COTORSION PAIRS IN C(R-Mod)

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ABSTRACT. In [**8**] Salce introduced the notion of a co-torsion pair $(\mathcal{A}, \mathcal{B})$ in the category of abelian groups. But his definitions and basic results carry over to more general abelian categories and have proved useful in a variety of settings. In this article we will consider complete cotorsion pairs $(\mathcal{C}, \mathcal{D})$ in the category $\mathbf{C}(R\text{-Mod})$ of complexes of left R-modules over some ring R. If $(\mathcal{C}, \mathcal{D})$ is such a pair, and if $\mathcal{C}$ is closed under taking suspensions, we will show when we regard $\mathbf{K}(\mathcal{C})$ and $\mathbf{K}(\mathcal{D})$ as subcategories of the homotopy category $\mathbf{K}(R\text{-}$Mod), then the embedding functors $\mathbf{K}(\mathcal{C}) \rightarrow \mathbf{K}(R\text{-Mod})$ and $\mathbf{K}(\mathcal{D}) \rightarrow \mathbf{K}(R\text{-Mod})$ have left and right adjoints, respectively. In finding examples of such pairs, we will describe a procedure for using Hoveys results in [**5**] to find a new model structure on $\mathbf{C}(R\text{-Mod})$.

1. Introduction. Let R be a ring, and let $\mathbf{C}(R\text{-Mod})$ denote the category of complexes of left R-modules. This category has enough injectives and projectives so we can compute derived functors. We let Ext^n denote the nth derived functor of Hom in the category of these complexes. We identify the elements of $\operatorname{Ext}^1(C, D)$ with the equivalence classes of short exact sequences

$$0 \longrightarrow D \longrightarrow U \longrightarrow C \longrightarrow 0$$

in $\mathbf{C}(R\text{-Mod})$.

If $C \in \mathbf{C}(R\text{-Mod})$, let $S(C)$ denote the suspension of the complex C. So $S(C)_n = C_{n+1}$ for all n, and the differential of $S(X)$ is d where d is the differential of C (with an appropriate change in subscripts). We then can define $S^k(C)$ for any $k \in \mathbf{Z}$. A class $\mathcal{C}$ of objects of $\mathbf{C}(R\text{-}$Mod) will be said to be closed under suspensions if $S^k(C) \in \mathcal{C}$ whenever $C \in \mathcal{C}$ and $k \in \mathbf{Z}$.

In later sections we will be concerned with the categories $\mathbf{C}(R\text{-Mod})$ and the homotopy categories $\mathbf{K}(R\text{-Mod})$. These categories have the same objects. To distinguish sets of morphisms, we will let $\operatorname{Hom}(C, D)$ denote the set of morphisms $C \to D$ in $\mathbf{C}(R\text{-Mod})$, and then we let $\operatorname{Hom}_{\mathbf{K}(R\text{-Mod})}(C, D)$ denote the morphisms $C \to D$ in $\mathbf{K}(R\text{-Mod})$.

We recall that, if $f : C \to D$ is a morphism in $\mathbf{C}(R\text{-Mod})$, then we have the mapping cone $c(f)$ of f. We have that $c(f)_n = D_n \oplus C_{n+1}$, and the differential d is such that $d(y, x) = (d(y) + f(x), d(x))$. We have the short exact sequence

$$0 \longrightarrow D \longrightarrow c(f) \longrightarrow S(C) \longrightarrow 0,$$

where the maps $D \to c(f)$ and $c(f) \to S(C)$ are given by $y \mapsto (y, 0)$ and $(y, x) \mapsto x$, respectively.

Given $f, g \in \operatorname{Hom}(C, D)$, we will let $f \cong g$ mean that f and g are homotopic. If we want to indicate the homotopy s, we write $f \overset{s}{\cong} g$.

A pair $(\mathcal{C}, \mathcal{D})$ of classes of objects of $\mathbf{C}(R\text{-Mod})$ is said to be a cotorsion pair in $\mathbf{C}(R\text{-Mod})$ if

$$\mathcal{D} = \mathcal{C}^\perp = \{D \mid \operatorname{Ext}^1(C, D) = 0 \text{ for all } C \in \mathcal{C}\}$$

and if

$$\mathcal{C} = {}^\perp\mathcal{D} = \{C \mid \operatorname{Ext}^1(C, D) = 0 \text{ for all } D \in \mathcal{D}\}.$$

For objects C and D of $\mathbf{C}(R\text{-Mod})$, the groups $\operatorname{Ext}^1(S^k(C), D)$ and $\operatorname{Ext}^1(C, S^k(D))$ are isomorphic. So if $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair, $\mathcal{C}$ is closed under suspensions if and only if $\mathcal{D}$ is closed under suspensions.

The cotorsion pair $(\mathcal{C}, \mathcal{D})$ will be said to be complete if, for every $X \in \mathbf{C}(R\text{-Mod})$, there are exact sequences

a) $0 \to D \to C \to X \to 0$

b) $0 \to X \to D' \to C' \to 0$

where $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{D}$.

If there is a set $\mathcal{S}$ of objects of $\mathbf{C}(R\text{-Mod})$ such that $\mathcal{S}^\perp = \mathcal{D}$ for some cotorsion pair $(\mathcal{C}, \mathcal{D})$, then the pair is said to be cogenerated by a set. If this is the case, then the pair is known to be complete; see

Theorem 3.2.1 of [**4**] where the proof is given for cotorsion pairs in the category R-Mod. It carries over directly to the category $\mathbf{C}(R\text{-Mod})$. The proof there shows that sequences a) and b) above can be chosen functorially. This is what is meant by saying the pair is functorially complete. So, when the pair $(\mathcal{C}, \mathcal{D})$ is cogenerated by a set, it is a functorially complete cotorsion pair.

