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AN INVERSION FORMULA OF RADON TRANSFORM ON THE PRODUCT HEISENBERG GROUP

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ABSTRACT. Let $\mathbf{H}_1^n$ be the n-direct product of the Heisenberg groups $\mathbf{H}_1$, and let $\mathcal{P}$ be the affine group of $\mathbf{H}_1^n$. Then $\mathcal{P}$ has a natural unitary representation on $L^2(\mathbf{H}_1^n)$. In this article, we present the inversion of the Radon transform on the product Heisenberg group by using inverse wavelet transform. In addition, we characterize a subspace of $L^2(\mathbf{H}_1^n)$ such that the inversion formula of the Radon transform holds in the weak sense.

1. Introduction. Wavelet analysis has many applications in pure and applied mathematics. The concept of continuous wavelet transform is deeply related to the theory of square integrable group representations, see [**1**]. In this viewpoint the theory of continuous wavelet analysis on the Heisenberg group has been established, see [**4, 8**]. It is known that the Radon transform on $\mathbf{R}^n$ is a very useful analysis tool. Recently, we find that a lot of authors deal with the inversion formula of the Radon transform by using inverse wavelet transforms. The first result in this area is due to Holschneider who considered the classical Radon transform on the two-dimensional plane, see [**7**]. Rubin [**11–13**] extended the result to k-dimensional Radon transforms on $\mathbf{R}^n$ and totally geodesic Radon transforms on the sphere and hyperbolic space. Later, He, Liu, Nessibi and Trimèche studied analogous problems on the Heisenberg group, see [**3, 9**], and the other cases, see [**5, 6**]. At the same time, we also find that Radon transforms can be applied to estimate the regularity for solutions of nonlinear Schrödinger equations, see [**10**]. So, in this paper, we introduce partial Radon transforms on the product Heisenberg group and discuss the

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inversion of the Radon transform on $\mathbf{H}_1^n$ by using the inverse wavelet transform, where $\mathbf{H}_1^n = \mathbf{H}_1 \times \mathbf{H}_1 \times \cdots \times \mathbf{H}_1$ is the product Heisenberg group with the manifold $R^n \times R^n \times R^n$, and $\mathbf{H}_1$ is the real three-dimensional Heisenberg group.

Let $\mathbf{H}_1^n = \{(x,y,t) : x,y,t \in R^n\}$. For $(x,y,t),\ (x',y',t') \in \mathbf{H}_1^n$, the multiplication law of $\mathbf{H}_1^n$ is given by

$$(x,y,t)(x',y',t') = (x+x', y+y', t+t'+2(yx'-xy')), \tag{1.1}$$

where "+" denotes the usual addition in Euclidean space and $(t+t'+2(yx'-xy'))_i = t_i+t'_i+2(y_ix'_i-x_iy'_i)$. The translation operator $T_{(x,\ y,\ t)}$ on $\mathbf{H}_1^n$ is defined by

$$\begin{aligned} T_{(x,\ y,\ t)} : (x',y',t') &\longmapsto (x,y,t)(x',y',t') \\ &= (x+x', y+y', t+t'+2(yx'-xy')). \end{aligned}$$

Also we define the dilation operator T_ρ on $\mathbf{H}_1^n$ as follows:

$$T_\rho : (x',y',t') \longmapsto (\sqrt{\rho}x', \sqrt{\rho}y', \rho t'),$$

where $\rho \in R^+ = \{x : x > 0\}$. Write

$$\mathcal{P} = \{(x,y,t,\rho) : (x,y,t) \in \mathbf{H}_1^n,\ \rho \in R^+\}. \tag{1.2}$$

Let $(x,y,t,\rho),\ (x',y',t',\rho') \in \mathcal{P}$. Then the multiplication law of $\mathcal{P}$ is given by

$$\begin{aligned} &(x,y,t,\rho)(x',y',t',\rho') \\ &\qquad = (x+\sqrt{\rho}x', y+\sqrt{\rho}y',\ t+\rho t'+2\sqrt{\rho}(yx'-xy'),\ \rho\rho'). \end{aligned}$$

Then $\mathcal{P}$ is a locally compact nonunimodular Lie group. The left and right Haar measures on $\mathcal{P}$ are, respectively, given by

$$d\mu_l(x,y,t,\rho) = \frac{dx\,dy\,dt\,d\rho}{\rho^{2n+1}}, \qquad d\mu_r(x,y,t,\rho) = \frac{dx\,dy\,dt\,d\rho}{\rho},$$

where $dx\,dy\,dt$ is the Haar measure of $\mathbf{H}_1^n$, which coincides with the Lebesgue measure of $R^n \times R^n \times R^n$. Let $f \in L^2(\mathbf{H}_1^n),\ (x,y,t,\rho) \in \mathcal{P}$. The unitary representation U of $\mathcal{P}$ on $L^2(\mathbf{H}_1^n)$ can be defined by

$$\begin{aligned} &U(x,y,t,\rho)f(x',y',t') \\ &\quad = \rho^{-n} f(\rho^{-1/2}(x'-x), \rho^{-1/2}(y'-y), \rho^{-1}(t'-t-2(yx'-xy'))). \end{aligned}$$

