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THE NUMBER OF SPANNING TREES IN SOME CLASSES OF GRAPHS

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ABSTRACT. In this paper, using properties of Chebyshev polynomials, we give explicit formulas for the number of spanning trees in some classes of graphs, including join of graphs, Cartesian product of graphs and nearly regular graphs.

1. Introduction. We use the terminology of Bondy and Murty [**4**]. All graphs in this paper are finite, undirected, and simple (i.e., without loops or multiple edges). We denote by $\tau(G)$ the number of spanning trees of a graph G.

A famous and classical result on the study of $\tau(G)$ is the following theorem, known as the *Matrix Tree theorem* [**9**]. But this theorem is not feasible for large graphs. The *Laplacian matrix* (also called *Kirchhoff matrix*) of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the degree matrix and the adjacency matrix of G, respectively.

Theorem 1.1. *For every connected graph G, $\tau(G)$ is equal to any cofactor of $L(G)$.*

The characteristic polynomial of a graph G is $\mathcal{P}_G(\lambda) = \det(\lambda I - A(G))$. Also we define $\mathcal{C}_G(\lambda) = \det(\lambda I - L(G))$.

The number of spanning trees of a connected graph G can be expressed in terms of the eigenvalues of $L(G)$. Since, by the definition, $L(G)$ is a real symmetric matrix, it therefore has n non-negative real eigenvalues, of which n is the number of vertices of G. In [**1**, Theorem 1], Anderson and Morley proved that the multiplicity of 0 as an eigenvalue of $L(G)$ equals the number of components of G. Therefore, the

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Laplacian matrix of a connected graph G has 0 as an eigenvalue with multiplicity one.

Theorem 1.2 [**6**]. *Suppose G is a connected graph with n vertices. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $L(G)$, with $\lambda_n = 0$. Then*

$$\tau(G) = \frac{(-1)^{n-1}}{n}\mathcal{C}'_G(\lambda)|_{\lambda=0} = \frac{1}{n}\lambda_1 \cdots \lambda_{n-1}.$$

Example 1.3. Consider the path P_n and the cycle C_n. It is known that the eigenvalues of $L(P_n)$ and $L(C_n)$ are $2 - 2\cos{(k\pi)}/n$ $(0 \le k \le n-1)$ and $2 - 2\cos{(2k\pi)}/n$ $(0 \le k \le n-1)$, respectively (see, e.g., [**2, 5**]). On the other hand, we know that $\tau(P_n) = 1$ and $\tau(C_n) = n$; therefore, by using Theorem 1.2, we obtain the well-known identities:

$$\prod_{k=1}^{n-1}\left(2 - 2\cos\frac{k\pi}{n}\right) = n \Longrightarrow \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}, \quad n \ge 2, \tag{1}$$

and

$$\prod_{k=1}^{n-1}\left(2 - 2\cos\frac{2k\pi}{n}\right) = n^2 \Longrightarrow \prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}, \quad n \ge 2. \tag{2}$$

There are many ways of combining graphs to produce new graphs. We now describe some binary operations defined on graphs.

The *union* of graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $H(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a *disjoint union*, denoted by $G + H$. The *join* of two graphs G and H, $G \vee H$, is obtained from the disjoint union of G and H by additionally joining every vertex of G to every vertex of H.

The join $W_n = C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with *n spokes*. Similarly, the join $\mathcal{F}_n = P_n \vee K_1$ of a path P_n and a single vertex is called a *fan*.

The *Cartesian product* of graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. The notation used for the Cartesian product reflects this fact. The Cartesian product $P_m \square P_n$ of two paths is the $(m \times n)$-*grid*. Also the Cartesian product $P_2 \square P_n$ $(n \geq 2)$ is called a *ladder*, and $P_2 \square C_n$ $(n \geq 3)$ is referred to as an n-*prism*.

In the next section, we review some properties of the well-known Chebyshev polynomials and then state some theorems that allow us to evaluate the number of spanning trees in join of graphs, Cartesian product of graphs and nearly regular graphs. Recall that a graph G is called *nearly k-regular* if all its vertices except one (referred to as an *exceptional vertex*) have degree k.