If we have sequences a) and b) as above for a complete pair $(\mathcal{C}, \mathcal{D})$, and if $\overline{C} \in \mathcal{C}$, then we have the exact sequence

$$\operatorname{Hom}(\overline{C}, C) \longrightarrow \operatorname{Hom}(\overline{C}, X) \longrightarrow \operatorname{Ext}^1(\overline{C}, D) = 0.$$

This gives that $C \to X$ is a $\mathcal{C}$-precover. Similarly, we get that $X \to D'$ is a $\mathcal{D}$-preenvelope.

If C is a complex, then $x \in C$ will mean that $x \in C_n$ for some unique $n \in \mathbf{Z}$. A cotorsion pair $(\mathcal{C}, \mathcal{D})$ is said to be hereditary if $\operatorname{Ext}^n(C, D) = 0$ for all $n \geq 1$ and all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. The pair is hereditary if and only if, for every short exact sequence $0 \to D' \to D \to D'' \to 0$ where $D', D \in \mathcal{D}$. Then $D'' \in \mathcal{D}$.

We will also consider cotorsion pairs $(\mathcal{A}, \mathcal{B})$ in the category R-Mod. The definitions and terminology are essentially the same as those above.

2. Two lemmas. In this and the following sections we will be concerned with $\mathbf{C}(R\text{-Mod})$ and $\mathbf{K}(R\text{-Mod})$ for some ring R. We prove two lemmas. The first of the two is known (see Lemma 3.2 of [**4**]). But, for the reader's convenience, we include the short proof.

Lemma 2.1. *If $f \in \operatorname{Hom}(C, D)$, then the short exact sequence $0 \longrightarrow D \longrightarrow c(f) \longrightarrow S(C) \longrightarrow 0$ is split exact if and only if $f \cong 0$.*

Proof. Assume that the sequence splits, and let $u : S(C) \to c(f)$ be a section. Then, for $x \in S(C)$, we have $u(x) = (s(x), x)$ for some $s(x) \in D$. Since u is a morphism of complexes, we have $(ds(x) + f(x), dx) = (sd(x), dx)$. So $f(x) = (ds(x) + sd(x))$. This shows that s provides the desired homotopy.

If, conversely, $f \overset{s}{\cong} 0$, and if we define u by $u(x) = (s(x), x)$, we check that u gives the desired section. $\square$

Corollary 2.2. *If $C, D \in \mathbf{C}(R\text{-Mod})$ and if $\operatorname{Ext}^1(S(C), D) = 0$, then every $f : C \to D$ is homotopic to 0.*

Proof. Immediate. ☐

The next lemma is our main result of this section.

Lemma 2.3.   *Given $f \in \operatorname{Hom}(C, D)$ and the associated exact sequence*
$$0 \longrightarrow D \longrightarrow c(f) \longrightarrow S(C) \longrightarrow 0,$$
the following are equivalent:

a) $D \to c(f)$ *admits a retraction in* $\mathbf{C}(R\text{-Mod})$.

b) $D \to c(f)$ *admits a retraction in* $\mathbf{K}(R\text{-Mod})$.

c) $c(f) \to S(C)$ *admits a section in* $\mathbf{C}(R\text{-Mod})$.

d) $c(f) \to S(C)$ *admits a section in* $\mathbf{K}(R\text{-Mod})$.

Proof. The equivalence of a) and c) is standard. Clearly, a)$\Rightarrow$ b) and c)$\Rightarrow$ d).

We now prove b) $\Rightarrow$ a). Let $r : c(f) \to D$ (r a morphism in $\mathbf{C}(R\text{-}$Mod)) give a retraction of $D \to c(f)$ in $\mathbf{K}(R\text{-Mod})$. Let t be the corresponding homotopy, i.e., for $y \in D$ we have $(dt+td)(y) = y r(y, 0)$. Define $c(f) \to D$ by $(y, x) \mapsto y + tf(x) + r(0, x)$ for $(y, x) \in c(f)$. If this map is a morphism of complexes, it gives the desired retraction. Since $d(y, x) = (dy + f(x), dx)$ we need to show that
$$d(y + tf(x) + r(0, x)) = dy + f(x) - tfd(x) - r(0, dx).$$
Canceling dy and using the fact that $df = fd$ in the term $tfd(x)$, we see that we are reduced to showing that
$$dtf(x) + tdf(x) = f(x) - dr(0, x) - r(0, dx).$$
But $dtf(x) + tdf(x) = (dt + td)(f(x)) = f(x) - r(f(x), 0)$. So now, canceling $f(x)$, we need that
$$r(f(x), 0) = dr(0, x) + r(0, dx).$$
Since $dr = rd$, we have
$$
\begin{aligned}
dr(0, x) + r(0, dx) &= rd(0, x) + r(0, dx) \\
&= r(f(x), dx) + r(0, dx) \\
&= r((f(x), 0) + (0, dx)) + r(0, dx) \\
&= r(f(x), 0).
\end{aligned}
$$

We now prove d) $\Rightarrow$ c). Let $s : S(C) \to c(f)$ be a section up to homotopy where $s(x) = (u(x), v(x))$. Let w be the associated homotopy. Then $-(dw + wd)(x) = x = v(x)$. We have

$$ds(x) = d(u(x), v(x)) = (du(x) + fv(x), vd(x)).$$

But s is a morphism and so $ds(x) = s(-dx) = (-ud(x), -vd(x))$. So we get $(du + ud)(x) = -fv(x)$ and $dv(x) = vd(x)$.

