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PARABOLIC SUBGROUPS OF COXETER GROUPS ACTING BY REFLECTIONS ON CAT(0) SPACES

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ABSTRACT. We consider a cocompact discrete reflection group W of a CAT(0) space X. Then W becomes a Coxeter group. In this paper, we study an analogy between the Davis-Moussong complex $\Sigma(W, S)$ and the CAT(0) space X and show several analogous results about the limit set of a parabolic subgroup of the Coxeter group W.

1. Introduction and preliminaries. The purpose of this paper is to study the limit set of a parabolic subgroup of a reflection group of a CAT(0) space. A metric space (X, d) is called a *geodesic space* if for each $x, y \in X$, there exists an isometric embedding $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such a ξ is called a *geodesic*). We say that an isometry r of a geodesic space X is a *reflection* of X, if

(1) r^2 is the identity of X,

(2) $\operatorname{Int} F_r = \varnothing$ for the fixed-point set F_r of r,

(3) $X \setminus F_r$ has exactly two convex components X_r^+ and X_r^-, and

(4) $rX_r^+ = X_r^-$ and $rX_r^- = X_r^+$,

where the fixed-point set F_r of r is called the *wall* of r. Let X_r^+ and X_r^- be the two convex connected components of $X \setminus F_r$, where X_r^+ contains a basepoint of X. An isometry group Γ of a geodesic space X is called a *reflection group*, if some set of reflections of X generates Γ.

Let Γ be a reflection group of a geodesic space X, and let R be the set of all reflections of X in Γ. Now we suppose that the action of Γ on X is proper, that is, $\{\gamma \in \Gamma \mid \gamma x \in B(x, N)\}$ is finite for any $x \in X$ and $N > 0$ (cf. [**2**, page131]). Then the set $\{F_r \mid r \in R\}$ is locally finite. Let C be a component of $X \setminus \bigcup_{r\in R} F_r$, which is called a *chamber*. Then $\Gamma C = X \setminus \bigcup_{r\in R} F_r$, $\Gamma\overline{C} = X$ and for each $\gamma \in \Gamma$, either $C \cap \gamma C = \varnothing$ or

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$C = \gamma C$. We say that Γ is a *cocompact discrete reflection group* of X if $\overline{C}$ is compact and $\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}$. Every Coxeter group is a cocompact discrete reflection group of some CAT(0) space.

A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \quad \text{for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \to \mathbf{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for any $s, t \in S$,

(2) $m(s,s) = 1$ for any $s \in S$, and

(3) $m(s,t) \geq 2$ for any $s, t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system*. Coxeter showed that a group Γ is a finite reflection group of some Euclidean space if and only if Γ is a finite Coxeter group. Every Coxeter system (W, S) induces the Davis-Moussong complex $\Sigma(W, S)$ which is a CAT(0) space ([**5, 6, 14**]). Then the Coxeter group W is a cocompact discrete reflection group of the CAT(0) space $\Sigma(W, S)$. It is known that a group Γ is a cocompact discrete reflection group of some geodesic space if and only if Γ is a Coxeter group ([**12**]).

Let W be a cocompact discrete reflection group of a CAT(0) space X, let R be the set of reflections in W, let C be a chamber, and let S be a *minimal* subset of R such that $C = \bigcap_{s \in S} X_s^+$ (i.e., $C \neq \bigcap_{s \in S \setminus \{s_0\}} X_s^+$ for any $s_0 \in S$). Then $\langle S \rangle C = X \setminus \bigcup_{r \in R} F_r = WC$, S generates W and the pair (W, S) is a Coxeter system ([**12**]). For a subset T of S, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. It is known that the pair (W_T, T) is also a Coxeter system.

Let X be a CAT(0) space, and let Γ be a group which acts properly by isometries on X.

The *limit set of* Γ (*with respect to* X) is defined as

$$L(\Gamma) = \overline{\Gamma x_0} \cap \partial X,$$

where $\overline{\Gamma x_0}$ is the closure of the orbit Γx_0 in $X \cup \partial X$ and x_0 is a point in X. We note that the limit set $L(\Gamma)$ is independent of the point $x_0 \in X$.