Let $Z^+ = \{0, 1, 2, \dots\}$, $k \in Z^+$, $x \in R$. The Hermite polynomials $\mathscr{H}_k(x)$ are defined by

$$\mathscr{H}_k(x) = (-1)^k e^{x^2} \left(\frac{d^k}{dx^k} e^{-x^2} \right). \tag{1.3}$$

Thus, the normalized Hermite function is given by

$$\phi_k(x) = (2^k \sqrt{\pi} k!)^{-1/2} e^{-1/2x^2} \mathscr{H}_k(x). \tag{1.4}$$

It is well known that the family $\{\phi_k(x) : \; k \in Z^+\}$ is an orthonormal basis for $L^2(R)$. Geller [**2**] developed the theory of Fourier analysis on the Heisenberg group. Now we state some preliminaries of Fourier analysis on the product Heisenberg group. For $\lambda \in R \setminus \{0\}$, $(x, y, t) \in \mathbf{H}_1$, let $\pi_\lambda(x, y, t)$ denote the Schrödinger representation of $\mathbf{H}_1$ which acts on $L^2(R)$ by

$$\pi_\lambda(x, y, t)\phi(\eta) = e^{i\lambda t - 2i\lambda xy + 4i\lambda \eta y} \phi(\eta - x).$$

Putting $\Lambda = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_j \neq 0, \; j = 1, 2, \dots, n\}$, $\lambda \in \Lambda$, we define the operator Π_λ on $\mathbf{H}_1^n$ by

$$\Pi_\lambda(x, y, t) = \pi_{\lambda_1}(x_1, y_1, t_1) \otimes \pi_{\lambda_2}(x_2, y_2, t_2) \otimes \cdots \otimes \pi_{\lambda_n}(x_n, y_n, t_n). \tag{1.5}$$

Let $k = (k_1, k_2, \dots, k_n) \in (Z^+)^n$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda$, $x = (x_1, x_2, \dots, x_n)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in R^n$. The n-dimensional Hermite functions denoted by Φ_k are then obtained by taking tensor products:

$$\Phi_k(x) = \prod_{j=1}^{n} \phi_{k_j}(x_j). \tag{1.6}$$

It is easy to see that the family $\{\Phi_k : k \in (Z^+)^n\}$ is a complete orthonormal basis for $L^2(R^n)$. Thus, $\Pi_\lambda(x, y, t)$ is the Schrödinger representation of $\mathbf{H}_1^n$ which acts on $L^2(R^n)$ by

$$\Pi_\lambda(x, y, t)\Phi_k(\eta) = \prod_{j=1}^{n} e^{i\lambda_j t_j - 2i\lambda_j x_j y_j + 4i\lambda_j \eta_j y_j} \phi_{k_j}(\eta_j - x_j). \tag{1.7}$$

The group Fourier transform of an L^1-function f is the operator-valued function on Λ which is defined by the Bochner-integral

$$\widehat{f}(\lambda)\varphi = \int_{\mathbf{H}_1^n} f(x,y,t)\Pi_\lambda(x,y,t)\varphi\,dx\,dy\,dt, \tag{1.8}$$

where $\lambda \in \Lambda$, $\varphi \in L^2(R^n)$. Let $f, g \in L^2(\mathbf{H}_1^n)$. The Plancherel identity is

$$\langle f, g\rangle_{L^2(\mathbf{H}_1^n)} = \frac{1}{\pi^{2n}} \int_\Lambda \mathrm{tr}\,(\widehat{g}(\lambda)^* \widehat{f}(\lambda))|\lambda|\,d\lambda, \tag{1.9}$$

where $\widehat{g}(\lambda)^*$ is the adjoint of $\widehat{g}(\lambda)$, $|\lambda| = |\lambda_1\lambda_2\cdots\lambda_n|$. Specifically, if $f = g$, then

$$\|f\|_{L^2(\mathbf{H}_1^n)} = \left\{\frac{1}{\pi^{2n}} \int_\Lambda \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|\,d\lambda\right\}^{1/2}, \tag{1.10}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm of operators. Let $\mathcal{S}(\mathbf{H}_1^n)$ denote the Schwartz space on $\mathbf{H}_1^n$. For $f \in \mathcal{S}(\mathbf{H}_1^n)$, then for all $(x,y,t) \in \mathbf{H}_1^n$, the Fourier inversion formula holds:

$$f(x,y,t) = \frac{1}{\pi^{2n}} \int_\Lambda \mathrm{tr}\,(\Pi_\lambda(x,y,t)^* \widehat{f}(\lambda))|\lambda|\,d\lambda. \tag{1.11}$$