We now claim that $x \mapsto (fw(x) + u(x), x)$ is the desired section. To get that the function commutes with differentials, we need that $dfw(x) + du(x) + f(x) = -fwd(x) - ud(x)$ or that $f((dw + wd)(x)) + (du + ud)(x) = -f(x)$. Since $(dw + wd)(x) = v(x) - x$ and since $(du + ud)(x) = -f(x)$, we see that the equality holds. $\square$
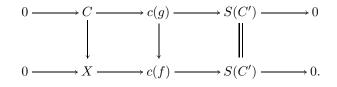
3. The existence of adjoints. The objective of this section is to prove that the adjoints mentioned in the abstract exist.

Proposition 3.1. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair in $\mathbf{C}(R\text{-Mod})$ where $\mathcal{C}$ is closed under taking suspensions. For $X \in \mathbf{C}(R\text{-Mod})$, let $0 \to D \to C \to X \to 0$ be exact where $C \in \mathcal{C}$ and $D \in \mathcal{D}$. If $C' \in \mathcal{C}$ and if $f_i \in \mathrm{Hom}\,(C', X)$ and $g_i \in \mathrm{Hom}\,(C', C)$ for $i = 1, 2$ are such that*

$$
\begin{array}{ccc}
 & C' & \\
{}^{g_i}\swarrow & & \downarrow{}^{f_i} \\
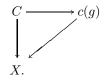C & \longrightarrow & X
\end{array}
$$

are commutative for $i = 1, 2$, then $f_1 \cong f_2$ if and only if $g_1 \cong g_2$.

Proof. If $g_1 \cong g_2$, then easily $f_1 \cong f_2$. For the converse, let $f = f_1 - f_2$ and $g = g_1 - g_2$. We see that we only need show that when $f \cong 0$ we have $g \cong 0$. With such an f and g, we get the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & c(g) & \longrightarrow & S(C') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & c(f) & \longrightarrow & S(C') & \longrightarrow & 0.
\end{array}
$$

Since $f \cong 0$, by Lemma 2.1 we get that the lower short exact sequence splits. A retraction $c(f) \to X$ provides us with a commutative diagram:

$$
\begin{array}{ccc}
C & \longrightarrow & c(g) \\
\big\downarrow & \swarrow & \\
X. &
\end{array}
$$

Since $\mathcal{C}$ is closed under extensions and suspensions, we have $c(g) \in \mathcal{C}$. Since $C \to X$ is a $\mathcal{C}$-precover, we get a lifting $c(g) \to C$. We now want to apply Lemma 2.3. So we want to argue that $c(g) \to C$ provides a retraction of $C \to c(g)$ in $\mathbf{K}(R\text{-Mod})$. For this, note that the difference of the composition $C \to c(g) \to C$ and the identity map id_C maps C into the kernel of $C \to X$, that is, into D. By Corollary 2.2, this difference (as a map into D) is homotopic to 0. But then the difference as a map into C is homotopic to 0. So, by Lemma 2.3, we get that the short exact sequence $0 \to C \to c(g) \to S(C') \to 0$ is split exact. So, by Lemma 2.1, we get that $g \cong 0$.   □

Corollary 3.2. *With the same notation,* $\mathrm{Hom}_{\mathbf{K}}(R\text{-Mod})(C', C) \to \mathrm{Hom}_{\mathbf{K}}(R\text{-Mod})(C', X)$ *is a bijection.*

Proof. We first note that the exact sequence $0 \to D \to C \to X \to 0$ gives the exact sequence $\mathrm{Hom}\,(C', C) \to \mathrm{Hom}\,(C', X) \to \mathrm{Ext}^1(C', D) = 0$. So $\mathrm{Hom}\,(C', C) \to \mathrm{Hom}\,(C', X)$ is surjective. This gives that $\mathrm{Hom}_{\mathbf{K}}(R\text{-Mod})(C', C) \to \mathrm{Hom}_{\mathbf{K}}(R\text{-Mod})(C', X)$ is surjective. Proposition 3.1 guarantees that this function is injective and so bijective.   □

We also have the duals of Proposition 3.1 and Corollary 3.2. Since the proofs will be dual proofs, we will just state the results.

Proposition 3.3. *Suppose that* $(\mathcal{C}, \mathcal{D})$ *is a complete cotorsion pair in* $\mathbf{C}(R\text{-Mod})$ *where* $\mathcal{D}$ *is closed under taking suspensions. For* $X \in \mathbf{C}(R\text{-Mod})$, *let* $0 \to X \to D \to C \to 0$ *be exact where* $C \in \mathcal{C}$ *and* $D \in \mathcal{D}$. *Then, if* $D' \in \mathcal{D}$ *and if* $f_i \in \mathrm{Hom}\,(X, D')$ *and* $g_i \in \mathrm{Hom}\,(D, D')$

for $i = 1, 2$ are such that

$$
\begin{array}{ccc}
X & \longrightarrow & D \\
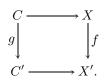f_i \downarrow & \swarrow g_i & \\
D' & &
\end{array}
$$

are commutative for $i = 1, 2$, then $f_1 \cong f_2$ if and only if $g_1 \cong g_2$.