2. Joins and Cartesian products. The starting point of our calculations is the following theorem.

Theorem 2.1 [**6**]. *Suppose $G_1, \ldots, G_k$, are graphs of order $n_1, \ldots, n_k$, respectively, and let $n_1 + \cdots + n_k = n$. For the disjoint union $G_1 + \cdots + G_k$ and the join $G_1 \vee \cdots \vee G_k$, we have*:

$$\mathcal{C}_{G_1+\cdots+G_k}(\lambda) = \prod_{i=1}^{k} \mathcal{C}_{G_i}(\lambda),$$

$$\mathcal{C}_{G_1\vee\cdots\vee G_k}(\lambda) = \lambda(\lambda - n)^{k-1} \prod_{i=1}^{k} \frac{\mathcal{C}_{G_i}(\lambda - n + n_i)}{\lambda - n + n_i}.$$

Now, by applying Theorems 1.2 and 2.1 we evaluate the number of spanning trees of the complete multipartite (or complete k-partite) graph $K_{n_1,\ldots,n_k}$, which is the main result of [**11**] and also studied in [**12**].

Theorem 2.2. *The number of spanning trees in the complete multipartite graph $K_{n_1,\ldots,n_k}$ of order n is equal to*:

$$\tau(K_{n_1,\ldots,n_k}) = n^{k-2} \prod_{i=1}^{k} (n - n_i)^{n_i - 1}.$$

Proof. Let $\mathcal{N}_m$ denote the empty graph of order m. Since $\mathcal{N}_m$ is the disjoint union of m copies of a single vertex, therefore $\mathcal{C}_{\mathcal{N}_m}(\lambda) = \lambda^m$. The complete multipartite graph $K_{n_1,\dots,n_k}$ is the join of graphs $\mathcal{N}_{n_1},\dots,\mathcal{N}_{n_k}$. Now, Theorem 2.1 implies that

$$\mathcal{C}_{K_{n_1,\dots,n_k}}(\lambda) = \lambda(\lambda - n)^{k-1}\prod_{i=1}^{k}(\lambda - n + n_i)^{n_i-1}.$$

Therefore, by Theorem 1.2,

$$\begin{aligned}\tau(K_{n_1,\dots,n_k}) &= \frac{(-1)^{n-1}}{n}\mathcal{C}'_{K_{n_1,\dots,n_k}}(\lambda)|_{\lambda=0}\\ &= \frac{(-1)^{n-1}}{n}(\lambda - n)^{k-1}\prod_{i=1}^{k}(\lambda - n + n_i)^{n_i-1}|_{\lambda=0}\\ &= n^{k-2}\prod_{i=1}^{k}(n - n_i)^{n_i-1}. \qquad \square\end{aligned}$$

Now, we review the properties of the Chebyshev polynomials (taken from [**8**]) that help us to derive explicit formulas for the number of spanning trees in some other classes of graphs.

The function $\cos n\theta$ is a Chebyshev polynomial function of $\cos\theta$. Specifically, for $n \geq 0$, $\cos n\theta = T_n(\cos\theta)$, where T_n is the *Chebyshev polynomial of the first kind*, defined by $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

If we change the initial conditions to be $U_0(x) = 1$ and $U_1(x) = 2x$, but keep the same recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

we get the *Chebyshev polynomials of the second kind.*

It is easy to show that, for all $n \geq 0$, $T_n(1) = 1$ and $U_n(1) = n + 1$, $T_n(-1) = (-1)^n$, $U_n(-1) = (-1)^n(n + 1)$.

Here we list a few intriguing identities satisfied by the Chebyshev polynomials:

(3)
$$T_n(x) = \frac{1}{2}((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n),$$

(4)
$$T_n(-x) = (-1)^n T_n(x),$$

(5)
$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}((x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}), \quad |x| \neq 1,$$

(6)
$$U_n(-x) = (-1)^n U_n(x),$$

(7)
$$U_n(x) = \prod_{k=1}^{n} \left(2x \pm 2\cos\frac{k\pi}{n+1}\right),$$

(8)
$$T_n(x) = U_n(x) - xU_{n-1}(x).$$

The lemma below gives us the characteristic polynomial of the path P_n and the cycle C_n in terms of Chebyshev polynomials.

