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m-ISOMETRIC WEIGHTED SHIFTS AND REFLEXIVITY OF SOME OPERATORS

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ABSTRACT. For a positive integer m, a bounded linear operator T on a Hilbert space $\mathcal{H}$ is called an m-isometry, if $\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0$. We characterize all m-isometric unilateral weighted shift operators that are not $m-1$-isometries in terms of their weight sequences. Then we prove the reflexivity of some classes of operators: (1) All non-negative integer powers of m-isometric unilateral weighted shifts. (2) The contractions whose spectrum are all the closed unit disc. (3) All non-negative integer powers of hyponormal m-isometries.

1. Introduction. Let T be a bounded linear operator on a complex separable Hilbert space $\mathcal{H}$, $\operatorname{Lat} T$ the lattice of all closed subspaces of $\mathcal{H}$ which are left invariant by T, and $\operatorname{Alglat} T$ the set of all operators that leave invariant every element of $\operatorname{Lat} T$. We denote by $W(T)$ the weakly closed algebra generated by T and the identity operator I. Since every polynomial in T belongs to $\operatorname{Alglat} T$, we always have $W(T)\subseteq \operatorname{Alglat}(T)$. If $W(T)$ is all of $\operatorname{Alglat} T$, then T is called a *reflexive operator*. The identity operator is a simple example of a reflexive operator.

Let us denote by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$, respectively, the spectrum, the point spectrum and the approximate point spectrum of T. Also, $\mathbf{D}$ and $\partial\mathbf{D}$ represent, respectively, the open unit disc and its boundary.

For a positive integer m, an operator T is called an m-isometry, if

$$(yx-1)^m(T):=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0;$$

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or equivalently,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k x\|^2 = 0, \tag{1}$$

for all $x \in \mathcal{H}$. The concept of m-isometric operators is, in some sense, a generalization of the concept of isometric operators. Agler [**1**] was the first who defined these operators. A 1-isometry is simply an isometry. m-isometries, especially isometries, have closed ranges. Indeed, if $\{x_n\}_n$ is any sequence in $\mathcal{H}$ so that $\{Tx_n\}_n$ converges to a vector y, then it is a Cauchy sequence, and so is $\{T^k x_n\}_n$ for each $k \geq 1$. This fact coupled with (1) implies that $\{x_n\}_n$ is a Cauchy sequence and so converges to some vector x. Consequently, $y = Tx$ and $\operatorname{ran} T$ is closed. They are also injective; furthermore, $\sigma(T) = \partial\mathbf{D}$ or $\overline{\mathbf{D}}$ (see [**2**, Lemma 1.21]). The covariance operator of an m-isometry T, denoted by Δ_T, is defined by

$$\Delta_T = \frac{1}{(m-1)!}(yx-1)^{m-1}(T).$$

It is known that Δ_T is a positive operator. Section 2 explores identifying m-isometric unilateral weighted shifts.

The reflexivity problem has been intensively studied by many authors. Deddens [**11**] showed that every isometry is reflexive. Later, Li and McCarthy [**15**] proved the reflexivity of any collection of commuting isometries. In Section 3, the reflexivity of some shift operators that are m-isometries will be discussed.

In [**5**, Theorem 5] it was shown that contraction operators whose essential spectrum in $\mathbf{D}$, $\sigma_e(T)$, is dominating for $\partial\mathbf{D}$ (i.e., almost every point of $\partial\mathbf{D}$ is a nontangential limit point of $\sigma_e(T) \cap \mathbf{D}$) are reflexive. Afterwards, in [**6**], the reflexivity of some contractions with rich spectrum was discussed. In fact, it was shown that, if T is a contraction on a Hilbert space $\mathcal{H}$ so that $I - T^*T$ is a trace class operator and $\sigma(T) = \overline{\mathbf{D}}$, then T is reflexive. At the end of the above-mentioned article, the authors posed a question. They inquire whether T is a contraction so that $\mathbf{D} \subseteq \sigma_p(T)$, is T necessarily reflexive? In this direction some developments have occurred in [**8**] by giving sufficient conditions for an arbitrary contraction to be reflexive. Especially, it is shown that if T is a contraction so that $\sigma(T)$ contains $\partial\mathbf{D}$, then either T is reflexive or has a nontrivial hyperinvariant subspace. In Section 4, we prove the reflexivity of a contraction whose spectrum is the closed unit disc $\overline{\mathbf{D}}$. The reflexivity of non-negative integer powers of hyponormal m-isometries is another discussion of the last section.