Also we say that (the action of) Γ is *convex-cocompact*, if there exists a compact subset K of X such that $\mathbf{R}_{x_0}(L(\Gamma)) \subset \Gamma K$ for some

$x_0 \in X$, where $\mathbf{R}_{x_0}(L(\Gamma))$ is the union of the images of all geodesic rays ξ issuing from x_0 with $\xi(\infty) \in L(\Gamma)$. We note that, for a group acting on a proper CAT(0) space, "convex-cocompactness" agrees with "geometrically finiteness" (cf., [**9, 10**]).

We first prove the following theorem in Section 2.

Theorem 1. *For each subset $T \subset S$,*

(1) *$W_T\overline{C}$ is convex (hence CAT(0)),*

(2) *the limit set $L(W_T)$ of W_T coincides with the boundary $\partial(W_T\overline{C})$, and*

(3) *the action of W_T on X is convex-cocompact.*

This theorem implies the following corollary ([**9, 10**]).

Corollary 2. *For each subset $T \subset S$, the following statements are equivalent*:

(1) $[W : W_T] < \infty$;

(2) $L(W_T) = \partial X$;

(3) $\mathrm{Int}_{\partial X} L(W_T) \neq \varnothing$.

In Section 3, we show the following theorem which is an analogue of Lemma 4.2 in [**8**].

Theorem 3. *Let $x_0 \in C$, and let $w \in W$. Then there exists a reduced representation $w = s_1 \cdots s_l$ such that*

$$d_H([x_0, wx_0], P_{s_1,\dots,s_l}) \le diam\,\overline{C},$$

where d_H is the Hausdorff distance and $P_{s_1,\dots,s_l} = [x_0, s_1x_0] \cup [s_1x_0, (s_1s_2)x_0] \cup \cdots \cup [(s_1 \cdots s_{l-1})x_0, wx_0]$.

Using this theorem, we can obtain the following corollaries by the same argument used in [**8, 11**].

Corollary 4. *For each subset $T \subset S$, the limit set $L(W_T)$ is W-invariant if and only if $W = W_{\widetilde{T}} \times W_{S\setminus\widetilde{T}}$.*

Here $W_{\widetilde{T}}$ is the *essential parabolic subgroup of* W_T (cf. [**8**]), that is, $W_{\widetilde{T}}$ is the minimum parabolic subgroup of finite index in (W_T, T).

We denote by $o(g)$ the order of an element g in the Coxeter group W. For $s_0 \in S$, we define $W^{\{s_0\}} = \{w \in W \mid \ell(ws) > \ell(w)$ for each $s \in S \setminus \{s_0\}\} \setminus \{1\}$. A subset A of a space Y is said to be *dense* in Y, if $\overline{A} = Y$. A subset A of a metric space Y is said to be *quasi-dense*, if there exists $N > 0$ such that each point of Y is N-close to some point of A.

Corollary 5. *Suppose that $W^{\{s_0\}}$ is quasi-dense in W with respect to the word metric and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$. Then there exists $\alpha \in \partial X$ such that the orbit $W\alpha$ is dense in ∂X.*

Corollary 6. *If the set*

$$\bigcup\{W^{\{s\}} \mid s \in S \text{ such that } o(st) = \infty \text{ for some } t \in S\}$$

is quasi-dense in W, then $\{w^\infty \mid w \in W$ such that $o(w) = \infty\}$ is dense in ∂X.

A subset T of S is said to be *spherical*, if W_T is finite.

Corollary 7. *Suppose that there exist a maximal spherical subset T of S and an element $s_0 \in S$ such that $o(s_0t) \geq 3$ for each $t \in T$ and $o(s_0t_0) = \infty$ for some $t_0 \in T$. Then*

(1) *$W\alpha$ is dense in ∂X for some $\alpha \in \partial X$, and*

(2) *$\{w^\infty \mid w \in W$ such that $o(w) = \infty\}$ is dense in ∂X.*

2. Convex-cocompactness of parabolic subgroups. Let W be a cocompact discrete reflection group of a CAT(0) space X, let C be a chamber containing a basepoint of X, let R be the set of reflections in W, and let S be a minimal subset of R such that $C = \bigcap_{s \in S} X_s^+$. (Then the pair (W, S) is a Coxeter system [**12**].) For each reflection r in W, F_r is the wall of r and X_r^+ and X_r^- are the two convex components of $X \setminus F_r$ such that $C \subset X_r^+$ and $C \cap X_r^- = \varnothing$. We note that F_r, $X_r^+ \cup F_r$ and $X_r^- \cup F_r$ are convex.