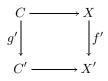**Corollary 3.4.** *With the same notation we have that $\mathrm{Hom}_{\mathbf{K}}(R\text{-Mod})$ $(D, D') \to \mathrm{Hom}_{\mathbf{K}}(R\text{-Mod})(X, D)$ is a bijection.*

Theorem 3.5. *If $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair in $\mathbf{C}(R\text{-Mod})$, and if $\mathcal{C}$ is closed under taking suspensions, then the embeddings $\mathbf{K}(\mathcal{C}) \to \mathbf{K}(R\text{-Mod})$ and $\mathbf{K}(\mathcal{D}) \to \mathbf{K}(R\text{-Mod})$ have right and left adjoints, respectively.*

Proof. A right adjoint T of $\mathbf{K}(\mathcal{C}) \to \mathbf{K}(R\text{-Mod})$ can be constructed as follows. For each $X \in \mathbf{C}(R\text{-Mod})$, we make a choice of an exact sequence $0 \to D \to C \to X \to 0$ in $\mathbf{C}(R)$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. We want to define a functor $T : \mathbf{K}(R\text{-Mod}) \to \mathbf{K}(\mathcal{C})$ so that $T(X) = C$. If $f : X \to X'$ is a morphism in $\mathbf{C}(R\text{-Mod})$, we let $[f]$ represent the corresponding morphism in $\mathbf{K}(R\text{-Mod})$. So $[f]$ consists of all $f' : X \to X'$ such that $f \cong f'$. We use the following procedure to define $T([f])$. We have the exact sequence $\mathrm{Hom}\,(C, C') \to \mathrm{Hom}\,(C, X') \to \mathrm{Ext}^1(C, D') = 0$. So this means that there is a $g \in \mathrm{Hom}\,(C, C')$ whose image in $\mathrm{Hom}\,(C, X')$ is the composition $C \to X \xrightarrow{f} X'$. So we have the commutative diagram:

$$
\begin{array}{ccc}
C & \longrightarrow & X \\
g \downarrow & & \downarrow f \\
C' & \longrightarrow & X'.
\end{array}
$$

For $f' \in [f]$ (so $f \cong f'$) we use the same argument and find a $g' : C \to C'$ so that the diagram

$$
\begin{array}{ccc}
C & \longrightarrow & X \\
g' \downarrow & & \downarrow f' \\
C' & \longrightarrow & X'
\end{array}
$$

is commutative. Then an application of Proposition 3.1 gives that $g \cong g'$. This means that we can define $T([f])$ to be $[g]$ with f and g as above. Then it can be quickly checked that T is an additive functor. Note that the maps $C \to X$ then become maps $T(X) \to X$ and give a natural transformation from T to the identity functor on $\mathbf{K}(R\text{-Mod})$.

Now we appeal to Corollary 3.2. This corollary says that, if $C' \in \mathcal{C}$ and if $0 \to D \to C \to X \to 0$ is as above, then $\operatorname{Hom}_{\mathbf{K}(R\text{-Mod})}(C', C) \to \operatorname{Hom}_{\mathbf{K}(R\text{-Mod})}(C', X)$ is a bijection. But $T(X) = C$ so we have the bijection

$$\operatorname{Hom}_{\mathbf{K}(R\text{-Mod})}(C', T(X)) \longrightarrow \operatorname{Hom}_{\mathbf{K}(R\text{-Mod})}(C', X).$$

From the definition of this map we see that it is natural in C'. From the natural transformation above, we see that it is natural in X. So this establishes that T is a right adjoint of the embedding functor $\mathbf{K}(\mathcal{C}) \to \mathbf{K}(R\text{-Mod})$. □

The definition of T in the above depends upon the choice of an exact sequence $0 \to D \to C \to X \to 0$ for every X. A different set of choices of such sequences will give a functor isomorphic to this T.

Remark 3.6. If D above is exact, then $T(X) = C \to X$ is a homology isomorphism. So if all such D are exact, the counit of the adjunction consists of homology isomorphisms. If all $C \in \mathcal{C}$ are exact, we get a similar claim about the unit of the left adjoint of $\Delta \to \mathbf{K}(R\text{-Mod})$.

4. Examples and applications. In this section we will give several complete cotorsion pairs in $\mathbf{C}(R\text{-Mod})$ where the components of the pair are closed under suspensions. So then we get the associated adjoint functors.

We now recall a method of Gillespie for creating cotorsion pairs in $\mathbf{C}(R\text{-Mod})$ from pairs in R-Mod. Here $\mathcal{E}$ is the class of exact complexes.

Proposition 4.1 [3, Propositions 4.4, 4.6]. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod which is cogenerated by a set, then $\mathbf{C}(\mathcal{B})$ and $\mathbf{C}(\mathcal{B}) \cap \mathcal{E}$ are the second components of cotorsion pairs in $\mathbf{C}(R\text{-Mod})$ which are cogenerated by sets.*

As noted in the introduction, the fact that these cotorsion pairs are cogenerated by sets implies that they are complete. Also, note that

classes $\mathbf{C}(\mathcal{B})$ and $\mathbf{C}(\mathcal{B}) \cap \mathcal{E}$ are closed under suspensions. We note that Gillespie's result for $\mathbf{C}(B) \cap \mathcal{E}$ in [**3**] required that $\mathcal{A}$ contain a generator of finite projective dimension. In this situation we even have a projective generator; namely, the ring as a left module over itself.

Examples. We will use the symbol R-Inj to denote the category of injective R-modules. Then we will use other variations of this notation.

(1) (R-Mod, R-Inj) is a cotorsion pair in R-Mod which is cogenerated by the set of R/I where I is a left ideal of R (the Baer criterion). Then, using Proposition 4.1 and Theorem 3.5 we see that $\mathbf{K}(R\text{-Inj}) \to \mathbf{K}(R\text{-Mod})$ and $\mathbf{K}((R\text{-Inj}) \cap \mathcal{E}) \to \mathbf{K}(R\text{-Mod})$ have left adjoints.

(2) In [**1**, Theorem 4.2 and Proposition 4.1] it was shown that $(\mathbf{C}(R\text{-Flat}) \cap \mathcal{E}, (\mathbf{C}(R\text{-Flat}) \cap \mathcal{E})^{\perp})$ is a cotorsion pair which is cogenerated by a set. The fact that $(\mathbf{C}(R\text{-Flat}), C(R\text{-Flat})^{\perp})$ is a cotorsion pair can be found in [**1**, Theorem 4.3]. This result, with different terminology, says that this pair is complete. The argument that it is complete was based on the fact that it was cogenerated by a set.