2. m-isometric unilateral weighted shifts. An operator T is called a unilateral weighted shift, if there exists an orthonormal basis $\{e_n : n \geq 0\}$ and a sequence $\{w_n\}_{n=0}^{\infty}$ of bounded complex numbers such that $Te_n = w_n e_{n+1}$ for all $n \geq 0$. The iterates of T are given by $T^0 = I$, and for $k > 0$,

$$T^k e_n = \left(\prod_{i=0}^{k-1} w_{n+i} \right) e_{n+k} \quad (n \geq 0).$$

It is known that T is an isometry if and only if $w_n \in \partial\mathbf{D}$, for all $n \geq 0$. Patel [**16**, Theorem 2.2] characterized all 2-isometric unilateral weighted shifts that are not isometries in terms of their weight sequences. In the following, we generalize the aforementioned theorem. This yields to the fact that $m-1$-isometries form a strict subclass of m-isometries. Before presenting the result, first we recall that a unilateral weighted shift T is unitarily equivalent to a weighted shift operator with a non-negative weight sequence. So we can assume that $w_n \geq 0$ for every $n \geq 0$. Furthermore, if T is injective, it can be assumed that $w_n > 0$ for every $n \geq 0$ ([**17**, page 52]).

Lemma 1. *Let T be an m-isometric unilateral weighted shift with weight sequence $\{w_n\}_{n=0}^{\infty}$, and put*

$$f(n) = (-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} {w_{n+i}}^2. \tag{2}$$

If $f(0) = 0$, then $f(n) = 0$ for all non-negative integers n.

Proof. We argue by using mathematical induction. The hypothesis shows that the result is valid for $n = 0$. Suppose that it is true for $n = j$. Since T is an m-isometry, equality (1) for $x = e_n$ states that, for every non-negative integer n,

$$(-1)^{m-1} + \sum_{k=1}^{m} (-1)^{m-k-1} \binom{m}{k} \prod_{i=0}^{k-1} {w_{n+i}}^2 = 0.$$

This, especially for $n = j$, implies that

$$(-1)^{m-1} + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} {w_{j+i}}^2 \right] + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k-1} \prod_{i=0}^{k-1} {w_{j+i}}^2 \right] - \prod_{i=0}^{m-1} {w_{j+i}}^2 = 0.$$

Multiplying both sides in ${w_j}^{-2}$, we get

$$(-1)^{m-2} + \sum_{k=2}^{m} (-1)^{m-k-1} \binom{m-1}{k-1} \prod_{i=1}^{k-1} {w_{j+i}}^2 = 0,$$

or

$$(-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} {w_{j+i+1}}^2 = 0.$$

Thus, the result is true for $n = j + 1$. Hence, $f(n) = 0$ for every non-negative integer n. □

Theorem 1. *Suppose that T is a unilateral weighted shift operator with weight sequence $\{w_n\}_{n=0}^{\infty}$. Then T is an m-isometry which is not an $m-1$-isometry if and only if the following hold for every non-negative integer n:*

(i)

$$(-1)^{m} + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \prod_{i=0}^{k-1} {w_{n+i}}^2 = 0;$$

(ii)

$$(-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} {w_{n+i}}^2 \neq 0.$$

Proof. First, note that

$$T^{*k}T^k e_n = \prod_{i=0}^{k-1} {w_{n+i}}^2 e_n, \quad n = 0, 1, 2, \dots . \tag{3}$$