The following lemmas are known.

Lemma 2.1 [**12**, Lemma 3.4]. *Let $w \in W$, and let $s \in S$. Then $\ell(w) < \ell(sw)$ if and only if $wC \subset X_s^+$.*

Lemma 2.2 [**7**, Lemma 1.3]. *Let $w \in W$, and let $T \subset S$. Then there exists a unique element of shortest length in the coset $W_T w$. Moreover, the following statements are equivalent*:

(1) *w is the element of shortest length in the coset $W_T w$*;

(2) *$\ell(sw) > \ell(w)$ for any $s \in T$*;

(3) *$\ell(vw) = \ell(v) + \ell(w)$ for any $v \in W_T$.*

We first show the following lemma.

Lemma 2.3. *Let $T \subset S$. Then $wX_s^+ = X_{wsw^{-1}}^+$ for any $w \in W_T$ and $s \in S \setminus T$.*

Proof. Let $T \subset S$, $w \in W_T$ and $s \in S \setminus T$. Then $\ell(sw^{-1}) > \ell(w^{-1})$. Hence $w^{-1}C \subset X_s^+$ by Lemma 2.1. Thus $C \subset wX_s^+$, i.e., $wX_s^+ = X_{wsw^{-1}}^+$. □

Using lemmas above, we prove the following theorem.

Theorem 2.4. *For each subset $T \subset S$,*

(1) *$W_T\overline{C}$ is convex* (*hence* CAT(0)),

(2) *the limit set $L(W_T)$ of W_T coincides with the boundary $\partial(W_T\overline{C})$, and*

(3) *the action of W_T on X is convex-cocompact.*

Proof. Let $T \subset S$. Then we show that

$$W_T\overline{C} = \bigcap\{\overline{X_{wsw^{-1}}^+} \mid w \in W_T,\ s \in S \setminus T\}.$$

For each $v, w \in W_T$ and $s \in S \setminus T$, $C \subset v^{-1}wX_s^+$ by Lemma 2.3. Hence $vC \subset wX_s^+ = X_{wsw^{-1}}^+$ by Lemma 2.3. Thus $v\overline{C} \subset \overline{X}_{wsw^{-1}}^+$ for any $v, w \in W_T$ and $s \in S \setminus T$, that is,

$$W_T\overline{C} \subset \bigcap\{\overline{X_{wsw^{-1}}^+} \mid w \in W_T,\ s \in S \setminus T\}.$$

To prove

$$W_T\overline{C} \supset \bigcap\{\overline{X^+_{wsw^{-1}}} \mid w \in W_T,\ s \in S \setminus T\},$$

we show that for each $v \in W \setminus W_T$, there exist $w \in W_T$ and $s \in S \setminus T$ such that $vC \subset X^-_{wsw^{-1}}$. Let $v \in W \setminus W_T$. By Lemma 2.2, there exists a unique element $x \in W_Tv$ of shortest length. Let $w = vx^{-1}$. Here we note that $w \in W_T$ and $\ell(v) = \ell(w) + \ell(x)$. Let $s \in S$ such that $\ell(sx) < \ell(x)$. By Lemma 2.2 (2), $s \in S \setminus T$. Then

$$\ell(sw^{-1}v) = \ell(sx) < \ell(x) = \ell(w^{-1}v).$$

Hence $w^{-1}vC \subset X^-_s$ by Lemma 2.1. By Lemma 2.3, $vC \subset wX^-_s = X^-_{wsw^{-1}}$. Therefore,

$$W_T\overline{C} = \bigcap\{\overline{X^+_{wsw^{-1}}} \mid w \in W_T,\ s \in S \setminus T\}.$$

Since $\overline{X^+_{wsw^{-1}}} = X^+_{wsw^{-1}} \cup F_{wsw^{-1}}$ is convex for any $w \in W_T$ and $s \in S \setminus T$, $W_T\overline{C}$ is convex. Hence $L(W_T) = \partial(W_T\overline{C})$ and the action of W_T on X is convex-cocompact. □

3. On geodesics and reduced representations. We give the following lemma which is an analogue of a result about Davis-Moussong complexes.