So now an application of Theorem 3.5 gives that $\mathbf{K}(R\text{-Flat}) \to \mathbf{K}(R\text{-Mod})$ and $\mathbf{K}((R\text{-Flat}) \cap \mathcal{E}) \to \mathbf{K}(R\text{-Mod})$ have right adjoints. The existence of the first of these two adjoints was proved by Neeman (see [**7**, Theorem 3.2]) but with different methods.

We now want to give another example analogous to that in (2) above. We first recall Kaplansky's theorem:

Theorem 4.2 [**6**, Theorem 1]. *If P is a projective module, then P is the direct sum of countably generated projective modules.*

The proof of Kaplansky's theorem carries over to projective complexes, i.e., to the projective objects in $\mathbf{C}(R\text{-Mod})$. But it is not true that every $P \in \mathbf{C}(R\text{-Proj})$ can be written as a direct sum of complexes with countably generated terms. To see that this is so, let M be a left R-module, and let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. If P is then the complex $\cdots \to P_2 \to P_1 \to P_0 \to 0$, we see that, if P were such a direct sum, then M would be the direct sum of countably generated modules. But it is certainly not true that every module has such a direct sum decomposition. For example, consider $M = \mathbf{Z}^{\mathbf{N}}$ as a module over $\mathbf{Z}$. However, there is something we can say about $P \in \mathbf{C}(R\text{-Proj})$. To do so, we need the next notion.

Definition 4.3. Given a complex $C \in \mathbf{C}(R\text{-Mod})$, by a filtration of C indexed by an ordinal λ we mean a family $(C_\alpha \mid \alpha \leq \lambda)$ of subcomplexes of C such that $C_0 = 0$, such that $C_\lambda = C$, such that $C_\alpha \subset C_{\alpha'}$ whenever $\alpha \leq \alpha' \leq \lambda$, and such that, for any limit ordinal $\beta \leq \lambda$ we have C_β is the union of the C_α with $\alpha < \beta$. If $\mathcal{S}$ is some class of complexes in $\mathbf{C}(R\text{-Mod})$, we say that this filtration is an $\mathcal{S}$-filtration if, for each $\alpha + 1 \leq \lambda$, we have that $C_{\alpha+1}/C_\alpha$ is isomorphic to an element of $\mathcal{S}$. By a filtration $(X_\alpha \mid \alpha \leq \lambda)$ of a set X, we will use the obvious modification of this notion for complexes.

Theorem 4.4. *If $P \in \mathbf{C}(R\text{-Proj})$ for some ring R, and if $\mathcal{S}$ is a set of representatives of complexes with all terms countably generated projective modules, then P has an $\mathcal{S}$-filtration.*

Proof. By Kaplansky's theorem each P_n can be written as a direct sum $P_n = \sum_{i \in I_n} P_n^i$ (direct) for some set I_n and where each P_n^i is countably generated and projective. We will construct the filtration $(P_n^\alpha \mid \alpha \leq \lambda)$ for some ordinal number λ by constructing a filtration $(I_n^\alpha \mid \alpha \leq \lambda)$ of I_n for each n. This filtration will be such that, if we let $P_n^\alpha = \sum_{i \in I_n^\alpha} P_n^i$ for each $n \in \mathbf{Z}$ and $\alpha \leq \lambda$, then for each α the terms P_n^α will give us a subcomplex P^α of P and then these in turn will give us the desired filtration of P.

We construct our filtrations of the sets I_n by transfinite induction. For $\alpha = 0$, we let $I_n^0 = \varnothing$ for each n. Then the corresponding subcomplex is $P^0 = 0$. The crucial step is now in constructing the subsets $I_n^1 \subset I_n$ for each n. To do this, we choose an arbitrary $m \in \mathbf{Z}$ and then choose a countably generated submodule $S \subset P_m$. Since S is countably generated, and since $S \subset \sum_{i \in I_m} P_m^i$, we can find a countable subset $I_m^1 \subset I_m$ with $S \subset \sum_{i \in I_m^1} P_m^i$. Now consider $d_m(\sum_{i \in I_m^1} P_m^i)$. Since this module is a countably generated submodule of P_{m+1}, it will in turn be contained in a sum $\sum_{i \in I_{m+1}^1} P_{m+1}^i$ for a countable subset I_{m+1}^1 of I_{m+1}. We then choose a countable subset $I_{m+2}^1 \subset I_{m+2}$ so that $d_{m+1}(\sum_{i \in I_{m+1}^1} P_{m+1}^i) \subset \sum_{i \in I_{m+2}^i} P_{m+2}^i$. Proceeding in this manner we choose countable subsets $I_n^1 \subset I_n$ for each $n \geq m$. Now, for $n < m$, let $I_n^1 = \varnothing$. Then, with $P_n^1 = \sum_{i \in I_n^1} P_n^i$, we get a subcomplex $P^1 \subset P$ with each term of P^1 countably generated and projective and such that $S \subset P_m^1$. This means that, if $P \neq 0$, we can choose $P^1 \neq 0$.

We now note that, by construction, $(P/P_1)_n = P_n/P_n^1 \cong \sum_{i \in I_n I_n^1} P_n^i$ (direct). So this means that we can use the same procedure to construct a subcomplex P^2/P^1 of P/P^1 where $P_n^2 = \sum_{i \in I_n^2} P_n^i$ for each n where $I_n^2 \supset I_n^1$ and where $I_n^2 I_n^1$ is countable. We can also require that, for some given m and countable submodule $T/P_m \subset P_m/P_m^1$, we have $T \subset P_m^2$. This means that we can choose $P^1 \subsetneqq P^2$ if $P^1 \neq P$.