Suppose that T is an m-isometry but not an $m-1$-isometry. Then using (3) the definition of an m-isometry shows that (i) holds. To prove (ii), assume, on the contrary, that n_0 is the smallest non-negative integer such that

$$(-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} {w_{n_0+i}}^2 = 0. \tag{4}$$

If $n_0 > 0$, then

$$(-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} {w_{n_0-1+i}}^2 \neq 0. \tag{5}$$

Furthermore, by applying (i) for $n_0 - 1$, we get

$$(-1)^{m} + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \prod_{i=0}^{k-1} {w_{n_0-1+i}}^2 = 0, \tag{6}$$

and so

$$(-1)^m + \binom{m}{1} (-1)^{m-1} {w_{n_0-1}}^2 + {w_{n_0-1}}^2 \sum_{k=2}^{m} (-1)^{m-k} \binom{m}{k} \prod_{i=0}^{k-2} {w_{n_0+i}}^2 = 0,$$

or

$$(-1)^m + \binom{m}{1} (-1)^{m-1} {w_{n_0-1}}^2 + {w_{n_0-1}}^2 \sum_{k=2}^{m} (-1)^{m-k} \binom{m-1}{k-1} \prod_{i=0}^{k-2} {w_{n_0+i}}^2 + {w_{n_0-1}}^2 \sum_{k=2}^{m} (-1)^{m-k} \binom{m-1}{k} \prod_{i=0}^{k-2} {w_{n_0+i}}^2 = 0,$$

in which $\binom{n}{r} = 0$ whenever $n < r$, by convention. Substitute (4) in the above equality to get

$$(-1)^m + \binom{m}{1}(-1)^{m-1}{w_{n_0-1}}^2 + {w_{n_0-1}}^2(-1)^m + {w_{n_0-1}}^2 \sum_{k=2}^{m}(-1)^{m-k}\binom{m-1}{k}\prod_{i=0}^{k-2}{w_{n_0+i}}^2 = 0,$$

or

$$(-1)^m + \binom{m-1}{1}(-1)^{m-1}{w_{n_0-1}}^2 + {w_{n_0-1}}^2 \sum_{k=2}^{m}(-1)^{m-k}\binom{m-1}{k}\prod_{i=0}^{k-2}{w_{n_0+i}}^2 = 0.$$

But, considering (5), the left side of the above identity is non-zero. Thus, we get a contradiction, and so $n_0 = 0$. Now, the preceding lemma implies that

$$(-1)^{m-1} + \sum_{k=1}^{m-1}(-1)^{m-k-1}\binom{m-1}{k}\prod_{i=0}^{k-1}{w_{n+i}}^2 = 0,$$

for all non-negative integers n, and so T is an $m-1$-isometry, which contradicts our hypothesis.

For the converse, suppose that (i) and (ii) are true. Since every vector x can be written as $x = \sum_{i=0}^{\infty}\alpha_i e_i$, an easy computation in light of condition (i) implies that (1) holds for any vector $x \in \mathcal{H}$. That is, T is an m-isometry. Furthermore, (ii) states that

$$\sum_{k=0}^{m-1}(-1)^{m-k-1}\binom{m-1}{k}\|T^k e_n\|^2 \neq 0,$$

for all n. Thus, T is not an $m-1$-isometry. □

Example. Let m be a positive integer, and define a unilateral weighted shift T by

$$Te_n = \sqrt{\frac{n+m}{n+1}}e_{n+1}, \quad n \geq 0.$$

Then T satisfies (i) and (ii) in the previous theorem. In fact, if for a real number α, we define $(\alpha)_0 = 1$, $(\alpha)_1 = \alpha$, and for $n \geq 2$,

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1),$$

then

$$\begin{aligned}
(n+1)_m&\left[(-1)^m + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\prod_{i=0}^{k-1} w_{n+i}^2\right] \\
&= (n+1)_m\left[(-1)^m + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\prod_{i=0}^{k-1}\frac{n+i+m}{n+i+1}\right] \\
&= (n+1)_m\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\frac{(n+m)_k}{(n+1)_k} \\
&= \sum_{k=0}^{m}\binom{m}{k}(n+m)_k(-n-m)_{m-k}.
\end{aligned}$$

