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A DENSITY CONDITION FOR INTERPOLATION ON THE HEISENBERG GROUP

BRADLEY CURREY AND AZITA MAYELI

ABSTRACT. We consider left invariant multiplicity free subspaces of $L^2(N)$ where N is the Heisenberg group. We prove a necessary and sufficient density condition in order that such subspaces possess the interpolation property with respect to a class of discrete subsets of N that includes the integer lattice. We exhibit a concrete example of a subspace that has interpolation for the integer lattice, and we also prove a necessary and sufficient condition for shift invariant subspaces to possess a singly-generated orthonormal basis of translates.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space of continuous functions on a topological space X for which point evaluation $f \mapsto f(x)$ is continuous, let Γ be a countable discrete subset of X, and let p be the restriction mapping $f \mapsto f|_\Gamma$ on $\mathcal{H}$. We say that $(\mathcal{H}, \Gamma)$ is a (Parseval) sampling pair if p is a constant multiple of an isometry of $\mathcal{H}$ into $\ell^2(\Gamma)$, and if p is surjective then we say that $(\mathcal{H}, \Gamma)$ has the interpolation property. In the present work we are interested in sufficient conditions in order that a pair $(\mathcal{H}, \Gamma)$ has the interpolation property, where $\mathcal{H}$ is a Hilbert space of functions on the Heisenberg group. Sampling has been studied in related settings in [**2, 3, 5**]. In [**5**] the author obtains sampling sets for Paley-Wiener functions on stratified Lie groups, while some of the results in [**2**] provide a characterization of Parseval sampling pairs for left invariant subspaces of $L^2(G)$ where G is any locally compact unimodular Type I group, in terms of the notion of admissibility. The work of [**3**] also provides quite general sufficient conditions for sampling and includes ideas from both of the preceding articles. The statement made in [**2**, page 47, also Theorem 6.4 (ii)] that, on the Heisenberg group, left-invariant sampling spaces associated to lattices never have the interpolation property, is particularly pertinent to the present work.

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Here we consider the interpolation property for a class of quasi-lattices $\Gamma_{\alpha,\beta}, \alpha, \beta > 0$ in the Heisenberg group N that we also consider in [**1**]. Let $\mathcal{H}$ be a left invariant subspace of $L^2(N)$ that is multiplicity free: the group Fourier transform of functions in $\mathcal{H}$ have rank at most one. In an explicit version of the group Plancherel transform, the dual $\widehat{N}$ is almost everywhere identified with $\mathbf{R} \setminus \{0\}$, the Plancherel measure on $\widehat{N}$ becomes $d\mu = |\lambda|\, d\lambda$, and there is a measurable subset E of $\mathbf{R} \setminus \{0\}$ such that $\mathcal{H}$ is naturally identified with $L^2(E \times \mathbf{R})$. With this identification, a left translation system $\{T_\gamma \psi : \gamma \in \Gamma_{\alpha,\beta}\}$ where $\psi \in \mathcal{H}$ becomes a *field* over E of Gabor systems in $L^2(\mathbf{R})$. We use this identification to show that $(\mathcal{H}, \Gamma_{\alpha,\beta})$ has the interpolation property exactly when the $\mu(E) = 1/\alpha\beta$.

The paper is organized as follows: after introducing some preliminaries, in Section 1 we collect relevant results from [**2, 3**] concerning admissibility and sampling for left invariant subspaces of $L^2(G)$, where G is unimodular and type I. The point is that admissibility is necessary for sampling, and (Theorem 1.5) a left invariant subspace is sampling if and only if it is admissible and its convolution projection generates a tight frame. We also recall the general fact that such a subspace has interpolation property if the aforementioned Parseval frame is actually orthonormal. In Section 2 we specialize to the Heisenberg group N and direct our attention to the interpolation property for multiplicity free subspaces with respect to the discrete subsets $\Gamma_{\alpha,\beta}$. In Theorem 2.4 we characterize such spaces that have the interpolation property by the density condition described above. In Example 2.6 we exhibit a multiplicity free subspace of $L^2(N)$ that has the interpolation property with respect to the sampling set $\Gamma_{1,1}$, which is in fact a lattice in N, thereby providing a counterexample to the aforementioned statement in [**2**]. Finally, with Theorem 2.8, we prove a necessary and sufficient condition for any shift invariant subspace to have the interpolation property.

1. Sampling spaces for unimodular groups. Let G be a locally compact unimodular, topological group that is type I and choose a Haar measure dx on G. For each $x \in G$, let T_x be the unitary left translation operator on $L^2(G)$. Let $\widehat{G}$ be the unitary dual of G, the set of equivalence classes of continuous unitary irreducible representations of G, endowed with hull-kernel topology and the Plancherel measure, μ. As is well known, for each $\lambda \in \widehat{G}$, there is a continuous unitary irreducible representation π_λ belonging to λ, acting in a Hilbert space $\mathcal{L}_\lambda$ with the following properties.

(1) For each $\phi \in L^1(G) \cap L^2(G)$, the weak operator integral

$$\pi_\lambda(\phi) := \int_N \phi(x)\pi_\lambda(x)\, dx$$

defines a trace-class operator on $\mathcal{L}_\lambda$.

(2) The group Fourier transform

$$\mathcal{F} : L^1(G) \cap L^2(G) \longrightarrow \int_{\hat{G}}^{\oplus} \mathcal{HS}(\mathcal{L}_\lambda)\, d\mu(\lambda)$$

defined by $\phi \mapsto \{\pi_\lambda(\phi)\} := \{\widehat{\phi}(\lambda)\}_{\lambda\in\widehat{G}}$ satisfies $\|\mathcal{F}(\phi)\| = \|\phi\|_2$ and has dense range. (Here $\mathcal{HS}(\mathcal{L}_\lambda)$ denotes the Hilbert space of Hilbert-Schmidt operators on $\mathcal{L}_\lambda$.)