Lemma 2.3 [**6**]. *For the path P_n, the cycle C_n, and the complete graph K_n, we have*:

$$\mathcal{P}_{P_n}(\lambda) = U_n\left(\frac{\lambda}{2}\right), \tag{9}$$

$$\mathcal{P}_{C_n}(\lambda) = 2\left(T_n\left(\frac{\lambda}{2}\right) - 1\right), \tag{10}$$

$$\mathcal{P}_{K_n}(\lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}. \tag{11}$$

Suppose G is a k-regular graph of order n. It is easy to see that

$$\mathcal{C}_G(\lambda) = (-1)^n \mathcal{P}_G(k-\lambda).$$

Thus, by using the lemma above we can evaluate $\mathcal{C}_{C_n}(\lambda)$ and $\mathcal{C}_{K_n}(\lambda)$. The eigenvalues of $L(P_n)$, as we have mentioned, are $2 - 2\cos(k\pi)/n$ $(0 \leq k \leq n-1)$; then, by applying (7), $\mathcal{C}_{P_n}(\lambda)$ also follows.

Lemma 2.4. *For the path P_n, the cycle C_n, and the complete graph K_n, we have*:

$$\mathcal{C}_{P_n}(\lambda) = \lambda U_{n-1}\left(\frac{\lambda - 2}{2}\right), \tag{12}$$

$$\mathcal{C}_{C_n}(\lambda) = 2\left(T_n\left(\frac{\lambda - 2}{2}\right) - (-1)^n\right), \tag{13}$$

$$\mathcal{C}_{K_n}(\lambda) = \lambda(\lambda - n)^{n-1}. \tag{14}$$

Now, we calculate the number of spanning trees in some special graphs.

Theorem 2.5.

$$\tau(K_m \vee P_n) = (m+n)^{m-1} U_{n-1}\left(\frac{m+2}{2}\right).$$

Proof. By Theorem 2.1 and Lemma 2.4:

$$\mathcal{C}_{K_m \vee P_n}(\lambda) = \lambda(\lambda - m - n)^m U_{n-1}\left(\frac{\lambda - m - 2}{2}\right).$$

Now applying Theorem 1.2 gives:

$$\begin{aligned}\tau(K_m \vee P_n) &= \frac{(-1)^{m+n-1}}{m+n}\mathcal{C}'_{K_m \vee P_n}(\lambda)|_{\lambda=0} \\ &= \frac{(-1)^{m+n-1}}{m+n}(\lambda - m - n)^m U_{n-1}\left(\frac{\lambda - m - 2}{2}\right)\Bigg|_{\lambda=0} \\ &= (m+n)^{m-1} U_{n-1}\left(\frac{m+2}{2}\right). \qquad \square\end{aligned}$$

By similar calculations, we can enumerate the number of spanning trees in some more cases:

Theorem 2.6.

$$\begin{aligned}
\tau(K_m \vee C_n) &= \frac{2}{m}(m+n)^{m-1}\left(T_n\left(\frac{m+2}{2}\right)-1\right),\\
\tau(P_m \vee C_n) &= \frac{2}{m}U_{m-1}\left(\frac{n+2}{2}\right)\left(T_n\left(\frac{m+2}{2}\right)-1\right),\\
\tau(P_m \vee P_n) &= U_{m-1}\left(\frac{n+2}{2}\right)U_{n-1}\left(\frac{m+2}{2}\right),\\
\tau(C_m \vee C_n) &= \frac{4}{mn}\left(T_m\left(\frac{n+2}{2}\right)-1\right)\left(T_n\left(\frac{m+2}{2}\right)-1\right).
\end{aligned}$$

Proof. To prove the first formula, by Theorem 2.1 and Lemma 2.4, we have:

$$\mathcal{C}_{K_m \vee C_n}(\lambda) = \frac{2\lambda}{\lambda - m}(\lambda - m - n)^m\left(T_n\left(\frac{\lambda - m - 2}{2}\right) - (-1)^n\right)$$