Using Vandermonde's theorem [**4**, page 340], the last equality in the above computation is $(0)_m = 0$. Furthermore, a similar argument shows that

$$(n+1)_{m-1}\left[(-1)^{m-1} + \sum_{k=1}^{m-1}(-1)^{m-k-1}\binom{m-1}{k}\prod_{i=0}^{k-1} w_{n+i}^2\right] = (1)_{m-1},$$

which is non-zero. Consequently, T is an m-isometry that is not an $m-1$-isometry.

Athavale [**3**] showed that, in general, not all m-isometries are isometries. The next result gives conditions under which an m-isometric unilateral weighted shift becomes an isometry.

Theorem 2. *Let $T \in B(\mathcal{H})$ be a unilateral weighted shift which is an m-isometry. If for some non-zero $x \in \mathcal{H}$,*

$$\|x\| = \|Tx\| = \cdots = \|T^{m-1}x\|,$$

then T is an isometry.

Proof. Suppose that $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $Te_n = w_n e_{n+1}$ for all $n \geq 0$. Put $x = \sum_{n=0}^{\infty} \beta_n e_n$. Then the hypothesis on x implies that

$$\begin{aligned} 0 &= \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \|T^k x\|^2 \\ &= \sum_{n=0}^{\infty} |\beta_n|^2 \left[(-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{n+i}{}^2 \right] \\ &= \sum_{n=0}^{\infty} |\beta_n|^2 \langle \Delta_T e_n, e_n \rangle. \end{aligned}$$

Since x is non-zero, $\beta_{n_0} \neq 0$ for some n_0. So the positivity of Δ_T implies that

(7)
$$\langle \Delta_T e_{n_0}, e_{n_0} \rangle = (-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{n_0+i}{}^2 = 0.$$

Thus, condition (ii) in Theorem 1 does not occur and so T must be an $m-1$-isometry. Now, applying an argument similar to the above and using Theorem 1, $m-1$ times, we conclude that T must be an isometry. ◻

Note that the preceding theorem is not valid, in general, for any operator T. For instance, the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

defines an operator T on $\mathbf{C}^2$ which is a 3-isometry but not a 2-isometry. Furthermore, if $x = (1, 0)$, then $\|x\| = \|Tx\| = \|T^2 x\|$.

3. Reflexivity of m-isometric unilateral weighted shifts. In the previous section, all m-isometric unilateral weighted shifts are characterized in terms of their weight sequences. In this section, we prove the reflexivity of these operators. The following result is basic in the proof of the next theorem.

Theorem 3 [**12**, Lemma 1]. *If T is an injective unilateral weighted shift operator and T^* has a non-zero eigenvalue, then for every positive integer n, the operator T^n is reflexive.*

Theorem 4. *Every non-negative integer power of an m-isometric unilateral weighted shift operator is reflexive.*

Proof. Let T be an m-isometric unilateral weighted shift. Since $T^*e_0 = 0$, $0 \in \sigma_p(T^*)$. Assume that $\sigma_p(T^*) = \{0\}$. Since T is a non-invertible m-isometry, $\sigma(T^*) = \sigma(T) = \overline{\mathbf{D}}$. Thus, $T^* - \lambda I$ is not surjective, for every nonzero scalar $\lambda \in \overline{\mathbf{D}}$. But it is easy to see that, in general, for every operator S,

$$\{\lambda : S^* - \lambda I \text{ is not surjective}\} = \sigma_{ap}(S)^* := \{\lambda : \overline{\lambda} \in \sigma_{ap}(S)\}.$$