So now we assume that β is some ordinal number and that, for each $n \in \mathbf{Z}$ and each $\alpha < \beta$, we have constructed subsets $I_n^\alpha \subset I_n$ satisfying:

1) $I_n^\alpha \subset I_n^{\alpha'}$ when $\alpha \leq \alpha' < \beta$,

2) $I_n^{\alpha+1} I_n^\alpha$ is countable when $\alpha + 1 < \beta$,

3) $I_n^\gamma = \cup_{\alpha < \gamma} I_n^\alpha$ when $\gamma < \beta$ is a limit ordinal,

4) If $P_n^\alpha = \sum_{i \in I_\alpha} P_n^i$ for $\alpha < \beta$, then these are the terms of a subcomplex P^α of P.

Now if β is a limit ordinal, we let $I_n^\beta = \cup_{\alpha < \beta} I_n^\alpha$ for each $n \in \mathbf{Z}$. If β is not a limit ordinal and if $\beta = \alpha + 1$, we construct each $I_n^{\alpha+1} \supset I_n$ just as we constructed the I_n^1 from the $I_n^0 = \varnothing$. But then it is clear that we can find an ordinal λ so that the corresponding subcomplex $P^\lambda = P$ (so then we can also assume that $I_n^\lambda = I_n$ for this λ and each n).

It is clear then that $(P_\alpha \mid \alpha \leq \lambda)$ is an $\mathcal{S}$-filtration of P. $\square$

Theorem 4.5. *If R is a ring, then $(\mathbf{C}(R\text{-Proj}), \mathbf{C}(R\text{-Proj})^\perp)$ is a cotorsion pair which is cogenerated by a set.*

Proof. We let $\mathcal{S}$ be a set of representatives of all complexes P of left R-modules such that each P_n is countably generated and projective. Then $\mathcal{S}$ contains a projective generator of $\mathbf{C}(R\text{-Mod})$. For, if we let $\overline{R}$ be the complex $\cdots 0 \to 0 \to R \xrightarrow{1} R \to 0 \to 0 \to \cdots$ with the two R's in the 1st and 0th place, then $\oplus_{n \in \mathbf{Z}} S^n(\overline{R})$ is such a projective generator in $\mathcal{S}$. In fact, this complex is free as a complex with exactly one base element of every possible degree. We then have the cotorsion pair $(^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ which is cogenerated by the set $\mathcal{S}$. We now use [**4**, Theorem 3.2.4]. That theorem was stated for modules, but the proof carries over directly to complexes where we let the projective generator above of $\mathbf{C}(R\text{-Mod})$ play the role of R in that theorem. We get that $^\perp(\mathcal{S}^\perp)$ consists of all complexes which are direct summands of complexes admitting an $\mathcal{S}$-filtration. By Theorem 4.4 above, we get that $\mathbf{C}(R\text{-Proj}) \subset^\perp (\mathcal{S}^\perp)$.

But, conversely, any complex P admitting an $\mathcal{S}$-filtration has all its terms projective. So then any summand of such a complex has all its terms projective. Hence, $^{\perp}(\mathcal{S}^{\perp}) \subset \mathbf{C}(R\text{-Proj})$. So we get that $(^{\perp}(\mathcal{S}^{\perp}), S^{\perp}) = (\mathbf{C}(R\text{-Proj}), (\mathbf{C}(R\text{-Proj}))^{\perp})$, and we have established our claim. $\square$

We now want to prove a theorem analogous to Theorem 4.4 above where we replace the class $\mathbf{C}(R\text{-Mod})$ with the class $\mathbf{C}(R\text{-Mod}) \cap \mathcal{E}$. To do so, we will use the following terminology. Given $P \in \mathbf{C}(R\text{-Proj})$ for some ring R, for each n let $P_n = \sum_{i \in I_n} P_n^i$ be a fixed direct sum decomposition of P_n as a direct sum of countably generated projective modules. We will call a subcomplex $P' \subset P$ a nice subcomplex (relative to these direct sum decompositions) if, for each n, we have $P'_n = \sum_{i \in I'_n} P_n^i$ for some subset $I'_n \subset I_n$. So we note that the complexes P^{α} in the proof above are all nice subcomplexes of the P of the theorem relative to the given direct sum decompositions. We also remark that any intersection and any sum of nice subcomplexes of P are nice subcomplexes of P.

Theorem 4.6. *Let $P \in \mathbf{C}(R\text{-Proj})$, and let κ be an infinite cardinal with $\kappa \geq |R|$. If T is a set of representatives of exact complexes with all terms projective and with the cardinality of each term at most κ, then P has a $\mathcal{T}$-filtration.*

Proof. The argument is analogous to but slightly more complicated than the argument in Theorem 4.4. In constructing the filtration, the crucial step is constructing $P^{\alpha+1}$ from P^{α}. Letting $P^0 = 0$, we show how to construct P^1. Then $P^{\alpha+1}$ will be constructed from P^{α} in a similar manner.

We construct P^1 as a union of a chain $Q^1 \subset Q^2 \subset Q^3 \subset \cdot$ of P where each of these Q^n are nice subcomplexes of P. To construct Q^1, we let $m \in \mathbf{Z}$ be arbitrary and $T \subset P^m$ be a submodule with $|T| \leq \kappa$. We choose a subset J_m^1 of I_m so that $T \subset \sum_{i \in J_M^1} P_m^i$. Then we choose subsets $J_n^1 \subset I_n$ for $n \geq m$ so $|J_m^1| \leq \kappa$ and so that $d_n(\sum_{i \in J_n^1} P_n^i) \subset \sum_{i \in J_{n+1}^1} P_{n+1}^i$ for all $n \geq m$. Then, we let $J_n^1 = \varnothing$ for $n < m$ and Q^1 be the complex whose nth term is $\sum_{i \in J_n^1} P_n^i$. Clearly, each term Q_p^1 of Q^1 has cardinality at most κ. We have a nice subcomplex all of whose terms are projective, but it is not necessarily exact.