2. Wavelet transforms. In this section we will establish the theory of continuous wavelet analysis on the product Heisenberg group $\mathbf{H}_1^n$. For $k \in (Z^+)^n$, let P_k be the orthogonal projection from $L^2(R^n)$ to the one-dimensional subspace spanned by Φ_k. Setting

$$H_k = \left\{f \in L^2(\mathbf{H}_1^n) : \widehat{f}(\lambda) = \widehat{f}(\lambda)P_k\right\}, \tag{2.1}$$

we can obtain

$$L^2(\mathbf{H}_1^n) = \bigoplus_{k\in(Z^+)^n} H_k. \tag{2.2}$$

For $h \in H_k$, $h \neq 0$, if h satisfies the admissibility condition:

$$\frac{1}{\|h\|_{L^2(\mathbf{H}_1^n)}^2} \int_{\mathbf{P}} \left|\langle h, U(x,y,t,\rho)h\rangle_{L^2(\mathbf{H}_1^n)}\right|^2 \frac{dx\,dy\,dt\,d\rho}{\rho^{2n+1}} < +\infty,$$

then we say that h is an admissible wavelet and write $h \in AW_k$.

Theorem 2.1. *Let $h \in H_k$. Then h is an admissible wavelet if there exists a positive constant C_h, such that for almost everywhere $\lambda \in \Lambda$,*

$$C_h = \int_{R^+} \left\| \widehat{h}(\rho\lambda) \right\|_{HS}^2 \frac{d\rho}{\rho} < +\infty. \tag{2.3}$$

Proof. Since

$$\int_{\mathbf{H}_1^n} \big\langle h, U(x,y,t,\rho)h \big\rangle_{L^2(\mathbf{H}_1^n)} \Pi_\lambda(x,y,t)\, dx\, dy\, dt = \rho^n \widehat{h}(\lambda)\widehat{h}(\rho\lambda)^*,$$

we have

$$\begin{aligned}
&\int_{\mathbf{P}} \left| \big\langle h, U(x,y,t,\rho)h \big\rangle_{L^2(\mathbf{H}_1^n)} \right|^2 \frac{dx\, dy\, dt\, d\rho}{\rho^{2n+1}} \\
&= \frac{1}{\pi^{2n}} \left(\int_{R^+} \int_{R^n} \|\widehat{h}(\lambda)\widehat{h}(\rho\lambda)^*\|_{HS} |\lambda|\, d\lambda \right) \frac{d\rho}{\rho} \\
&= \frac{1}{\pi^{2n}} \int_{R^+} \left(\int_{R^n} \operatorname{tr} \left(\widehat{h}(\lambda)\widehat{h}(\lambda)^*\widehat{h}(\rho\lambda)\widehat{h}(\rho\lambda)^*\right) |\lambda|\, d\lambda \right) \frac{d\rho}{\rho} \\
&= \frac{1}{\pi^{2n}} \int_{R^+} \left(\int_{R^n} \big\langle \widehat{h}(\rho\lambda)\widehat{h}(\rho\lambda)^*\Phi_k, \widehat{h}(\lambda)\widehat{h}(\lambda)^*\Phi_k \big\rangle_{L^2(R^n)} |\lambda|\, d\lambda \right) \frac{d\rho}{\rho} \\
&= \left(\int_{R^+} \|\widehat{h}(\rho\lambda)\|_{HS}^2 \frac{d\rho}{\rho} \right) \left(\frac{1}{\pi^{2n}} \int_{R^n} \|\widehat{h}(\lambda)\|_{HS}^2 |\lambda|\, d\lambda \right) \\
&= C_h \|h\|_{L^2(\mathbf{H}_1^n)}^2.
\end{aligned}$$

This is our desired result. □

Let $h \in AW_k$, $f \in H_k$, be the continuous wavelet transform of f with respect to h in H_k defined by

$$(W_h f)(x,y,t,\rho) = \langle f, U(x,y,t,\rho)h \rangle_{L^2(\mathbf{H}_1^n)}. \tag{2.4}$$

Theorem 2.2. *Let $f \in H_k \cap \mathcal{S}(\mathbf{H}_1^n)$, $h \in AW_k \cap \mathcal{S}(\mathbf{H}_1^n)$. Then for all $(x', y', t') \in \mathbf{H}_1^n$, the following inversion is valid:*

$$f(x', y', t') = \frac{1}{C_h} \int_{R^+} \int_{\mathbf{H}_1^n} (W_h f)(x, y, t, \rho) U(x, y, t, \rho) h(x', y', t') \frac{dx\, dy\, dt\, d\rho}{\rho^{2n+1}}. \tag{2.5}$$

If $h \in AW_k$, $f \in H_k$, then the above formula holds in the weak sense.