By the Ridge theorem [**17**], $\sigma_{ap}(T) = \{\lambda : r_1(T) \leq |\lambda| \leq 1\}$, where $r_1(T) = \lim_{n\to\infty}(\inf\{\|T^n x\| : \|x\| = 1\})^{1/n}$. Hence, $r_1(T) = 0$ and so T^* is not surjective. On the other hand, since $\ker(T) = 0$, $\operatorname{ran}(T^*)$ is dense in $\mathcal{H}$. But $\operatorname{ran}(T)$ and so $\operatorname{ran}(T^*)$ is closed; thus, T^* is surjective, which is a contradiction. Hence, $\sigma_p(T^*)$ contains a nonzero element. Now, Theorem 3 implies that T^k is reflexive for every $k \geq 1$. The result also clearly holds for $k = 0$. □

4. Some other reflexive operators. Recall that a unilateral weighted shift operator T satisfying the hypotheses of Theorem 2 is an isometry. Isometric operators form a subclass of a wider collection of operators called contractions. It is known that every contraction m-isometry is an isometry (see [**13**, Corollary 1], or [**7**, Corollary 2.4]), and so is reflexive. To characterize some other reflexive contractions, first we bring some terminology.

Let $T \in B(\mathcal{H})$ and K be a compact subset of $\mathbf{C}$. By $\|f\|_K$ we mean $\sup\{|f(x)| : x \in K\}$. The set K is said to be a spectral set for T if $\sigma(T) \subseteq K$ and $\|f(T)\| \leq \|f\|_K$ for every rational function f with poles off K. If $\sigma(T)$ is a spectral set, we say that T is a von Neumann operator. Subnormal operators are well-known examples of von Neumann operators [**9**, Proposition 9.2]. Conway and Dudziak [**10**] have shown that every von Neumann operator is reflexive. Our approach in the next result is also based upon the reflexivity of von Neumann operators.

Theorem 5. *If $T \in B(\mathcal{H})$ is a contraction so that $\sigma(T) = \overline{\mathbf{D}}$, then T is a reflexive operator.*

Proof. The hypotheses imply that $\|T\| = 1$. Let f be a rational function with poles off $\overline{\mathbf{D}}$. Choose $R > 0$ so that $\overline{\mathbf{D}} \subseteq D(0, R) := \{z : |z| < R\}$, and f is analytic on $D(0, R)$. Suppose that $\sum_{k=0}^{\infty} a_k z^k$ denotes the power series representation of f. Then the sequence of its partial sums, defined by $p_n(z) = \sum_{k=0}^{n} a_k z^k$, converges uniformly to f on compact subsets of $D(0, R)$. By von Neumann's inequality [**14**, Problem 229],

$$\|p_n(T)\| \leq \sup\{|p_n(z)| : z \in \overline{\mathbf{D}}\}.$$

Thus, using Riesz functional calculus we get

$$\|f(T)\| \leq \sup\{|f(z)| : z \in \overline{\mathbf{D}}\}.$$

Hence, T is a von Neumann operator and so is reflexive. □

The following open problem, posed in [**6**], can now be answered affirmatively as a direct consequence of the above theorem.

Problem. Assume that T is a contraction such that every $\lambda \in \overline{\mathbf{D}}$ is an eigenvalue for T. Is T necessarily reflexive?

Some other reflexive operators can be found among hyponormal operators. The authors have shown in [**12**] that a hyponormal operator T with $\sigma(T) = \{z : |z| \leq r(T)\}$, where $r(T)$ is the spectral radius of T, is a von Neumann operator and so is reflexive.

Theorem 6. *Every non-negative integer power of a hyponormal m-isometry T is an isometry, hence, reflexive.*

Proof. Since T is hyponormal, $r(T) = \|T\|$, and since T is an m-isometry, $\sigma(T) = \overline{\mathbf{D}}$ or $\partial\mathbf{D}$, and so $r(T) = 1$. Hence, T is a contraction m-isometry and so is an isometry [**7, 13**]. This, in turn, implies that T^n is reflexive for $n \geq 0$. □

Remark. In [**18**] it is shown that every subnormal m-isometry is an isometry. Since subnormal operators are hyponormal, the proof of the previous theorem generalizes this result.

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