To construct Q^2, we pick any $p \in \mathbf{Z}$. We have $|Q_p^1| \leq \kappa$, and so $|Z_p(Q^1)| \leq \kappa$. Since P is exact, we have that $d_{p+1}(P_{p+1}) \supset Z_p(Q^1)$. So there is a submodule $U \subset P_{p+1}$ with $|U| \leq \kappa$ such that $d_{p+1}(U) \supset Z_p(Q^1)$. Now we find a nice subcomplex C of P with $C_{p+1} \supset U$ and where the corresponding subsets of each I_n have cardinality at most κ. Now let $Q^2 = Q^1 + C$. Then Q^2 is a nice subcomplex of P with the corresponding subsets of I_n still having cardinality at most κ. Also we have that $d_{p+1}(Q_{p+1}^2) \supset Z_p(Q^1)$. We then construct Q^3 from Q^2 as we constructed Q^2 from Q^1 but with perhaps a different p. If we then continue and construct $Q^1 \subset Q^2 \subset Q^3 \subset \cdots$, but making sure that a given $p \in \mathbf{Z}$ is used infinitely many times in constructing Q^{n+1} from Q^n, we see that if we let $P^1 = \cup_{n=1}^{\infty} Q^n$, we get a nice subcomplex of P with the corresponding subsets of each I_n having cardinality at most κ. We also have that, by construction of P^1, every cycle of P^1 is a boundary of P^1 and so P^1 is exact. By construction, $P_m^1 \supset T$. Now the rest of the argument for the existence of our desired filtration follows the pattern of the proof in Theorem 4.4 above. $\square$

Theorem 4.7. *If R is a ring, then $(\mathbf{C}(R\text{-Proj}) \cap \mathcal{E}, (\mathbf{C}(R\text{-Proj}) \cap \mathcal{E})^{\perp})$ is a cotorsion pair which is cogenerated by a set.*

The proof of this theorem is like that of the proof of Theorem 4.5 but with the set S of that theorem replaced by the set $\mathcal{T}$.

We now get other examples of adjoint functors.

The functors $\mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Mod})$ and
$\mathbf{K}((R\text{-Proj}) \cap \mathcal{E}) \longrightarrow \mathbf{K}(R\text{-Mod})$ have right adjoints.

5. Hovey pairs. For a ring R, we again let $\mathcal{E} \subset \mathbf{C}(R\text{-Mod})$ be the class of exact complexes. In [**2**, page 28, the Main theorem] it was noted that $(^{\perp}\mathcal{E}, \mathcal{E})$ is a cotorsion pair where $^{\perp}\mathcal{E}$ is the class of DG-projective complexes. So $P \in {}^{\perp}\mathcal{E}$ if and only if each P_n is projective and if every morphism $P \to E$ with $E \in \mathcal{E}$ is homotopic to 0. We also need the fact that this cotorsion pair is cogenerated by a set. So, using this notation we have:

Lemma 5.1. $(^{\perp}\mathcal{E}, \mathcal{E})$ *is cogenerated by a set of complexes.*

Proof. We again let $\overline{R}$ be the complex $\cdots \to 0 \to R \xrightarrow{1} R \to 0 \to \cdots$ with the two Rs in the 1st and 0th places. Then R is a projective

object in $\mathbf{C}(R\text{-Mod})$ (in fact, it is free with a single generator of degree 1). We let $\underline{R}$ be the complex $\cdots \to 0 \to R \to 0 \to \cdots$ with the single R in the 0th place. Then $\underline{R}$ is a subcomplex of R and the quotient $\overline{R}/\underline{R}$ is $S(R)$, so we have the short exact sequence $0 \to R \to R \to S(R) \to 0$. So we can use this partial projective resolution of $S(\underline{R})$ to compute $\text{Ext}^1(S(\underline{R}), C)$ for any $C \in \mathbf{C}(R\text{-Mod})$. Clearly, $\text{Hom}(\underline{R}, C) \cong Z_0(C)$ and a morphism $f : \underline{R} \to C$ has an extension $g : \overline{R} \to C$ if and only if $x = f_0(1) \in Z_0(C)$ is a boundary in C. This gives that $\text{Ext}^1(S(\underline{R}), C) \cong H_0(C)$. More generally, we have that $\text{Ext}^1(S^{k+1}(\underline{R}), C) \cong H_k(C)$ for any $k \in \mathbf{Z}$. So this gives that a complex C is exact if and only if $\text{Ext}^1(S^k(\underline{R}), C) = 0$ for all k, and so we get that $\mathcal{S}^\perp = \mathcal{E}$ where $\mathcal{S}$ is the set of complexes $S^k(\underline{R})$ for $k \in \mathbf{Z}$. $\square$

Definition 5.2. If $\mathcal{C}$ and $\mathcal{D}$ are two classes of complexes of left R-modules, we will say that $(\mathcal{C} \cap \mathcal{E}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{D} \cap \mathcal{E})$ form a Hovey pair if each of these pairs is a functorially complete cotorsion pair in $\mathbf{C}(R\text{-Mod})$.