(3) For each $x \in G$, $\mathcal{F}(T_x\phi) = \pi_\lambda(x)\widehat{\phi}(\lambda)$, $\lambda \in \widehat{G}$.

A closed subspace $\mathcal{H}$ of $L^2(G)$ is said to be *left invariant* if $T_x(\mathcal{H}) \subset \mathcal{H}$ holds for all $x \in G$. Let $\mathcal{H}$ be a left invariant subspace of $L^2(G)$, and let $P : L^2(G) \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}$. Then there is a unique (up to μ-almost everywhere equality) measurable field $\{\widehat{P}_\lambda\}_{\lambda\in\widehat{G}}$ of orthogonal projections where $\widehat{P}_\lambda$ is defined on $\mathcal{L}_\lambda$, and so that

$$\widehat{(P\phi)}(\lambda) = \widehat{\phi}(\lambda)\widehat{P}_\lambda$$

holds for μ-almost everywhere $\lambda \in \widehat{G}$. Set $m_{\mathcal{H}}(\lambda) = \operatorname{rank}(\widehat{P}_\lambda)$. Then the spectrum of $\mathcal{H}$ is the set $\Sigma(\mathcal{H}) = \operatorname{supp}(m_{\mathcal{H}})$. A left invariant subspace $\mathcal{H}$ of $L^2(G)$ is said to be multiplicity free if $m_{\mathcal{H}}(\lambda) \le 1$ almost everywhere: if $\mathcal{H}$ is left invariant and $m_{\mathcal{H}}(\lambda) = 1$ almost everywhere, then we will say that $\mathcal{H}$ is multiplicity one.

Recall that $\psi \in \mathcal{H}$ is said to be admissible (with respect to the left regular representation) if the operator V_ψ defined by $V_\psi(\phi) = \phi * \psi^*$ defines an isometry of $\mathcal{H}$ into $L^2(G)$. For convenience we recall [**2**, Theorem 4.22] for unimodular groups.

Theorem 1.1. *Let $\mathcal{H}$ be a closed left invariant subspace of $L^2(G)$ with the associated measurable projection field $\{\widehat{P}_\lambda\}$. Then $\mathcal{H}$ has admissible vectors if and only if the map $\lambda \mapsto \operatorname{rank}(\widehat{P}_\lambda)$ is finite and μ-integrable.*

For an example of $\mathcal{H}$ and an admissible vector we refer the interested reader to [**4**]. Different examples have also been presented in this work. Gleaning from results in [**2**], we have the following.

Proposition 1.2. *Let $\mathcal{H}$ be a closed left invariant subspace of $L^2(G)$ with G unimodular. Then the following are equivalent.*

(a) *$\mathcal{H}$ has an admissible vector.*

(b) *There is a left invariant subspace $\mathcal{K}$ of $L^2(G)$ and $\eta \in \mathcal{K}$ such that $\phi \mapsto \phi * \eta^*$ is an isometric isomorphism of $\mathcal{K}$ onto $\mathcal{H}$.*

(c) *There is a unique self-adjoint convolution idempotent $S \in \mathcal{H}$ such that $\mathcal{H} = L^2(G) * S$.*

(d) *The function $m_{\mathcal{H}}$ is integrable over $\widehat{G}$ with respect to Plancherel measure μ.*

Proof. Let (a) hold and ψ be an admissible vector for $\mathcal{H}$. Then $V_\psi^* V_\psi$ is an isometry and hence the identity on $\mathcal{H}$. Take $\mathcal{K} = V_\psi(\mathcal{H})$. Then V_ψ^* is an isometric isomorphism of $\mathcal{K}$ onto $\mathcal{H}$. To prove (b), we only need to show that V_ψ^* acts by $V_\psi^* f = f * \psi$. For this, let $f \in \mathcal{H} * \psi$ and $g \in L^1(G) \cap L^2(G)$. Then

$$\langle V_\psi^* f, g\rangle = \langle f, V_\psi g\rangle = \langle f, g * \psi^*\rangle = \langle f * \psi, g\rangle. \tag{1.1}$$

Now $\eta = \psi^*$ applies for (b).

Suppose that (b) holds. Define $V_\eta(\phi) = \phi * \eta^*$ for $\phi \in \mathcal{K}$. Then $V_\eta V_\eta^*$ is a projection of $L^2(N)$ onto $\mathcal{H}$. Since V_η^* is bounded, then by an analogous computation in (1.1), we have $V_\eta^* = V_{\eta^*}$, and hence $V_\eta V_\eta^*(\phi) = \phi * (\eta * \eta^*)$. Now set $S = \eta * \eta^* = V_\eta(\eta)$. Evidently, S belongs to $\mathcal{H}$, is self-adjoint and is a convolution idempotent, and the projection onto $\mathcal{H}$ is given by convolution with S.

Suppose that (c) holds. Then S itself is an admissible vector in $\mathcal{H}$, so (d) follows from Theorem 1.1. Finally, Theorem 1.1 says that (d) implies (a). $\square$

We will say that a left invariant subspace $\mathcal{H}$ is *admissible* if it satisfies one of the conditions of Proposition 1.2. Function S of condition (c) is called the reproducing kernel for $\mathcal{H}$. Note that, in this case, the associated projection field $\{\widehat{P}_\lambda\}$ is just the group Fourier transform of S.