3. The subspaces $\mathcal{S}_{*,2}^{(l,2-l)}(\mathbf{H}_1^n)$ and $\mathcal{S}_{\mathcal{R}^{(l,2-l)}}(\mathbf{H}_1^n)$ of $\mathcal{S}(\mathbf{H}_1^n)$. In [**14**], Strichartz gave the definition of Radon transform $\mathcal{R}$ on the Heisenberg group and showed a fact that, even if a function f is very well behaved, the Radon transform $\mathcal{R}(f)$ may not be sufficiently deceasing at infinity. On the other hand, we can verify that if $f \neq g$ then $\mathcal{R}(f) \neq \mathcal{R}(g)$. Thus we naturally hope to find a subspace of Schwartz functions space on which the Radon transform is a bijection. When one considers the problems of radial functions on the Heisenberg group, the fundamental manifold is the Laguerre hypergroup $\mathbf{K} = [0, +\infty) \times R$. Nessibi and Trimèche [**9**] defined a subspace $\mathcal{S}_{*,2}(\mathbf{K})$ of Schwartz function space on which the Radon transform is a bijection. He [**3**] gave another subspace $\mathcal{S}_{\mathcal{R}_\alpha}(\mathbf{K})$ of $\mathcal{S}(\mathbf{K})$ which is equivalent to $\mathcal{S}_{*,2}(\mathbf{K})$. Without loss of generality, we may assume that the number n of Cartesian product $\mathbf{H}_1^n$ is 2. We are now in a position to define the Radon transforms $\mathcal{R}^{(l,2-l)}$ $(l = 1, 2)$, and introduce subspaces $\mathcal{S}_{*,2}^{(l,2-l)}(\mathbf{H}_1^n)$ and $\mathcal{S}_{\mathcal{R}^{(l,2-l)}}(\mathbf{H}_1^n)$. Our results are analogous to those in [**3**].

We say that $\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)$, a subspace of $\mathcal{S}(\mathbf{H}_1^2)$, consists of all functions g in $\mathcal{S}(\mathbf{H}_1^2)$, satisfying

(1) For all $x, y \in R^2$, $t_2 \in R$, $j_1 \in Z^+$, $\int_R t_1^{j_1} g(x, y, t)\, dt_1 = 0$;

(2) For all $x, y \in R^2$, $t_1 \in R$, $j_2 \in Z^+$, $\int_R t_2^{j_2} g(x, y, t)\, dt_2 = 0$;

(3) For all $x, y \in R^2$, $(j_1, j_2) \in (Z^+)^2$, $\int_{R\times R} t_1^{j_1} t_2^{j_2} g(x, y, t)\, dt_1\, dt_2 = 0$.

That is, $\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)$ can be denoted by

$$\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2) = \left\{ g \in \mathcal{S}(\mathbf{H}_1^2) : g \text{ satisfies (1), (2) and (3) defined above} \right\}. \tag{3.1}$$

Similarly, $\mathcal{S}_{*,2}^{(1,1)}(\mathbf{H}_1^2)$ is defined by

$$\mathcal{S}_{*,2}^{(1,1)}(\mathbf{H}_1^2) = \Big\{ g \in \mathcal{S}(\mathbf{H}_1^2) : \int_R t_2^{j_2} g(x,y,t)\, dt_2 = 0 \quad \text{for all } x, y \in R^2, t_1 \in R, j_2 \in Z^+ \Big\}. \tag{3.2}$$

Let $u' = (0, u_2)$, $v' = (0, v_2)$, $t' = (0,0)$. We define the Radon transform $\mathcal{R}^{(1,1)}$ on $\mathbf{H}_1^2$ by

(3.3)

$$\begin{aligned} \mathcal{R}^{(1,1)}(f)(x,y,t) &= \int_{R\times R} f((x,y,t)(u',v',t'))\, du_2\, dv_2 \\ &= \int_{R\times R} f(x^\star, y^\star, (t_1, t_2 + 2(y_2u_2 - x_2v_2)))\, du_2\, dv_2, \end{aligned}$$

where $x^\star = (x_1, u_2)$, $y^\star = (y_1, v_2)$. $\mathcal{R}^{(1,1)}$ is called the Radon transform with respect to the center variable t_2; here we call it the partial Radon transform. Similarly, we may define the Radon transform with respect to the center variable t_1. Let $u' = (u_1, u_2)$, $v' = (v_2, v_2)$, $t' = (0,0)$, the Radon transform $\mathcal{R}^{(2,0)}$ on $\mathbf{H}_1^2$ can be stated by

