\gamma(b - \eta) + \delta] \int_a^\eta (s - a + \beta)[Q(s, y, y', \dots, y^{(n-2)}) \\
&\qquad\qquad - P(s, y, y', \dots, y^{(n-1)})]\, ds \\
&\qquad + \gamma(\eta - a + \beta) \int_\eta^b (b - s)[Q(s, y, y', \dots, y^{(n-2)})
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - P(s,y,y',\dots,y^{(n-1)})]\,ds\\
&\quad + \delta\int_\eta^b [\alpha(\eta-s)+\beta-a+s]\\
&\qquad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds\Big\}\\
&\ge \frac{\gamma(\eta-a+\beta)}{d}\int_\eta^b (b-s)(q(s)-p_1(s))f(y(s))\,ds\\
&> p\frac{\gamma(\eta-a+\beta)}{Md}\int_\eta^b (b-s)(q(s)-p_1(s))\,ds\\
&\ge p,
\end{aligned}
$$

using (3.3) and hypothesis (i).

Now we shall show that condition (ii) of Theorem 3.1 is satisfied. Since $y \in \partial\mathcal{P}(\theta,q)$, from (3.5) we have that $0 \le y^{(n-2)}(t) \le \|y\| \le q/k$ for $t \in [a,b]$. By (2.4) and (2.8), we have $y(t) \in [0,(q(b-a)^{n-2})/k(n-2)!]$, for $t \in [a,b]$. Thus,

$$
\begin{aligned}
\theta(Ay) &= \max_{t\in[\eta,b]} (Ay)^{(n-2)}(t)\\
&= \max_{t\in[\eta,b]}\Big\{-\int_a^t (t-s)[Q(s,y,y',\dots,y^{(n-2)})\\
&\qquad\qquad - P(s,y,y',\dots,y^{(n-1)})]\,ds\\
&\quad + \frac{t(\alpha-\gamma)-[\alpha(\delta+b)+\gamma(\beta-a)]}{d}\\
&\quad \times \int_a^\eta (\eta-s)[Q(s,y,y',\dots,y^{(n-2)})\\
&\qquad\qquad - P(s,y,y',\dots,y^{(n-1)})]\,ds\\
&\quad + \frac{t(1-\alpha)+(\alpha\eta+\beta-a)}{d}\int_a^b (b+\delta-s)\\
&\quad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds\Big\}\\
&\le \max_{t\in[\eta,b]}\Big\{\frac{t\alpha+\gamma a}{d}\int_a^\eta (\eta-s)(q_1(s)-p(s))f(y(s))\,ds\\
&\qquad + \frac{t+\alpha\eta+\beta}{d}\int_a^b (b+\delta-s)(q_1(s)-p(s))f(y(s))\,ds\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{b\alpha+\gamma a}{d}\int_a^\eta (\eta-s)(q_1(s)-p(s))f(y(s))\,ds \\
&\quad + \frac{b+\alpha\eta+\beta}{d}\int_a^b (b+\delta-s)(q_1(s)-p(s))f(y(s))\,ds \\
&\le \frac{\alpha(b+\eta)+\gamma a+b+\beta}{d}\int_a^b (b+\delta-s)(q_1(s)-p(s))f(y(s))\,ds \\
&< qm\frac{\alpha(b+\eta)+b+\gamma a+\beta}{d}\int_a^b (b+\delta-s)(q_1(s)-p(s))\,ds \\
&= q
\end{aligned}
$$

by hypothesis (ii) and (3.2).