Definition 1.3. Let Γ be any countable discrete subset of G, and let $\mathcal{H}$ be a left invariant subspace of $L^2(G)$ consisting of continuous functions. We shall call $(\mathcal{H}, \Gamma)$ a sampling pair if $S \in \mathcal{H}$ and $c = c_{\mathcal{H},\Gamma} > 0$ exist such that, for all $\phi \in \mathcal{H}$,

$$\|\phi\|^2 = \frac{1}{c} \sum_{\gamma \in \Gamma} |\phi(\gamma)|^2, \tag{1.2}$$

and

$$\phi = \sum_{\gamma} \phi(\gamma)\ T_\gamma S, \tag{1.3}$$

where the sum (1.3) converges in L^2.

Following [**2**], we say that S is a sinc-type function. It is well known that the sum above (1.3) converges uniformly as well as in L^2 (see [**3**, Remark 2.4]).

Recall that a system $\{\psi_j\}_{j \in J}$ of functions in a separable Hilbert space $\mathcal{H}$ is a tight frame for $\mathcal{H}$ if, for some $c > 0$,

$$c\,\|g\|^2 = \sum_{j \in J} |\langle g, \psi_j \rangle|^2$$

holds for every $g \in \mathcal{H}$. A Parseval frame is a tight frame for which $c = 1$. Now suppose that $\mathcal{H}$ is closed left invariant admissible with reproducing kernel S and that Γ is a countable discrete subset such that relation (1.2) holds for all $\phi \in \mathcal{H}$. Then the identity $\phi(x) = \phi * S(x) = \langle \phi, T_x S \rangle$ makes it clear that $\{T_\gamma S\}_{\gamma \in \Gamma}$ is a tight frame with frame bound c and hence that $\mathcal{H}$ is a sampling space with sinc-type function $(1/c)S$ (see [**3**, Corollary 2.3].

To characterize sampling spaces, we only need to observe that each such subspace is necessarily admissible.

Theorem 1.4. *Let $\mathcal{H}$ be a left invariant subspace consisting of continuous functions, and suppose that for some countable discrete* *subset* Γ *of* G, (1.2) *holds for all* $\phi \in \mathcal{H}$. *Then* $\mathcal{H}$ *is admissible and hence a sampling space.*

Proof. This is an immediate consequence of [**2**, Theorem 2.56]; see also [**3**, Theorem 2.2]. □

Hence we have the following equivalent conditions for left invariant subspaces.

Theorem 1.5. *Let* $\mathcal{H}$ *be a left invariant subspace, and let* Γ *be a countable discrete subset of* G. *Then the following are equivalent.*

(i) $\mathcal{H}$ *is admissible and* $\{T_\gamma S\}_{\gamma\in\Gamma}$ *is a tight frame for* $\mathcal{H}$ *with frame bound* c, *where* S *is its reproducing kernel.*

(ii) $\mathcal{H}$ *consists of continuous functions and the map* $A : \mathcal{H} \to \ell^2(\Gamma)$ *defined by* $A(\phi) = \{(1/\sqrt{c})\phi(\gamma)\}$ *is isometry.*

Of course, if $\mathcal{H}$ satisfies one of the conditions of the preceding theorem, then $(\mathcal{H}, \Gamma)$ is a sampling pair with sinc-type function $1/cS$. Next we turn to the question of interpolation.

Definition 1.6. We say a sampling pair $(\mathcal{H}, \Gamma)$ has an interpolation property if the isometry map A is also surjective.

The following is a consequence of Theorem 1.5 and standard frame theory.

Theorem 1.7. *Let* $(\mathcal{H}, \Gamma)$ *be a sampling pair. Then a sinc-type function* S *exists for* $\mathcal{H}$ *for which the following equivalent properties hold*: $(\mathcal{H}, \Gamma)$ *has the interpolation property if and only if* $\{(1/\sqrt{c})T_\gamma S\}_{\gamma\in\Gamma}$ *is an orthonormal basis for* $\mathcal{H}$.

Proof. By Theorem 1.5, $\mathcal{H}$ is admissible and denoting its convolution projection by S, we have that $\{(1/\sqrt{c})T_\gamma S\}_{\gamma\in\Gamma}$ is a Parseval frame for $\mathcal{H}$, and the isometry A is the associated analysis operator. If $\{(1/\sqrt{c})T_\gamma S\}_{\gamma\in\Gamma}$ is an orthonormal basis, then of course A is surjective. On the other hand, if isometry A is surjective, then A is unitary, and if

δ_γ denotes the canonical basis element in $\ell^2(\Gamma)$, then $\|(1/\sqrt{c})T_\gamma S\| = \|A^*\delta_\gamma\| = \|\delta_\gamma\| = 1$. ◻

In the following section we describe a class of subspaces of the Heisenberg group that admit sampling with the interpolation property.

2. The Heisenberg group and multiplicity free subspaces. We now assume that $G = N$ is the Heisenberg group: as a topological space N is identified with $\mathbf{R}^3$, and we let N have the group operation

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2).$$

We recall some basic facts about harmonic analysis on N. Put $\Lambda = \mathbf{R} \setminus \{0\}$. For $x \in N$, $\lambda \in \Lambda$, we define the unitary operator $\pi_\lambda(x)$ on $L^2(\mathbf{R})$ by

$$(\pi_\lambda(x) f)(t) = e^{2\pi i \lambda x_3} e^{-2\pi i \lambda x_2 t} f(t - x_1), \quad f \in L^2(\mathbf{R}).$$

Then $x \mapsto \pi_\lambda(x)$ is an irreducible representation of N (the Schrödinger representation), and for $\lambda \neq \lambda'$, the representations π_λ and $\pi_{\lambda'}$ are inequivalent. With respect to the Plancherel measure on $\widehat{N}$, almost every member of $\widehat{N}$ is realized as above and the group Fourier transform takes the explicit form

$$\mathcal{F} : L^2(N) \longrightarrow \int_\Lambda^\oplus \mathcal{HS}\left(L^2(\mathbf{R})\right) |\lambda| \, d\lambda.$$