Lemma 3.1. *Let $w \in W$, let $w = s_1 \cdots s_l$ be a reduced representation, and let $T = \{s_1, \ldots, s_l\}$. Then*

$$\overline{C} \cap w\overline{C} = \bigcap_{t \in T}(F_t \cap \overline{C}) = \bigcap_{t \in T}(t\overline{C} \cap \overline{C}) = \bigcap_{v \in W_T} v\overline{C}.$$

Proof. Let $y \in \overline{C} \cap w\overline{C}$. Since $\ell(s_1w) < \ell(w)$, $wC \subset X^-_{s_1}$ by Lemma 2.1. Then

$$y \in \overline{C} \cap w\overline{C} \subset \overline{X^+_{s_1}} \cap \overline{X^-_{s_1}} = F_{s_1}.$$

Hence $s_1 y = y$ and

$$y = s_1 y \in s_1(\overline{C} \cap w\overline{C}) = s_1\overline{C} \cap (s_2 \cdots s_l)\overline{C},$$

i.e., $y \in \overline{C} \cap (s_2 \cdots s_l)\overline{C}$. By iterating the above argument, $s_i y = y$ for any $i \in \{1, \ldots, l\}$, that is, $ty = y$ for any $t \in T$. Hence $y \in \bigcap_{t \in T}(F_t \cap \overline{C})$. Thus $\overline{C} \cap w\overline{C} \subset \bigcap_{t \in T}(F_t \cap \overline{C})$.

Since $F_t \cap \overline{C} = t\overline{C} \cap \overline{C}$ for any $t \in T$, $\bigcap_{t \in T}(F_t \cap \overline{C}) = \bigcap_{t \in T}(t\overline{C} \cap \overline{C})$.

Let $y \in \bigcap_{t \in T}(F_t \cap \overline{C})$. Then $ty = y$ for any $t \in T$. Since T generates W_T, $vy = y$ for any $v \in W_T$. Hence $y = vy \in v\overline{C}$ for each $v \in W_T$. Thus $\bigcap_{t \in T}(F_t \cap \overline{C}) \subset \bigcap_{v \in W_T} v\overline{C}$.

It is obvious that $\bigcap_{v \in W_T} v\overline{C} \subset \overline{C} \cap w\overline{C}$, since $1, w \in W_T$. □

Lemma 3.2. *Let $w \in W$, let $w = s_1 \cdots s_l$ be a reduced representation, and let $T = \{s_1, \ldots, s_l\}$. Then $\overline{C} \cap w\overline{C} \neq \varnothing$ if and only if W_T is finite.*

Proof. Suppose that $\overline{C} \cap w\overline{C} \neq \varnothing$. Then $\bigcap_{v \in W_T} v\overline{C} \neq \varnothing$ by Lemma 3.1. Hence W_T is finite because the action of W on X is proper.

Suppose that W_T is finite. Then W_T acts on the CAT(0) space $W_T\overline{C}$ by Theorem 2.4. By [**2**, Corollary II.2.8(1)], there exists a fixed-point $y \in W_T\overline{C}$ such that $vy = y$ for any $v \in W_T$. Then $y \in \overline{C} \cap w\overline{C}$ which is non-empty. □

By the proof of [**8**, Lemma 4.2], we can obtain the following theorem from Lemma 2.1, 3.1 and 3.2.

Theorem 3.3. *Let $x_0 \in C$, and let $w \in W$. Then there exists a reduced representation $w = s_1 \cdots s_l$ such that*

$$d_H([x_0, wx_0], P_{s_1, \ldots, s_l}) \leq \operatorname{diam} \overline{C},$$

where d_H is the Hausdorff distance and $P_{s_1, \ldots, s_l} = [x_0, s_1x_0] \cup [s_1x_0, (s_1s_2)x_0] \cup \cdots \cup [(s_1 \cdots s_{l-1})x_0, wx_0]$.

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