Now applying Theorem 1.2 gives:

$$\begin{aligned}
\tau(K_m \vee C_n) &= \frac{(-1)^{m+n-1}}{m+n}\mathcal{C}'_{K_m \vee C_n}(\lambda)|_{\lambda=0}\\
&= \frac{2(-1)^{m+n-1}}{(m+n)(\lambda-m)}(\lambda-m-n)^m\\
&\quad \times \left(T_n\left(\frac{\lambda-m-2}{2}\right)-(-1)^n\right)\Bigg|_{\lambda=0}\\
&= \frac{2}{m}(m+n)^{m-1}\left(T_n\left(\frac{m+2}{2}\right)-1\right).
\end{aligned}$$

In order to prove the second formula, by Theorem 2.1 and Lemma 2.4, again, we have:

$$\begin{aligned}
\mathcal{C}_{P_m \vee C_n}(\lambda) &= \frac{2\lambda(\lambda-m-n)}{\lambda-m}U_{m-1}\left(\frac{\lambda-n-2}{2}\right)\\
&\quad \times \left(T_n\left(\frac{\lambda-m-2}{2}\right)-(-1)^n\right).
\end{aligned}$$

Now applying Theorem 1.2 gives:

$$\begin{aligned}\tau(P_m \vee C_n) &= \frac{(-1)^{m+n-1}}{m+n}\mathcal{C}'_{P_m \vee C_n}(\lambda)|_{\lambda=0} \\ &= \frac{2(-1)^{m+n-1}(\lambda-m-n)}{(m+n)(\lambda-m)}U_{m-1}\Big(\frac{\lambda-n-2}{2}\Big) \\ &\quad \times \Big(T_n\Big(\frac{\lambda-m-2}{2}\Big)-(-1)^n\Big)\Big|_{\lambda=0} \\ &= \frac{2}{m}U_{m-1}\Big(\frac{n+2}{2}\Big)\Big(T_n\Big(\frac{m+2}{2}\Big)-1\Big).\end{aligned}$$

The proofs of the other formulas are similar. □

By the same method, we get $\tau(K_m \vee K_n) = (m+n)^{m+n-2}$, which is nothing but Cayley's formula.

Our machinery gives the formulas in the corollary below which have also appeared in [**3**].

Corollary 2.7. *The number of spanning trees of fan $\mathcal{F}_n$ and wheel W_n are*:

$$\begin{aligned}\tau(\mathcal{F}_n) &= U_{n-1}\Big(\frac{3}{2}\Big) \\ &= F_{2n} = \frac{1}{\sqrt{5}}\Big(\Big(\frac{3+\sqrt{5}}{2}\Big)^n - \Big(\frac{3-\sqrt{5}}{2}\Big)^n\Big), \\ \tau(W_n) &= 2\Big(T_n\Big(\frac{3}{2}\Big)-1\Big) \\ &= L_{2n}-2 = \Big(\frac{3+\sqrt{5}}{2}\Big)^n + \Big(\frac{3-\sqrt{5}}{2}\Big)^n - 2,\end{aligned}$$

where F_n and L_n denote the Fibonacci and Lucas numbers, respectively. That is, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$ with $F_1 = F_2 = 1$, and $L_{n+2} = L_{n+1} + L_n$, for $n \geq 1$ with $L_1 = 1$ and $L_2 = 3$.

Proof. By Theorem 2.6, we have:

$$\begin{aligned}\tau(\mathcal{F}_n) &= \tau(P_n \vee P_1)\\ &= U_0\left(\frac{n+2}{2}\right)U_{n-1}\left(\frac{3}{2}\right) = U_{n-1}\left(\frac{3}{2}\right) = F_{2n},\\ \tau(W_n) &= \tau(C_n \vee P_1)\\ &= 2U_0\left(\frac{n+2}{2}\right)\left(T_n\left(\frac{3}{2}\right)-1\right)\\ &= 2\left(T_n\left(\frac{3}{2}\right)-1\right) = L_{2n}-2. \qquad \square\end{aligned}$$

Now, we study the number of spanning trees in Cartesian products of graphs. The key theorem here is the following.