Let $\mathcal{H}$ be a multiplicity free subspace of $L^2(N)$, $E = \Sigma(\mathcal{H})$ the spectrum of $\mathcal{H}$, P the projection onto $\mathcal{H}$, and let $\{\widehat{P}_\lambda\}$ be the associated measurable field of projections. We have a measurable field $e = \{e_\lambda\}_{\lambda \in \Lambda}$ where each e_λ belongs to $L^2(\mathbf{R})$, where $(\lambda \mapsto \|e_\lambda\|) = \mathbf{1}_E$, and where $\widehat{P}_\lambda = e_\lambda \otimes e_\lambda$ (i.e., $\mathcal{K}_\lambda = \mathbf{C} e_\lambda$) for $\lambda \in E$. Thus, the image of $\mathcal{H}$ under the group Fourier transform is

$$\widehat{\mathcal{H}} = \int_E^\oplus L^2(\mathbf{R}) \otimes e_\lambda |\lambda| \, d\lambda, \tag{2.1}$$

where e_λ is regarded as an element of $\overline{L^2(\mathbf{R})}$. Hence, $\mathcal{H}$ is isomorphic with

$$\int_E^\oplus L^2(\mathbf{R}) |\lambda| \, d\lambda \tag{2.2}$$

via the unitary isomorphism V_e defined on $\mathcal{H}$ by $\{V_e\eta(\lambda)\}_{\lambda\in E}$ where $\eta \in \mathcal{H}$ and

$$V_e\eta(\lambda) = \widehat{\eta}(\lambda)\,(e_\lambda)\,, \text{ almost everywhere } \lambda \in E.$$

We identify the direct integral (2.2) with $L^2(E\times\mathbf{R})$ in the obvious way, where it is understood that E carries the measure $|\lambda|\,d\lambda$. Note that if we write $\widehat{\eta}(\lambda) = \{f_\lambda \otimes e_\lambda\}_\lambda$, then $V_e\eta(\lambda) = f_\lambda$. For a fixed unit vector field $e = \{e_\lambda\}$ we will say that V_e is the reducing isomorphism for $\mathcal{H}$ associated with the vector field e. Note that, given a multiplicity-free subspace $\mathcal{H}$, the unit vector field $e = \{e_\lambda\}$ is essentially unique: if $e' = \{e'_\lambda\}$ is another measurable unit vector field for which (2.1) holds, then there is a measurable unitary complex-valued function $c(\lambda)$ on E such that $e'_\lambda = c(\lambda)e_\lambda$ holds for almost every λ. Finally, given any subset E of Λ and measurable field $e = \{e_\lambda\}_{\lambda\in\Lambda}$ with $(\lambda \mapsto \|e_\lambda\|) = \mathbf{1}_E$, the subspace

$$\mathcal{H}_e = \Big\{\phi \in L^2(N) : \text{Range}\,(\widehat{\phi}(\lambda)^*) \subset \mathbf{C}e_\lambda,\ \text{a.e. } \lambda\Big\}$$

is multiplicity free with spectrum E and associated vector field e.

Let Γ be a countable discrete subset of N. If $V_e : \mathcal{H} \to L^2(E\times\mathbf{R})$ is a reducing isomorphism and $\psi \in \mathcal{H}$ with $g = V_e\psi$, then the system $\mathcal{T}(\psi,\Gamma) = \{T_{(k,l,m)}\psi\}_{(k,l,m)\in\Gamma}$ is obviously equivalent with the system $\widehat{\mathcal{T}}(g,\Gamma) = \{\widehat{T}_{k,l,m}g\}_{(k,l,m)\in\Gamma}$ through the isomorphism V_e, where

$$\widehat{T}_{k,l,m}g(\lambda,t) = e^{2\pi i\lambda m}e^{-2\pi i\lambda lt}\; g(\lambda, t-k).$$

In [**1**], the discrete subsets $\Gamma_{\alpha,\beta} = \alpha\mathbf{Z}\times\beta\mathbf{Z}\times\mathbf{Z}$ for positive integers α and β are considered and, when $\Gamma = \Gamma_{\alpha,\beta}$, then we denote the above function systems by $\mathcal{T}(\psi,\alpha,\beta)$ and $\widehat{\mathcal{T}}(g,\alpha,\beta)$, respectively. For $\lambda \in \Lambda$ fixed, $\widehat{T}_{k,l,0}$ defines a unitary (Gabor) operator on $L^2(\mathbf{R})$ in the obvious way which we denote by $\widehat{T}^\lambda_{k,l}$. For $u \in L^2(\mathbf{R})$, set $\mathcal{G}(u,\alpha,\beta,\lambda) = \{\widehat{T}^\lambda_{k,l}u\}_{(k,l,0)\in\Gamma_{\alpha,\beta}}$. We say that $g \in L^2(E\times\mathbf{R})$ is a Gabor field over E with respect to $\Gamma_{\alpha,\beta}$ if, for almost every $\lambda \in E$, $\mathcal{G}(|\lambda|^{1/2}g(\lambda,\cdot),\alpha,\beta,\lambda)$ is a Parseval frame for $L^2(\mathbf{R})$. If g is a Gabor field over E with respect to $\Gamma_{\alpha,\beta}$, then standard Gabor theory implies that $\||\lambda|^{1/2}g(\lambda,\cdot)\|^2 = \alpha\beta|\lambda| \le 1$.

The following is an easy but significant extension of part of [**1**, Proposition 2.3].