$$\begin{aligned} \mathcal{R}^{(2,0)}(f)(x,y,t) &= \int_{R^4} f((x,y,t)(u',v',t'))\, du_1\, dv_1\, du_2\, dv_2 \\ &= \int_{R^4} f(x^*, y^*, (t_1 + 2(y_1u_1 - x_1v_1), t_2 \\ &\qquad + 2(y_2u_2 - x_2v_2)))\, du_1\, dv_1\, du_2\, dv_2, \end{aligned} \tag{3.4}$$

where $x^* = (u_1, u_2)$, $y^* = (v_1, v_2)$. We shall introduce a subspace $\mathcal{S}_{\mathcal{R}^{(l,2-l)}}(\mathbf{H}_1^2)$ of $\mathcal{S}(\mathbf{H}_1^2)$ such that the Radon transform $\mathcal{R}^{(l,2-l)}$ $(l = 1, 2)$ is a bijection.

Now, we write

$$\mathcal{F}_3(f)(x,y,(t_1,\lambda_2)) = \int_R f(x,y,t) e^{i\lambda_2 t_2} dt_2.$$

Then we get

$$\begin{aligned}
&\left(\widehat{\mathcal{R}^{(1,1)}(f)}(\lambda)\Phi_k\right)(\eta)\\
&= \int_{\mathbf{H}_1^2} \mathcal{R}^{(1,1)}(f)(x,y,t)\Pi_\lambda(x,y,t)\Phi_k(\eta - x)\,dx\,dy\,dt\\
&= \int_{R\times R} \mathcal{F}_3(f)(x^\star, y^\star, (t_1, \lambda_2))\\
&\left(\int_{R\times R} e^{-2i\lambda_2 x_2 y_2 + 4i\lambda_2\eta_2 y_2 - 2i\lambda_2(y_2 u_2 - x_2 v_2)}\phi_{k_2}(\eta_2 - x_2)\,dx_2\,dy_2\right)\\
&\quad \times \left(\int_{\mathbf{H}_1} e^{i\lambda_1 t_1 - 2i\lambda_1 x_1 y_1 + 4i\lambda_1\eta_1 y_1}\phi_{k_1}(\eta_1 - x_1)\,dx_1\,dy_1\,dt_1\right)du_2\,dv_2.
\end{aligned}$$

Let $\widehat{\phi}_{k_2}$ denote the Fourier transform of ϕ_{k_2} on R. Then we have

$$\begin{aligned}
&\int_{R\times R} e^{-2i\lambda_2 x_2 y_2 + 4i\lambda_2\eta_2 y_2 - 2i\lambda_2(y_2 u_2 - x_2 v_2)}\phi_{k_2}(\eta_2 - x_2)\,dx_2\,dy_2\\
&\qquad = \int_R e^{2i\lambda_2\eta_2 v_2 - 2i\lambda_2 y_2 u_2 + 2i\lambda_2\eta_2 y_2}\\
&\qquad\quad \times \left(\int_R \phi_{k_2}(\eta_2 - x_2)e^{-2i\lambda_2(\eta_2 - x_2)(v_2 - y_2)}\,dx_2\right)dy_2\\
&\qquad = e^{i\lambda_2\eta_2 v_2 - 2i\lambda_2 v_2(u_j - \eta_2)}\\
&\qquad\quad \times \int_R \widehat{\phi_{k_2}}(2\lambda_2(v_2 - y_2))e^{2i\lambda_2(v_2 - y_2)(u_2 - \eta_2)}\,dy_2\\
&\qquad =|\lambda_2|^{-1}\pi e^{-2i\lambda_2 v_2 u_2 + 4i\lambda_2\eta_2 v_2}\phi_{k_2}(u_2 - \eta_2).
\end{aligned}$$

On the other hand, by the recursion formula of Hermite polynomials we then obtain

$$\Phi_k(\eta_1, -\eta_2) = (-1)^{|k_2|}\Phi_k(\eta_1, \eta_2).$$

Consequently,

$$
\begin{aligned}
&\left(\widehat{\mathcal{R}^{(1,1)}(f)}(\lambda)\right)\Phi_k)(\eta)\\
&=\int_{\mathbf{H}_1^2}\mathcal{R}^{(1,1)}(f)(x,y,t)\Pi_\lambda(x,,y,t)\Phi_k(\eta-x)\,dx\,dy\,dt\\
&=\int_{R\times R}\mathcal{F}_3(f)(x^\star,y^\star,(t_1,\lambda_2)(|\lambda_2|^{-1}\pi e^{-2i\lambda_2v_2u_2+4i\lambda_2v_2}\phi_{k_2}(u_2-\eta_2)))\\
&\quad\times\left(\int_{\mathbf{H}_1}e^{i\lambda_1t_1-2i\lambda_1x_1y_1+4i\lambda_1\eta_1y_1}\phi_{k_1}(\eta_1-x_1)\,dx_1\,dy_1\,dt_1\right)du_2\,dv_2\\
&=|\lambda_2|^{-1}\pi\int_{\mathbf{H}_1^2}f(x^\star,y^\star,t)e^{i\lambda_2t_2-2i\lambda_2u_2v_2+4i\lambda_2\eta_2v_2}\\
&\qquad\qquad\times e^{i\lambda_1t_1-2i\lambda_1x_1y_1+4i\lambda_1\eta_1y_1}\\
&\qquad\qquad\qquad\times\phi_{k_1}(\eta_1-x_1)\phi_{k_2}(u_2-\eta_2)\,dx^\star dy^\star dt\\
&=(-1)^{|k_2|}\pi|\lambda_2|^{-1}(\widehat{f}(\lambda)\Phi_k)(\eta).
\end{aligned}
$$