Finally, using hypothesis (iii) and (3.3), we shall show that condition (i) of Theorem 3.1 is satisfied. Since $y \in \partial\mathcal{P}(\psi, r)$, from (3.5) we have that $\min_{t\in[\eta,b]} y^{(n-2)}(t) = r$ and $r \le \|y\| \le r/k$. By the concavity of $(Ay)^{(n-2)}$,

$$\psi(Ay) = \min_{t\in[\eta,b]} (Ay)^{(n-2)}(t) = \min\{(Ay)^{(n-2)}(\eta), (Ay)^{(n-2)}(b)\}.$$

Now, we suppose $\psi(Ay) = (Ay)^{(n-2)}(\eta)$. By using (2.8) and (2.9), we get $y(t) \in [(r(\eta-a)^{n-2})/(n-2)!, (r(b-a)^{n-2})/k(n-2)!]$, for $t \in [\eta, b]$. Thus,

$$
\begin{aligned}
\psi(Ay) &= -\int_a^\eta (\eta-s)[Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&\quad + \frac{\eta(\alpha-\gamma)-[\alpha(\delta+b)+\gamma(\beta-a)]}{d} \\
&\quad \times \int_a^\eta (\eta-s)[Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&\quad + \frac{\eta(1-\alpha)+(\alpha\eta+\beta-a)}{d}\int_a^b (b+\delta-s) \\
&\qquad\qquad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&= -\frac{b-a+\beta+\delta}{d}\int_a^\eta (\eta-s) \\
&\qquad\qquad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&\quad + \frac{\eta-a+\beta}{d}\int_a^b (b+\delta-s)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&= \frac{1}{d}\int_a^\eta [(s-a)(b-\eta+\delta)+\beta(b+\delta)] \\
&\qquad\qquad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&\quad + \frac{\eta-a+\beta}{d}\int_\eta^b (b+\delta-s) \\
&\qquad\qquad \times [Q(s,y,y',\dots,y^{(n-2)}) - P(s,y,y',\dots,y^{(n-1)})]\,ds \\
&\geq \frac{\eta-a+\beta}{d}\int_\eta^b (b+\delta-s)(q(s)-p_1(s))f(y(s))\,ds \\
&> r\frac{\eta-a+\beta}{Md}\int_\eta^b (b+\delta-s)(q(s)-p_1(s))\,ds \\
&\geq r.
\end{aligned}
$$

Now, we suppose that $\psi(Ay) = (Ay)^{(n-2)}(b)$. Then

$$
\begin{aligned}
\psi(Ay) &\geq \frac{\gamma(\eta-a+\beta)}{d}\int_\eta^b (b-s)(q(s)-p_1(s))f(y(s))\,ds \\
&> r\frac{\gamma(\eta-a+\beta)}{Md}\int_\eta^b (b-s)(q(s)-p_1(s))\,ds \\
&\geq r.
\end{aligned}
$$

This completes the proof. □

4. Existence of three positive solutions. In this section, using Theorem 4.1, the Leggett-Williams fixed-point theorem, we prove the existence of at least three positive solutions of the TPBVP (1.1).

Theorem 4.1 [**7**]. *Let $\mathcal{P}$ be a cone in the real Banach space E. Set*

$$\mathcal{P}_r := \{x \in \mathcal{P} : \|x\| < r\}$$

and

$$\mathcal{P}(\psi, p, q) := \{x \in \mathcal{P} : p \leq \psi(x),\ \|x\| \leq q\}.$$

Let $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$ be a completely continuous operator and ψ a nonnegative continuous concave functional on $\mathcal{P}$ with $\psi(x) \le \|x\|$ for all $x \in \overline{\mathcal{P}_r}$. Suppose that there exists a $0 < p < q < s \le r$ such that the following conditions hold:

(i) $\{x \in \mathcal{P}(\psi, q, s) : \psi(x) > q\} \neq \{\ \}$ *and* $\psi(Ax) > q$ *for all* $x \in \mathcal{P}(\psi, q, s)$;

(ii) $\|Ax\| < p$ *for* $\|x\| \le p$;

(iii) $\psi(Ax) > q$ *for* $x \in \mathcal{P}(\psi, q, r)$ *with* $\|Ax\| > s$.