Proposition 2.1. *Let E be a measurable subset of Λ and $g \in L^2(E \times \mathbf{R})$ such that $\widehat{\mathcal{T}}(g,\alpha,\beta)$ is a Parseval frame for $L^2(E \times \mathbf{R})$. Then g is a Gabor field over E with respect to $\Gamma_{\alpha,\beta}$.*

Proof. Write $E = \cup E_j$ where each E_j is translation congruent with a subset of $[0,1]$, and let $g_j = g|_{E_j}$. Then, for each j, $\widehat{\mathcal{T}}(g_j,\alpha,\beta)$ is a Parseval frame for $L^2(E_j \times \mathbf{R})$. Hence, by [**1**, Proposition 2.3], g_j is a Gabor field over E_j. Hence, g is a Gabor field over E. □

The "only part" of [**1**, Proposition 2.3] is also true for any subset E provided the orthogonality of some coefficient operators holds.

Proposition 2.2. *Let $E \subset \Lambda$ and $g \in L^2(E \times \mathbf{R})$ such that g is a Gabor field for $L^2(\mathbf{R})$ over E for $\Gamma_{\alpha,\beta}$. Write $E = \dot\cup_{j\in J} E_j$ where each E_j is translation congruent with a subset of $[0,1]$, and let $g_j = g|_{E_j}$. For any j, let C_j be the coefficient operator defined from $L^2(E \times \mathbf{R})$ into $\ell^2(\Gamma_{\alpha,\beta})$ by*

$$C_j : f \longrightarrow \{\langle \widehat{T}_\gamma g_j, f\rangle\}_\gamma, \tag{2.3}$$

and assume that, for each $j \neq j'$, Range$(C_j) \subset$ Range$(C_{j'})^\perp$. Then $\widehat{\mathcal{T}}(g,\alpha,\beta)$ is a Parseval frame for $L^2(E \times \mathbf{R})$.

Proof. By [**1**, Proposition 2.3], for any $j \in \mathbf{Z}$ the system $\widehat{\mathcal{T}}(g_j,\alpha,\beta)$ is a Parseval frame for $L^2(E_j \times \mathbf{R})$. Therefore, with the orthogonality assumption, for any $f \in L^2(E \times \mathbf{R})$, one has

$$\begin{aligned} \sum_\gamma \left|\langle \widehat{T}_\gamma g, f\rangle\right|^2 &= \sum_\gamma \left|\sum_j \langle \widehat{T}_\gamma g_j, f\rangle\right|^2 \\ &= \sum_j \sum_\gamma |\langle \widehat{T}_\gamma g_j, f\rangle|^2 = \|f\|^2, \end{aligned} \tag{2.4}$$

and, hence, $\widehat{\mathcal{T}}(g,\alpha,\beta)$ is a Parseval frame for $L^2(E \times \mathbf{R})$. □

The above observations together with Theorem 1.5 now give the following.

Theorem 2.3. *Let E be a measurable subset of Λ, and let $e = \{e_\lambda\}_{\lambda\in\Lambda}$ be a measurable field of unit vectors in $L^2(\mathbf{R})$ such that $(\lambda \mapsto \|e_\lambda\|) = \mathbf{1}_E$. The following are equivalent.*

(i) *E has finite Plancherel measure and $\widehat{\mathcal{T}}((1/\sqrt{c})e, \alpha, \beta)$ is a Parseval frame for $L^2(E \times \mathbf{R})$.*

(ii) *$(\mathcal{H}_e, \Gamma_{\alpha,\beta})$ is a sampling pair with the sinc-type function $S = (1/c)V_e^{-1}(e)$.*

Moreover, if the above conditions hold, then $(1/\sqrt{c})e$ is a Gabor field over E, and E is included in the interval $[-1/\alpha\beta, 1/\alpha\beta]$.

We now have a precise density criterion for the interpolation property in this situation.

Theorem 2.4. *Let $\mathcal{H}$ be a multiplicity free subspace of $L^2(N)$ with $E = \Sigma(\mathcal{H})$. Suppose that, for some $\alpha, \beta > 0$, $(\mathcal{H}, \Gamma_{\alpha,\beta})$ is a sampling pair with $c = c_{\mathcal{H},\Gamma_{\alpha,\beta}}$. Then $c = 1/\alpha\beta$. Moreover, $(\mathcal{H}, \Gamma_{\alpha,\beta})$ has the interpolation property if and only if $\mu(E) = 1/\alpha\beta$. Hence if $(\mathcal{H}, \Gamma_{\alpha,\beta})$ has the interpolation property, then $\alpha\beta \leq 1$.*

Proof. By Theorem 1.5, $\mathcal{H}$ is admissible; let S be the associated reproducing kernel, and let V_e be a reducing isomorphism, where $e = \{e_\lambda\}$ is an $L^2(\mathbf{R})$-vector field with $(\lambda \mapsto \|e_\lambda\|) = \mathbf{1}_E$, so that $V_e(S) = e$. It follows from the above that $\{(1/\sqrt{c})T_\gamma S\}_{\gamma\in\Gamma_{\alpha,\beta}}$ is a Parseval frame for $\mathcal{H}$, $\widehat{\mathcal{T}}(1/\sqrt{c})e, \alpha, \beta)$ is a Parseval frame for $L^2(E\times\mathbf{R})$, and $(1/\sqrt{c})e$ is a Gabor field over E. Hence, for almost every $\lambda \in E$, we have

$$\left\| |\lambda|^{1/2} \frac{1}{\sqrt{c}} e_\lambda \right\|^2 = \alpha\beta|\lambda|,$$

and the relation $c = 1/\alpha\beta$ follows immediately. Now $\mathcal{H}$ has the interpolation property if and only if $\{(1/\sqrt{c})T_\gamma S\}_{\gamma\in\Gamma_{\alpha,\beta}}$ is an orthonormal basis for $\mathcal{H}$, if and only if $\|(1/\sqrt{c})S\|^2 = 1$. But

$$\left\| \frac{1}{\sqrt{c}} S \right\|^2 = \alpha\beta \|S\|^2 = \alpha\beta \|V_e(S)\|^2$$
$$= \alpha\beta \int_E \|e_\lambda\|^2 |\lambda| \, d\lambda$$
$$= \alpha\beta \int_E |\lambda| \, d\lambda = \alpha\beta\mu(E).$$