Hovey proved ([**5**, Theorem 2.2]) that every such pair gives rise to a model structure on $\mathbf{C}(R\text{-Mod})$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{D}$ is the class of fibrant objects and $\mathcal{E}$ is the class of trivial objects of the model structure. Our aim here is to provide an example of a Hovey pair. All cotorsion pairs here will be in the category $\mathbf{C}(R\text{-Mod})$. We note that we are considering a special case of Hovey's results. We are taking the $\mathcal{P}$ of his paper to be the class of all short exact complexes and the class of trivial objects of his paper to be the class of exact complexes.

Lemma 5.3. *If $(\mathcal{C}, \mathcal{D}')$ is a cotorsion pair and $(\mathcal{U}, \mathcal{V})$ is a complete and hereditary cotorsion pair in $\mathbf{C}(R\text{-Mod})$ and if $\mathcal{U} \subset \mathcal{C}$, then when $(\mathcal{C} \cap \mathcal{V})^\perp = \mathcal{D}$, we have $\mathcal{D}' = \mathcal{D} \cap \mathcal{V}$.*

Proof. Since $\mathcal{C} \cap \mathcal{V} \subset \mathcal{C}$, we have $\mathcal{D}' = \mathcal{C}^\perp \subset (\mathcal{C} \cap \mathcal{V})^\perp = \mathcal{D}$. Since $\mathcal{U} \subset \mathcal{C}$, we have $\mathcal{D}' = \mathcal{C}^\perp \subset \mathcal{U}^\perp = \mathcal{V}$. Hence, $\mathcal{D}' \subset \mathcal{D} \cap \mathcal{V}$. Let $D \in \mathcal{D} \cap \mathcal{V}$. We want to show that $D \in \mathcal{D}' = \mathcal{C}^\perp$, i.e., that $\text{Ext}^1(C, D) = 0$ for all $C \in \mathcal{C}$. Since $(\mathcal{U}, \mathcal{V})$ is complete, for any $C \in \mathcal{C}$, we have an exact sequence

$$0 \longrightarrow C \longrightarrow V \longrightarrow U \longrightarrow 0$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $C \in \mathcal{C}$ and $U \in \mathcal{U} \subset \mathcal{C}$, we have $V \in \mathcal{C}$

($\mathcal{C}$ is closed under extensions). This gives that $\text{Ext}^1(V, D) = 0$. But we have the exact

$$\text{Ext}^1(V, D) \longrightarrow \text{Ext}^1(C, D) \longrightarrow \text{Ext}^2(U, D).$$

But $\text{Ext}^2(U, D) = 0$ since $U \in \mathcal{U}$, $D \in \mathcal{D}$ and since $(\mathcal{U}, \mathcal{V})$ is hereditary. So $\text{Ext}^1(C, D) = 0$. This gives that $D \in \mathcal{D}'$. So we have $\mathcal{D}' = \mathcal{D} \cap \mathcal{V}$. $\square$

A dual argument gives the next result.

Lemma 5.4. *If $(\mathcal{C}', \mathcal{D})$ is a cotorsion pair and $(\mathcal{U}, \mathcal{V})$ is a complete and hereditary cotorsion pair in $\mathbf{C}(R\text{-Mod})$, and if $\mathcal{V} \subset \mathcal{D}$, then when $\mathcal{C} = ^{\perp}(\mathcal{D} \cap \mathcal{U})$, we have $\mathcal{C}' = \mathcal{C} \cap \mathcal{U}$.*

Theorem 5.5. *For any ring R there is a model structure on $\mathbf{C}(R\text{-Mod})$ where $\mathbf{C}(R\text{-Proj})$ is the class of cofibrant objects.*

Proof. By Theorems 4.5 and 4.7, we have that $(\mathbf{C}(R\text{-Proj}), (\mathbf{C}(R\text{-Proj}))^{\perp})$ and $(\mathbf{C}(R\text{-Proj}) \cap \mathcal{E}, (\mathbf{C}(R\text{-Proj}) \cap \mathcal{E})^{\perp}$ are cotorsions pairs which are cogenerated by sets. So, as noted in the introduction, both pairs are functorially complete. We want to argue that they form a Hovey pair as in Definition 5.2 with $\mathcal{C} = \mathbf{C}(R\text{-Proj})$ and with $\mathcal{D} = (\mathbf{C}(R\text{-Proj}) \cap \mathcal{E})^{\perp}$. So, with this notation, we need that $(\mathbf{C}(R\text{-Proj}))^{\perp} = \mathcal{D} \cap \mathcal{E} = (\mathbf{C}(R\text{-Proj}) \cap \mathcal{E})^{\perp} \cap \mathcal{E}$.

To get this equality we will appeal to Lemma 5.3 with $(\mathbf{C}(R\text{-Proj}), (\mathbf{C}(R\text{-Proj}))^{\perp})$ being the pair $(\mathcal{C}, \mathcal{D}')$ of that lemma and with $(^{\perp}\mathcal{E}, \mathcal{E})$ being the pair $(\mathcal{U}, \mathcal{V})$ of that lemma. This latter pair is complete by Lemma 5.1. It is also hereditary. For, if $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of complexes where E' and E are exact, then E'' is also exact. Since the $P \in^{\perp} \mathcal{E}$ are the DG-projective complexes, we have $\mathcal{U} = ^{\perp} \mathcal{E} \subset \mathbf{C}(R\text{-Proj}) = \mathcal{C}$. Now, applying Lemma 5.3, we get that $\mathcal{D}' = \mathcal{D} \cap \mathcal{V}$. But this equality just says that $\mathcal{D}' = (\mathbf{C}(R\text{-Proj}))^{\perp} = \mathcal{D} \cap \mathcal{V} = (\mathbf{C}(R\text{-Proj}) \cap \mathcal{E})^{\perp} \cap \mathcal{E}$. So, finally, an appeal to [**5**, Theorem 2.2] gives the conclusion about the model structure on $\mathbf{C}(R\text{-Proj})$. $\square$

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