Theorem 2.8 [**2**]. *The Laplacian eigenvalues of the Cartesian product $G\,\square\,H$, are precisely the numbers*

$$\lambda_i(G) + \lambda_j(H),$$

for $i = 1, 2, \ldots, |V(G)|$ *and* $j = 1, 2, \ldots, |V(H)|$.

Now we get the number of spanning trees of the *complete prism* $K_n \,\square\, P_m$.

Theorem 2.9. *For any* $m, n \geq 2$,

$$\tau(K_n \,\square\, P_m) = n^{n-2}\left(U_{m-1}\left(\frac{n+2}{2}\right)\right)^{n-1}.$$

Proof. Since the eigenvalues of $L(K_n)$ by Lemma 2.4 are $0, n, n, \ldots, n$, and the eigenvalues of $L(P_m)$ are $2 - 2\cos(k\pi)/m$ $(0 \leq k \leq m-1)$, therefore by Theorems 1.2 and 2.8,

$$\tau(K_n \,\square\, P_m) = \frac{1}{mn}\prod_{k=1}^{m-1}\left(2 - 2\cos\frac{k\pi}{m}\right)\left(\prod_{k=0}^{m-1}\left(n+2-2\cos\frac{k\pi}{m}\right)\right)^{n-1}.$$

By identity (1),

$$\prod_{k=1}^{m-1}\left(2-2\cos\frac{k\pi}{m}\right)=m.$$

Now, applying (7) implies the theorem. $\square$

Similarly, we obtain the number of spanning trees of the $(m\times n)$-grid $P_m\,\square\,P_n$, and *complete cyclic prism* $K_n\,\square\,C_m$.

Theorem 2.10.

$$\tau(P_m\,\square\,P_n)=4^{(m-1)(n-1)}\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}\left(\sin^2\frac{i\pi}{2m}+\sin^2\frac{j\pi}{2n}\right),$$

$$\tau(C_m\,\square\,C_n)=mn4^{(m-1)(n-1)}\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}\left(\sin^2\frac{i\pi}{m}+\sin^2\frac{j\pi}{n}\right),$$

$$\tau(P_m\,\square\,C_n)=n4^{(m-1)(n-1)}\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}\left(\sin^2\frac{i\pi}{2m}+\sin^2\frac{j\pi}{n}\right),$$

$$\tau(K_m\,\square\,K_n)=m^{m-2}n^{n-2}(m+n)^{(m-1)(n-1)},$$

$$\tau(K_n\,\square\,C_m)=\frac{m2^{n-1}}{n}\left(T_m\left(\frac{n+2}{2}\right)-1\right)^{n-1}.$$

Proof. The eigenvalues of $L(K_n)$, by Lemma 2.4, are $0,n,n,\dots,n$, and the eigenvalues of $L(P_m)$ and $L(C_m)$ are $2-2\cos(k\pi/m)$ $(0\le k\le m-1)$ and $2-2\cos(2k\pi/m)$ $(0\le k\le m-1)$, respectively. Therefore, by direct application of Theorems 1.2 and 2.8 and identities (1) and (2), the proofs of these formulas follow easily. $\square$

The first and latter formulas also appeared in [**10, 3**], respectively.

We now derive the number of spanning trees of the ladder $P_2\,\square\,P_n$, and the n-prism $P_2\,\square\,C_n$, which was also proved in [**3**].

Corollary 2.11. *The number of spanning trees of the ladder $P_2\,\square\,P_n$ and the n-prism $P_2\,\square\,C_n$ are*:

$$\tau(P_2\,\square\,P_n)=U_{n-1}(2)=\frac{\sqrt{3}}{6}((2+\sqrt{3})^n-(2-\sqrt{3})^n),$$

$$\tau(P_2\,\square\,C_n)=n(T_n(2)-1)=\frac{n}{2}((2+\sqrt{3})^n+(2-\sqrt{3})^n-2).$$