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{\mathcal{P}_r}$ satisfying:

$$\|x_1\| < p, \ \psi(x_2) > q, \qquad p < \|x_3\| \text{ with } \psi(x_3) < q.$$

Let $\mathcal{P}$ be a cone in the Banach space $\mathcal{B} = \mathcal{C}[a, b]$ by

$$\mathcal{P} = \{y \in \mathcal{B} : y^{(n-2)}(t) \text{ concave down and } y^{(n-2)}(t) \ge 0 \text{ on } [a, b]\}.$$

Again define the continuous concave functional $\psi : \mathcal{P} \to [0, \infty)$ to be $\psi(y) := \min_{t\in[\eta,b]} y^{(n-2)}(t)$, m, M as in (3.2) and (3.3), respectively, and the operator $A : \mathcal{P} \to \mathcal{B}$ by (2.5).

Theorem 4.2. *Assume that* (H1)–(H3) *hold, and* $0 \le \alpha < [b - \gamma\eta + \delta + (\gamma - 1)(a - \beta)]/(b - \eta + \beta)$, $0 < \gamma < (b - a + \delta + \beta)/(\eta - a + \beta)$, $\beta - \alpha(b - \eta + \delta) > 0$. *Suppose that there exist constants* $0 < p < q < q/k < r$ *such that*

(D1) $f(y) < mp$ *for* $t \in [a, b]$ *and* $y \in [0, (p(b - a)^{n-2})/(n - 2)!]$,

(D2) $f(y) > q/M$ *for* $t \in [\eta, b]$ *and* $y \in [(q(\eta - a)^{n-2})/(n - 2)!, (q(b - a)^{n-2})/k(n - 2)!]$,

(D3) $f(y) \le mr$ *for* $t \in [a, b]$ *and* $y \in [0, r(b - a)^{n-2}/(n - 2)!]$,

where k, m and M are as defined in (2.2), (3.2) *and* (3.3), *respectively. Then the boundary value problem* (1.1) *has at least three positive solutions, y_1, y_2 and y_3, satisfying*

$$\|y_1\| < p, \ \min_{t\in[\eta,b]} (y_2)(t) > q, \qquad p < \|y_3\| \text{ with } \min_{t\in[\eta,b]} (y_3)(t) < q.$$

Proof. Since $(Ay)^{(n)}(t) = -[Q(t, y, \dots, y^{(n-2)}) - P(t, y, \dots, y^{(n-1)})]$ for $t \in [a, b]$, together with (H2), (H3), Lemma 2.2 and Lemma 2.3, we see that $(Ay)^{(n)}(t) \le 0$ and $(Ay)^{(n-2)}(t) \ge 0$, for $t \in [a, b]$. Thus, $A : \mathcal{P} \to \mathcal{P}$. Moreover, A is completely continuous. For all $y \in \mathcal{P}$, we have $\psi(y) \le \|y\|$. If $y \in \overline{\mathcal{P}_r}$, then $\|y\| \le r$. By using (2.8), we get $y(t) \in [0, (r(b-a)^{n-2})/(n-2)!]$ for $t \in [a, b]$. Assumption (D3) implies $f(y(t)) \le mr$ for $t \in [a, b]$. We arrive at

$$\begin{aligned}(Ay)^{(n-2)}(t) &\le \frac{\alpha(b+\eta)+\gamma a+b+\beta}{d}\int_a^b (b+\delta-s)(q_1(s)-p(s))f(y(s))ds \\ &\le mr\frac{\alpha(b+\eta)+\gamma a+b+\beta}{d}\int_a^b (b+\delta-s)(q_1(s)-p(s))ds \\ &= r.\end{aligned}$$

Thus, $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$.

The remaining conditions of Theorem 4.1 will now be shown to be satisfied.

By (D1) and an argument similar to the above, we can get that $A : \overline{\mathcal{P}_p} \to \mathcal{P}_p$. Hence, condition (ii) of Theorem 4.1 is satisfied.