This proves the first part of the theorem. Now, if $(\mathcal{H}, \Gamma_{\alpha,\beta})$ has the interpolation property, then, since $E \subseteq [-1/\alpha\beta, 1/\alpha\beta]$, we have

$$1/\alpha\beta = \int_E |\lambda| \, d\lambda \leq \int_{[-1/\alpha\beta, 1/\alpha\beta]} |\lambda| \, d\lambda = 1/(\alpha\beta)^2. \qquad \square$$

We now construct an example of a sampling pair with the interpolation property. We assume that $\alpha = \beta = 1$; note that, in this case, the interpolation property is equivalent with $E = [-1, 1]$. In light of Theorems 2.3 and 2.4, it is evident that, in order to construct an example $(\mathcal{H}, \Gamma_{1,1})$ with $\mathcal{H}$ multiplicity free, it is enough to construct a measurable field of $L^2(\mathbf{R})$-vectors $\{e_\lambda\}$ such that $(\lambda \to \|e_\lambda\|) = \mathbf{1}_{[-1,1]}$ and such that e generates a Heisenberg frame for $L^2([-1, 1] \times \mathbf{R})$. The following technical lemma is helpful in the construction of such a function e.

Lemma 2.5. *Let $e \in L^2([-1, 1] \times \mathbf{R})$ be such that e is a Gabor field over $[-1, 1]$ with respect to $\Gamma_{\alpha,\beta}$, and such that the orthogonality condition*

$$\sum_{k,l} \langle f(\lambda - 1, \cdot), e_{k,l,0}(\lambda - 1, \cdot)\rangle \overline{\langle f(\lambda, \cdot), e_{k,l,0}(\lambda, \cdot)\rangle} = 0 \tag{2.5}$$

holds for all $\lambda \in (0, 1]$ and for all $f \in L^2([-1, 1] \times \mathbf{R})$. Then the system $\widehat{\mathcal{T}}(e, \Gamma_{\alpha,\beta})$ is a Parseval frame for $L^2([-1, 1] \times \mathbf{R})$.

Proof. Suppose that e is a Gabor field satisfying (2.5), and let $f \in L^2([-1, 1] \times \mathbf{R})$. By Proposition 2.3 of [**1**] and the Parseval identity for Fourier series, we have

$$\begin{aligned}\int_0^1 \|f(\lambda-1,\cdot)\|^2|\lambda-1|\,d\lambda \\ &= \sum_{k,l,m}\left|\int_0^1 \langle f(\lambda-1,\cdot), e_{k,l,m}(\lambda-1,\cdot)\rangle|\lambda-1|\,d\lambda\right|^2 \\ &= \sum_{k,l}\int_0^1 |\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle|\lambda-1||^2\ d\lambda\end{aligned}$$

and, similarly,

$$\int_0^1 \|f(\lambda,\cdot)\|^2|\lambda|\,\lambda = \sum_{k,l}\int_0^1 |\langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle|\lambda||^2\,d\lambda.$$

Hence,

$$\begin{aligned}\|f\|^2 &= \int_0^1 \|f(\lambda-1,\cdot)\|^2|\lambda-1|\,d\lambda \\ &\quad + \int_0^1 \|f(\lambda,\cdot)\|^2|\lambda|\,d\lambda \\ &= \int_0^1 \sum_{k,l}\Big(\big|\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle|\lambda-1|\big|^2 \\ &\quad + \big|\langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle|\lambda|\big|^2\Big)\,d\lambda.\end{aligned} \tag{2.6}$$

But (2.5) implies that, for $\lambda \in (0,1]$,

$$\begin{aligned}\sum_{k,l}\Big(\big|\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle|\lambda-1|\big|^2 \\ &+ \big|\langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle|\lambda|\big|^2\Big) \\ &= \sum_{k,l}\Big(\Big|\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle|\lambda-1| \\ &\quad + \langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle|\lambda|\Big|^2\Big).\end{aligned}$$

Combining the preceding with (2.6) and applying the Parseval identity for Fourier series again, we have

$$\begin{aligned}\|f\|^2 &= \sum_{k,l} \int_0^1 \Big| \langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle |\lambda-1| \\ &\qquad + \langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle |\lambda| \Big|^2 \, d\lambda \\ &= \sum_{k,l}\sum_m \Big| \int_0^1 \big(\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle |\lambda-1| \\ &\qquad + \langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle |\lambda|\big) e^{-2\pi i\lambda m} \, dl \Big|^2 \\ &= \sum_{k,l,m} \Big| \int_{-1}^1 \langle f(\lambda,\cdot), e_{k,l,m}(\lambda,\cdot)\rangle |\lambda| \, d\lambda \Big|^2 .\end{aligned}$$

This proves the claim. □

By virtue of Lemma 2.5, it is sufficient to construct a function e with the properties in the preceding lemma.

Example 2.6. For $\lambda \in (0,1]$, put

$$e_\lambda = \mathbf{1}_{[(1/\lambda)-1,(1/\lambda)]} \quad \text{and} \quad e_{\lambda-1} = \mathbf{1}_{[-1,0]}.$$

Then e defined by $e(\lambda,t) = e_\lambda(t)$ for $\lambda \in (0,1]$ and $e(\lambda,t) = \mathbf{1}_{[-1,0]}(t)$ for $\lambda \in [-1,0)$ is a Gabor field over $[-1,1]$ with respect to $\Gamma_{1,1}$.