Proof. Using Theorem 2.10 we have:

$$\begin{aligned}\tau(P_2 \,\square\, P_n) &= 4^{n-1}\prod_{j=1}^{n-1}\left(\frac{1}{2}+\sin^2\frac{j\pi}{2n}\right) = U_{n-1}(2)\\ &= \frac{\sqrt{3}}{6}((2+\sqrt{3})^n-(2-\sqrt{3})^n),\\ \tau(P_2 \,\square\, C_n) &= \tau(K_2 \,\square\, C_n) = n(T_n(2)-1)\\ &= \frac{n}{2}((2+\sqrt{3})^n+(2-\sqrt{3})^n-2). \qquad \square\end{aligned}$$

3. Nearly regular graphs. In this section, we prove a theorem for enumerating the number of spanning trees in nearly regular graphs. First, we present a theorem for k-regular graphs.

Theorem 3.1 [**6**]. *Suppose G is a connected k-regular graph with n vertices. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G, with $\lambda_n = k$. Then*

$$\tau(G) = \frac{1}{n}\prod_{i=1}^{n-1}(k-\lambda_i) = \frac{1}{n}\mathcal{P}'_G(k).$$

Theorem 3.2. *Suppose G is a connected nearly k-regular graph. Then*

$$\tau(G) = \mathcal{P}_H(k),$$

where H is the subgraph of G obtained by removing the exceptional vertex.

Proof. By the matrix tree theorem, $\tau(G)$ is equal to any cofactor of $L(G)$. Now we take the cofactor of the diagonal element corresponding to the exceptional vertex of G. Hence, the theorem follows. $\square$

Example 3.3. A wheel W_n is a nearly 3-regular graph. If we remove the exceptional vertex (called the *hub*), we obtain the cycle C_n. The characteristic polynomial of the cycle C_n, by Lemma 2.3, is

$$\mathcal{P}_{C_n}(\lambda) = 2\left(T_n\left(\frac{\lambda}{2}\right)-1\right).$$

Therefore, by Theorem 3.2,

$$\tau(W_n) = 2\left(T_n\left(\frac{3}{2}\right) - 1\right),$$

as we already obtained.

Let G be a plane graph. Denote its *dual* by G^*.

Lemma 3.4 [**7**, Lemma 14.3.3]. *Let G be a connected plane graph. Then the graphs G and G^* have the same number of spanning trees.*

Example 3.5. Consider the fan $\mathcal{F}_n$. Replace any edge on the *rim* by the path P_{k+1} ($k \geq 1$), and denote the graph obtained by $\mathcal{F}_{n,k}$. The dual $\mathcal{F}^*_{n,k}$ is nearly $(k+2)$-regular. If we remove the exceptional vertex of $\mathcal{F}^*_{n,k}$, then we obtain the path P_{n-1}. The characteristic polynomial of the path P_n by Lemma 2.3 is

$$\mathcal{P}_{P_n}(\lambda) = U_n\left(\frac{\lambda}{2}\right).$$

Consequently, by Theorem 3.2 and Lemma 3.4,

$$\tau(\mathcal{F}_{n,k}) = \tau(\mathcal{F}^*_{n,k}) = U_{n-1}\left(\frac{k+2}{2}\right).$$

Example 3.6. Consider the wheel W_n. Replace any edge on the rim by the path P_{k+1} ($k \geq 1$), and denote the graph obtained by $W_{n,k}$. The dual $W^*_{n,k}$ is nearly $(k+2)$-regular. If we remove the exceptional vertex of $W^*_{n,k}$, then we obtain the cycle C_n. Similar to the example above,

$$\tau(W_{n,k}) = \tau(W^*_{n,k}) = 2\left(T_n\left(\frac{k+2}{2}\right) - 1\right).$$

Example 3.7. Place n k-gons in a row, such that each two consecutive k-gons have a side in common. Denote this graph by $G_{n,k}$.

The dual $G^*_{n,k}$ is nearly k-regular. If we remove the exceptional vertex of $G^*_{n,k}$, then we obtain the path P_n. As above,

$$\tau(G_{n,k}) = \tau(G^*_{n,k}) = U_n\left(\frac{k}{2}\right).$$

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