From the above discussion, we can get

$$|\lambda_2|(\widehat{R^{(1,1)}(f)}(\lambda)\Phi_k)(\eta)=(-1)^{|k_2|}\pi(\widehat{f}(\lambda)\Phi_k)(\eta).$$

It should now be clear that

$$\widehat{\mathcal{R}^{(1,1)}(f)}(\lambda)=(-1)^{|k_2|}\pi|\lambda_2|^{-1}\widehat{f}(\lambda). \tag{3.5}$$

For $j\geq 0$, we define the operator $\mathcal{R}^{(1,1)^{j+1}}=\mathcal{R}^{(1,1)}\mathcal{R}^{(1,1)^j}$, where $\mathcal{R}^{(1,1)^0}=I$ is the identity operator. And $\mathcal{R}^{(1,1)^{-j}}$ is the inverse operator of $\mathcal{R}^{(1,1)^j}$. Let Z be the set of all integers. Thus for every $j\in Z$, the operator $\mathcal{R}^{(l,2-l)^j}$ has the definite meaning. It is easy to verify that for $f\in H_k$,

$$\widehat{\mathcal{R}^{(1,1)^j(f)}}(\lambda)=(-1)^{j|k_2|}\pi^j|\lambda_2|^{-j}\widehat{f}(\lambda). \tag{3.6}$$

Similarly, we have

$$\widehat{\mathcal{R}^{(2,0)^j}}(f)(\lambda)=(-1)^{j|k|}\pi^{2j}|\lambda_1\lambda_2|^{-j}\widehat{f}(\lambda), \tag{3.7}$$

where $|k| = k_1 + k_2$. Using the Plancherel formula on $\mathcal{R}^{(l,2-l)^j}$, we can define a subspace $\mathcal{S}_{\mathcal{R}}^{(l,2-l)}(\mathbf{H}_1^2)$ of $\mathcal{S}(\mathbf{H}_1^2)$ as follows:

$$
\begin{aligned}
(3.8)\quad \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2) = \Big\{ f \in \mathcal{S}(\mathbf{H}_1^2) : \|\mathcal{R}^{(1,1)^j}(f)\|^2_{L^2(\mathbf{H}_1^2)} \\
= \pi^{2j-4} \int_{(R^+)^2} |\lambda_1||\lambda_2|^{-2j+1} \|\widehat{f}(\lambda)\|^2_{HS}\, d\lambda \\
< +\infty \text{ for all } j \in Z \Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9)\quad \mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2) = \Big\{ f \in \mathcal{S}(\mathbf{H}_1^2) : \|\mathcal{R}^{(2,0)^j}(f)\|^2_{L^2(\mathbf{H}_1^2)} \\
= \pi^{4j-4} \int_{(R^+)^2} |\lambda_1\lambda_2|^{-2j+1} \|\widehat{f}(\lambda)\|^2_{HS}\, d\lambda \\
< +\infty \text{ for all } j \in Z \Big\}.
\end{aligned}
$$

Our principal goal in this section is to show the following results:

$$
\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2) = \mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2), \qquad \mathcal{S}_{*,2}^{(1,1)}(\mathbf{H}_1^2) = \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2).
$$

Let $m = (m_1, m_2) \in (Z^+)^2$,

$$
D_t^m f(x, y, t^0) = \frac{\partial^{|m|}}{\partial t_1^{m_1} \partial t_2^{m_2}} f(x, y, t)|_{(x,y,t^0)},
$$
$$
D_t^m f(x, y, tz) = \frac{\partial^{|m|}}{\partial t_1^{m_1} \partial t_2^{m_2}} f(x, y, t)|_{(x,y,tz)},
$$

where $tz = (t_1z_1, t_2z_2)$, $z = (z_1, z_2) \in R^2$, $t^0 = (t_1^0, t_2^0)$, and at least one of t_1^0, t_2^0 equals to 0. Let $g(x, y, t) \in \mathcal{S}(\mathbf{H}_1^2)$, $\widetilde{g}(x, y, \lambda)$ denote the Fourier transform of $g(x, y, t)$ with respect to t, i.e.,