Proof. We compute that, for any $f \in L^2([-1,1]\times\mathbf{R})$ and for $\lambda \in (0,1]$,

$$\begin{aligned}&\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle \\ &\qquad = \int_{\mathbf{R}} f(\lambda-1,t) e^{2\pi i(\lambda-1)lt} \mathbf{1}_{[-1,0]}(t-k)\, dt \\ &\qquad = \int_{I_k^{\lambda-1}} \left(\left(\frac{1}{1-\lambda}\right) f\left(\lambda-1, \frac{s}{\lambda-1}\right)\right) e^{2\pi i l s}\, ds,\end{aligned}$$

and, similarly,

$$\begin{aligned}\langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle &= \int_{\mathbf{R}} f(\lambda,\cdot)e^{2\pi i\lambda lt}\mathbf{1}_{[(1/\lambda)-1,(1/\lambda)]}(t-k)\,dt\\ &= \int_{I_k^\lambda}\left(\frac{1}{\lambda}f\left(\left(\lambda,\frac{s}{\lambda}\right)\right)\right)e^{2\pi ils}ds\end{aligned}$$

where $I_k^{\lambda-1} = [-(1-\lambda)k, -(1-\lambda)k+(1-\lambda)]$ and $I_k^\lambda = [1+\lambda k-\lambda, 1+\lambda k]$. It is easily seen that, for each k,

$$I_k^{\lambda-1}\cap I_k^\lambda = \varnothing \quad\text{and}\quad (I_k^{\lambda-1}+k)\cup I_k^\lambda = [\lambda k, \lambda k+1].$$

Hence, for each k, the sequences $\{\langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle : l\in\mathbf{Z}\}$ and $\{\langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle : l\in\mathbf{Z}\}$ are Fourier coefficients for orthogonal functions and we have

$$\sum_l \langle f(\lambda-1,\cdot), e_{k,l,0}(\lambda-1,\cdot)\rangle\overline{\langle f(\lambda,\cdot), e_{k,l,0}(\lambda,\cdot)\rangle} = 0.$$

Thus, equation (2.5) holds for e. □

Since the vector field $e=\{e_\lambda\}$ is compactly supported, one does not expect that the inverse Fourier image is well localized. We show this explicitly in the following, where we compute the inverse group Fourier transform in terms of ordinary Fourier transforms. For a function $f\in L^1(\mathbf{R})$, put $\widehat{f}(s)=\int_{\mathbf{R}} f(t)e^{2\pi ist}\,dt$ and $\check{f}(s)=\int_{\mathbf{R}} f(t)e^{-2\pi ist}dt$.

Example 2.7. Let $e=\{e_\lambda\}$ be the unit vector field from the preceding example, and let $S\in L^2(N)$ be the function for which $V_e(S)=e$. For each $x\in\mathbf{R}$, define the intervals $I_{x,\lambda}$ and J_x by

$$\begin{aligned}I_{x,\lambda} &= \left[-\frac{1}{\lambda}-1,\frac{1}{\lambda}\right]\bigcap\left(\left[-\frac{1}{\lambda}-1,\frac{1}{\lambda}\right]+x\right),\\ J_x &= [-1,0]\cap([-1,0]+x).\end{aligned}$$

Then $S=S_0+S_1$, where $S_0(x,y,z)=\check{F}_{x,y}(z)$ and $S_1(x,y,z)=\check{G}_{x,y}(z)$, and where $G_{x,y}(\lambda)=\lambda\mathbf{1}_{[0,1]}(\lambda)\widehat{\mathbf{1}}_{I_{x,\lambda}}(\lambda y)$ and $F_{x,y}(\lambda)=-\lambda\mathbf{1}_{[-1,0]}(\lambda)\widehat{\mathbf{1}}_{J_x}(\lambda y)$. In particular, S vanishes outside the strip $U=$

$\{(x, y, z) : |x| < 1\}$ and S_0 and S_1 are given by sinc-type expressions. For example, for $(x, y, z) \in U$ and $y \neq 0$, $z \neq 0$, then

$$S_0(x, y, z) = \begin{cases} 1/2\pi iy((e^{2\pi i(z-xy)} - 1)/ \\ \quad (2\pi i(z - xy)) - (e^{2\pi i(z+y)} - 1)/(2\pi i(z + y))), \\ \qquad \text{if } -1 < x < 0,\ z \neq xy,\ y \neq -z, \\ 1/2\pi iy((e^{2\pi iz} - 1)/ \\ \quad (2\pi iz) - (e^{2\pi i(z+y(1-x))} - 1)/(2\pi i(z + y(1 - x)))), \\ \qquad \text{if } 0 < x < 1,\ x \neq -y(1 - x). \end{cases}$$

Proof. We have

$$\begin{aligned} S(x, y, z) &= \int_\Lambda \langle e_\lambda, \pi_\lambda(x, y, z)e_\lambda\rangle |\lambda|\, d\lambda \\ &= -\int_\Lambda \mathbf{1}_{[-1,0]}(\lambda)\langle \mathbf{1}_{[-1,0]}, \pi_\lambda(x, y, z)\mathbf{1}_{[-1,0]}\rangle \lambda\, d\lambda \\ &\quad + \int_\Lambda \mathbf{1}_{[0,1]}(\lambda)\langle \mathbf{1}_{[(1/\lambda)-1,(1/\lambda)]}, \pi_\lambda(x, y, z)\mathbf{1}_{[(1/\lambda)-1,(1/\lambda)]}\rangle\, \lambda\, d\lambda \\ &= -\int_\Lambda \mathbf{1}_{[-1,0]}(\lambda) \int_{\mathbf{R}} e^{-2\pi i\lambda z} e^{2\pi i\lambda yt} \mathbf{1}_{[-1,0]}\mathbf{1}_{[-1,0]}(t - x)\, dt\lambda\, d\lambda \\ &\quad + \int_\Lambda \mathbf{1}_{[0,1]}(\lambda) \\ &\quad \times \int_{\mathbf{R}} e^{-2\pi i\lambda z} e^{2\pi i\lambda yt} \mathbf{1}_{[(1/\lambda)-1,(1/\lambda)]}\mathbf{1}_{[(1/\lambda)-1,(1/\lambda)]}(t-x)\, dt\lambda\, d\lambda \\ &= -\int_\Lambda \mathbf{1}_{[-1,0]}(\lambda)e^{-2\pi i\lambda z}\left(\int_{\mathbf{R}} e^{2\pi i\lambda yt}\mathbf{1}_{J_x}(t)\, dt\right)\lambda\, d\lambda \\ &\quad + \int_\Lambda \mathbf{1}_{[0,1]}(\lambda)e^{-2\pi i\lambda z}\left(\int_{\mathbf{R}} e^{2\pi i\lambda yt}\mathbf{1}_{I_{x,\lambda}}(t)\, dt\right)\lambda\, d\lambda \\ &= \int_\Lambda F_{x,y}(\lambda)e^{-2\pi i\lambda z}\, d\lambda + \int_\Lambda G_{x,y}(\lambda)e^{-2\pi i\lambda z}\, d\lambda. \end{aligned}$$