We shall show (i) of Theorem 4.1. Choose $y_{\mathcal{P}}(t) \equiv q/k$ for $t \in [a, b]$, where k is given in (2.2). Then $y_{\mathcal{P}} \in \mathcal{P}(\psi, q, q/k)$ and $\psi(y_{\mathcal{P}}) = \psi(q/k) > q$, so that $\{y \in \mathcal{P}(\psi, p, q/k) : \psi(y) > q\} \ne \{\ \}$. For $y \in \mathcal{P}(\psi, q, q/k)$, we have $q \le y^{(n-2)}(t) \le q/k$, $t \in [a, b]$. By using (2.8) and (2.9), we get $y(t) \in [(q(\eta-a)^{n-2})/(n-2)!, (q(b-a)^{n-2})/k(n-2)!]$ for $t \in [\eta, b]$. Combining with (D2), we get

$$f(y) \ge q/M.$$

By the concavity of $(Ay)^{(n-2)}$, there are two cases: either $\psi(Ay) = (Ay)^{(n-2)}(\eta)$, or $\psi(Ay) = (Ay)^{(n-2)}(b)$.

First, suppose $\psi(Ay) = (Ay)^{(n-2)}(\eta)$. Then,

$$\begin{aligned}\psi(Ay) = (Ay)^{(n-2)}(\eta) &\ge \frac{\eta-a+\beta}{d}\int_\eta^b (b+\delta-s)(q(s)-p_1(s))f(y(s))\,ds \\ &> q\frac{\eta-a+\beta}{Md}\int_\eta^b (b+\delta-s)(q(s)-p_1(s))\,ds \\ &\ge q.\end{aligned}$$

Now we suppose that $\psi(Ay) = (Ay)^{(n-2)}(b)$. Then,

$$\begin{aligned}\psi(Ay) &= (Ay)^{(n-2)}(b)\\ &\geq \frac{\gamma(\eta-a+\beta)}{d}\int_\eta^b (b-s)(q(s)-p_1(s))f(y(s))\,ds\\ &> q\frac{\gamma(\eta-a+\beta)}{Md}\int_\eta^b (b-s)(q(s)-p_1(s))\,ds\\ &\geq q.\end{aligned}$$

Finally, we shall show that condition (iii) of Theorem 4.1 holds. We suppose that $y \in \mathcal{P}(\psi, q, r)$ with $\|Ay\| > q/k$. Then Lemma 2.3 and the definition of ψ yield

$$\psi(Ay) = \min_{t\in[\eta,b]} (Ay)^{(n-2)}(t) \geq k\|Ay\| > \frac{kq}{k} = q.$$

Thus, all conditions of Theorem 4.1 are satisfied. It implies that the TPBVP (1.1) has at least three positive solutions y_1, y_2, y_3 with

$$\|y_1\| < p,\ \psi(y_2) > q, \qquad p < \|y_3\| \text{ with } \psi(y_3) < q. \qquad \square$$

5. Examples.

Example 5.1. Let us introduce an example to illustrate the applicability of Theorem 3.2. Consider the TPBVP:

$$(5.1)\qquad \begin{cases} y'''(t) + (g(t,y,y')+2)(220e^y)/(e^y+2640e^{18}) \\ \quad = g(t,y,y')(220e^y)/(e^y+2640e^{18}), & t\in[0,3],\\ y(0)=0,\\ 1/5y'(1)+y''(0)=y'(0),\\ 1/2y'(1)-2y''(3)=y'(3).\end{cases}$$

Then $a = 0$, $\eta = 1$, $b = 3$, $\alpha = 1/5$, $\beta = 1$, $\delta = 2$, $\gamma = 1/2$, $P(t,y,y',y'') = g(t,y,y')(220e^y)/(e^y+2640e^{18})$ and $Q(t,y,y') = (g(t,y,y')+2)(220e^y)/(e^y+2640e^{18})$, where the map $u \in \mathbf{R}^2 \to g(t,u) \in \mathbf{R}$ is continuous for all $t \in [0,3]$.