$$
\widetilde{g}(x, y, \lambda) = \int_{R^2} g(x, y, t) e^{i\lambda \cdot t} dt,
$$

where $\lambda \cdot t = \lambda_1 t_1 + \lambda_2 t_2$. We claim that $g \in \mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)$ is equivalent to $D_\lambda^m \widetilde{g}(x, y, \lambda)|_{g(x,y,\lambda^0)} = 0$ for all $m \in (Z^+)^2$, where $\lambda^0 = (\lambda_1^0, \lambda_2^0)$ and at least one of λ_1^0, λ_2^0 equals to 0.

Theorem 3.1. *Let* $f \in \mathcal{S}(\mathbf{H}_1^2)$. *Then* $D_t^m f(x,y,t^0) = 0$ *for all* $m = (m_1, m_2) \in (Z^+)^2$ *if and only if* $f(x,y,t) = t_1^{r_1} t_2^{r_2} g_r(x,y,t)$, *where* $g_r(x,y,t) \in \mathcal{S}(\mathbf{H}_1^2)$, $r = (r_1, r_2) \in (Z^+)^2$, $D_t^m g_r(x,y,t^0) = 0$.

Proof. For convenience, D_{t_j} $(j = 1, 2)$ is used to denote $\partial/(\partial t_j)$. Obviously,

$$f(x,y,t) = t_1 \int_0^1 D_{t_1} f(x,y,(t_1 z_1, t_2))\, dz_1 + f(x,y,(0,t_2^0)).$$

Taking $m = (0, 0, \dots, 0)$ in the condition $D_t^m f(x,y,t^0) = 0$, we get $f(x,y,t^0) = 0$. Thus $f(x,y,t)$ can be represented in the form:

$$f(x,y,t) = t_1 \int_0^1 D_{t_1} f(x,y,(t_1 z_1, t_2))\, dz_1.$$

Set $g_{(1,0)}(x,y,t) = \int_0^1 D_{t_1} f(x,y,(t_1 z_1, t_2))\, dz_1$. We shall prove that $g_{(1,0)} \in \mathcal{S}(\mathbf{H}_1^2)$. In fact,

$$\begin{aligned}
&\int_0^1 D_{t_1} f(x,y,(t_1 z_1, t_2))\, dz_1 \\
&\quad = \frac{1}{t_1} \int_0^{t_1} D_{t_1} f(x,y,(z_1, t_2))\, dz_1 \\
&\quad = \frac{1}{t_1} \left[\int_0^{\infty} D_{t_1} f(x,y,(z_1, t_2))\, dz_1 - \int_{t_1}^{\infty} D_{t_1} f(x,y,(z_1, t_2))\, dz_1 \right] \\
&\quad = - \int_{t_1}^{\infty} D_{t_1} f(x,y,(z_1, t_2))\, dz_1 \\
&\quad = O\left(\frac{1}{t^l} \right)
\end{aligned}$$

as $|t| \to +\infty$ for all $l \in (Z^+)^2$. This implies that $g_{(1,0)} \in \mathcal{S}(\mathbf{H}_1^2)$. Similarly, we can get

$$f(x,y,t) = t_2 \int_0^1 D_{t_2} f(x,y,(t_1, t_2 z_2))\, dz_2,$$

where $g_{(0,1)}(x,y,t) = \int_0^1 D_{t_2} f(x,y,(t_1, t_2 z_2))\, dz_2 \in \mathcal{S}(\mathbf{H}_1^2)$. Observing the following formula

$$\begin{aligned}
f(x,y,t) = t_1 t_2 \int_0^1 \int_0^1 D_t^{(1,1)} f(x,y,(t_1 z_1, t_2 z_2))\, dz_1 dz_2 - f(x,y,(0,0)) \\
+ [f(x,y,(0,t_2^0)) + f(x,y,(t_1^0, 0))]
\end{aligned}$$

together with $f(x,y,t^0)=0$, we get

$$f(x,y,t)=t_1t_2\int_0^1\int_0^1 D_t^{(1,1)}f(x,y,(t_1z_1,t_2z_2))\,dz_1\,dz_2.$$