The explicit expression for G is now an elementary calculation. □

We conclude this section with a necessary and sufficient condition for the generator of a Heisenberg orthonormal basis for any arbitrary shift-invariant spaces. Let $g \in L^2(\Lambda \times \mathbf{R})$ and define the closed subspace

$\mathcal{S}(g,\alpha,\beta)$ of $L^2(\Lambda\times\mathbf{R})$ by

$$\mathcal{S}(g,\alpha,\beta)=\overline{\mathrm{sp}}\big(\widehat{\mathcal{T}}(g,\alpha,\beta)\big).$$

For each $(\lambda,t)\in\Lambda\times\mathbf{R}$, put

$$\Theta_k^g(\lambda,t):=\sum_{\substack{l'\in 1/\beta\mathbf{Z}\\ l''\in\mathbf{Z}}} g\left(\lambda-l'',\frac{t-l'}{\lambda-l'}-k\right)\overline{g}\left(\lambda-l'',\frac{t-l'}{\lambda-l'}\right). \tag{2.7}$$

Then we have the following

Theorem 2.8. *$\widehat{\mathcal{T}}(g,\alpha,\beta)$ is an orthonormal basis for $\mathcal{S}(g,\alpha,\beta)$ if and only if*

$$\Theta_k^g(\lambda,t)=\delta_k \quad \textit{almost everywhere } (\lambda,t).$$

Proof. For convenience, we consider the case $\alpha=\beta=1$; the proof for general α and β can be adapted. For each $\gamma=(k,l,m)\in\Gamma_{1,1}$, the function

$$(\lambda,t)\longmapsto e^{2\pi i\lambda m}e^{-2\pi i\lambda lt}g(\lambda,t-k)\overline{g}(\lambda,t)|\lambda|$$

is absolutely integrable, and we can apply periodization and Fubini's theorem to calculate

$$\begin{aligned}
\langle\widehat{T}_\gamma g,g\rangle &= \int_\Lambda\int_{\mathbf{R}} e^{2\pi i\lambda m}e^{-2\pi i\lambda lt}g(\lambda,t-k)\overline{g}(\lambda,t)|\lambda|\,dt\,d\lambda\\
&= \int_\Lambda\int_{\mathbf{R}} e^{2\pi i\lambda m}e^{-2\pi ilt}g(\lambda,t/\lambda-k)\overline{g}(\lambda,t/\lambda)\,dt\,d\lambda\\
&= \int_\Lambda e^{2\pi i\lambda m}\sum_{l'\in\mathbf{Z}}\int_0^1 e^{-2\pi ilt}g\left(\lambda,\frac{t-l'}{\lambda-k}\right)\overline{g}\left(\lambda,\frac{t-l'}{\lambda}\right)dt\,d\lambda\\
&= \int_0^1\int_0^1 e^{2\pi i\lambda m}e^{-2\pi ilt}\sum_{l''}\sum_{l'\in\mathbf{Z}}\\
&\qquad\qquad \times g\left(\lambda-l',\frac{t-l'}{\lambda-l'}-k\right)\overline{g}\left(\lambda-l'',\frac{t-l'}{\lambda-l''}\right)dt\,d\lambda\\
&= \int_0^1\int_0^1 e^{2\pi i\lambda m}e^{-2\pi ilt}\Theta_k^g(\lambda,t)\,dt\,d\lambda.
\end{aligned}$$

Suppose that $\widehat{\mathcal{T}}(g,\alpha,\beta)$ is an orthonormal basis for $\mathcal{S}(g,\alpha,\beta)$. Note that Θ_k^g is a $(1,1)$-periodic integrable function on $\mathbf{T}\times\mathbf{T}$. If $k\neq 0$, then $\widehat{\Theta_k^g}(m,l)=0$ for all integers m and l, and hence $\Theta_k^g\equiv 0$. If $k=0$, then $\widehat{\Theta_0^g}(m,l)=0$ holds for all $(m,l)\neq 0$ while $\widehat{\Theta_0^g}(0,0)=1$. Hence, $\Theta_0^g\equiv 1$.

On the other hand, if $\Theta_k^g(\lambda,t)=\delta_k$ almost everywhere (λ,t), then the above reasoning can be reversed to show that the system $\widehat{\mathcal{T}}(g,\alpha,\beta)$ is orthonormal. □

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Department of Mathematics and Computer Science, Saint Louis University, St. Louis, MO 63103
Email address: curreybn@slu.edu

Department of Mathematics and Computer Science, Queensborough C. College, CUNY, 222-05 56th Ave., Bayside, NY 11364
Email address: amayeli@qcc.cuny.edu