Taking $f(y) = (220e^y)/(e^y + 2640e^{18})$, we get

$$\frac{Q(t,y,y')}{f(y)} = g(t,y,y') + 2 \quad \text{and} \quad \frac{P(t,y,y',y'')}{f(y)} = g(t,y,y').$$

Hence, we may choose

$$q(t) = g(t,y,y') + 1, \qquad q_1(t) = g(t,y,y') + 2,$$

and

$$p(t) = p_1(t) = g(t,y,y').$$

Clearly, f is continuous and increasing on $[0,\infty)$. We can also show

$$0 < \alpha(b - \eta + \delta) = \frac{3}{5} \le b - \gamma\eta + \delta + (\gamma - 1)(a - \beta) = \frac{3}{2},$$
$$0 < \gamma(\eta - a + \beta) = 1 < b - a + \delta + \beta = 5,$$
$$\alpha(b - \eta + \delta) = \frac{4}{5} < \beta = 1.$$

By (2.2), (3.2) and (3.3), we get $k = 1/6$, $m = 1/12$, $M = 10/21$. If we take $p = 1/10^{10}$, $q = 1$ and $r = 100$, then

$$0 < p < q < q/k < r.$$

It is clear that (i), (ii) and (iii) are satisfied. Thus, by Theorem 3.2, the TPBVP (5.1) has at least two positive solutions y_1 and y_2 with

$$\frac{1}{10^{10}} < \eta(y_1) \quad \text{with } \theta(y_1) < 1 \text{ and } 1 < \theta(y_2) \text{ with } \psi(y_2) < 100.$$

Example 5.2. Let us introduce an example to illustrate Theorem 4.2. Consider the TPBVP:

$$\text{(5.2)} \qquad \begin{cases} y''(t) + (g(t,y) + 1)2000y^2/(y^2 + 100) \\ \quad = g(t,y)2000y^2/(y^2 + 100), & t \in [1,2], \\ 1/10y(5/3) + 1/4y'(1) = y(1), \\ 1/3y(5/3) - 3/4y'(2) = y(2). \end{cases}$$

Then $a = 1$, $\eta = 5/3$, $b = 2$, $\alpha = 1/10$, $\beta = 1/4$, $\delta = 3/4$, $\gamma = 1/3$, $P(t, y, y') = g(t, y)2000y^2/(y^2 + 100)$ and $Q(t, y) = (g(t, y) + 1)$ $2000y^2/(y^2 + 100)$, where $g : [1, 2] \times \mathbf{R} \to \mathbf{R}$ is continuous with respect to y for all $t \in [1, 2]$.

Taking $f(y) = 2000y^2/(y^2 + 100)$, we get

$$\frac{Q(t,y)}{f(y)} = g(t,y) + 1 \quad \text{and} \quad \frac{P(t,y,y')}{f(y)} = g(t,y).$$

Hence, we may choose

$$q(t) = g(t,y) + 1, \qquad q_1(t) = g(t,y) + 2,$$

and

$$p(t) = p_1(t) = g(t,y).$$

Clearly, f is continuous and increasing on $[0, \infty)$. We can also show

$$\begin{aligned} 0 < \alpha(b - \eta + \delta) &= \frac{13}{120} \le b - \gamma\eta + \delta + (\gamma - 1)(a - \beta) = \frac{61}{36}, \\ 0 < \gamma(\eta - a + \beta) &= \frac{11}{36} < b - a + \delta + \beta = 2, \\ \alpha(b - \eta + \delta) &= \frac{13}{120} < \beta = \frac{1}{4}. \end{aligned}$$

By (2.2), (3.2) and (3.3), we get $k = 1/9$, $m = 1142/2655$, $M = 55/5139$. If we take $p = 1/50$, $q = 10$ and $r = 5022$, then

$$0 < p < q < q/k < r.$$

It is clear that (D1), (D2) and (D3) are satisfied. Thus, by Theorem 4.2, the TPBVP (5.2) has at least three positive solutions y_1, y_2 and y_3 with

$$\|y_1\| < \frac{1}{50}, \; \psi(y_2) > 10, \qquad \frac{1}{50} < \|y_3\| \text{ with } \psi(y_3) < 10.$$

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