Write

$$g_{(1,1)}(x,y,t)=\int_0^1\int_0^1 D_t^{(1,1)}f(x,y,(t_1z_1,t_2z_2))\,dz_1\,dz_2;$$

we shall show that $g_{(1,1)}(x,y,t)\in\mathcal{S}(\mathbf{H}_1^2)$. From the above discussion we can see that

$$\begin{aligned}
\int_0^1\int_0^1 & D_t^{(1,1)}f(x,y,(t_1z_1,t_2z_2))\,dz_1\,dz_2\\
&=\frac{1}{t_1t_2}\int_0^{t_1}\int_0^{t_2} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\\
&=\frac{1}{t_1t_2}\Big[\int_0^{\infty}\int_0^{\infty} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\\
&\qquad-\int_0^{t_1}\int_0^{\infty} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\\
&\qquad-\int_0^{\infty}\int_0^{t_2} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\\
&\qquad-\int_{t_1}^{\infty}\int_{t_2}^{\infty} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\Big]\\
&=-\frac{1}{t_1t_2}\Big[\int_0^{t_1}\int_0^{\infty} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\\
&\qquad+\int_0^{\infty}\int_0^{t_2} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\\
&\qquad+\int_{t_1}^{\infty}\int_{t_2}^{\infty} D_t^{(1,1)}f(x,y,(z_1,z_2))\,dz_1\,dz_2\Big]\\
&=O\Big(\frac{1}{t^l}\Big)
\end{aligned}$$

as $|t|\to+\infty$ for all $l\in(Z^+)^2$. Interchanging the differential operator D_t^m and integral together with $D_t^m f(x,y,t^0)=0$ for all $m\in(Z^+)^2$, we can prove that $D_t^m g_r(x,y,t^0)=0$ for all $m\in(Z^+)^2$, where $r=(r_1,r_2)$, $r_1,r_2\in\{0,\ 1\}$. Evidently, $f(x,y,t^0)=t_1^{r_1}t_2^{r_2}g_r(x,y,t)$ can be shown in the same way by induction, and $D_t^m g_r(x,y,t^0)=0$ for all $m\in(Z^+)^2$. The sufficiency is obvious. □

Theorem 3.2.

$$\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)=\mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2). \tag{3.10}$$

Proof. Let $f\in\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)$. By

$$\int_R t_1^{j_1}g(x,y,t)\,dt_1=0,\qquad \int_R t_2^{j_2}g(x,y,t)\,dt_2=0$$

together with $\int_{R^2}t_1^{j_1}t_2^{j_2}g(x,y,t)\,dt_1\,dt_2=0$ for all $(j_1,j_2)\in(Z^+)^2$, we can deduce that $D_\lambda^m\widetilde{f}(x,y,\lambda^0)=0$ for all $m\in(Z^+)^2$, where

$$\widetilde{f}(x,y,\lambda)=\int_{R^2}f(x,y,t)e^{i\lambda\cdot t}\,dt.$$

Clearly, $\widetilde{f}(x,y,\lambda)\in\mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$. By Theorem 2.2, we can get

$$\widetilde{f}(x,y,\lambda)=\lambda_1^{r_1}\lambda_2^{r_2}\widetilde{g}(x,y,\lambda)$$

for all $k\in(Z^+)^2$. Taking the group Fourier transform for $g(x,y,t)$ on $\mathbf{H}_1^2$, then we get $\widehat{f}(\lambda)=\lambda_1^{r_1}\lambda_2^{r_2}\widehat{g}_r(\lambda)$. Obviously, we have $\widehat{f}(\lambda)=\lambda_1^{r'_1}\lambda_2^{r'_2}\widehat{g}_{r'}(\lambda)$ for all $r'\in(Z^+)^2$. By formula (1.10) we have

$$\int_{R^2}|\widehat{f}(\lambda)|^2|\lambda_1|^{-2r'_1}|\lambda_2|^{-2r'_2}|\lambda|\,d\lambda=\int_{R^2}|\widehat{g}_{r'}(\lambda)|^2|\lambda|\,d\lambda=\|g_{r'}\|^2<+\infty$$

for all $r'\in(Z^+)^2$, which implies $f\in\mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$. On the other hand, if $f\in\mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$, we can deduce that $D_\lambda^m\widetilde{f}(x,y,\lambda^0)=0$ for all $m\in(Z^+)^2$ and $x,y\in R^2$. It follows that $f\in\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)$. Thus Theorem 3.2 is proved. □

In the next section, we will show that $\mathcal{R}^{(2,0)}$ on $\mathcal{S}_{*,2}^{(2,0)}(\mathbf{H}_1^2)$ is a bijection, as is the Radon transform $\mathcal{R}^{(1,1)}$ on $\mathcal{S}_{*,2}^{(1,1)}(\mathbf{H}_1^2)$.

4. Inversion of the radon transforms. In the above sections, we investigate the theory of continuous wavelet transforms and define the Radon transforms $\mathcal{R}^{(l,2-l)} (l = 1, 2)$. We also introduce subspaces $\mathcal{S}_{\mathcal{R}^{(l,2-l)}}(\mathbf{H}_1^2)$ and $\mathcal{S}_{*,2}^{(l,2-l)}(\mathbf{H}_1^2)$. This section is devoted to the explicit inversion of the Radon transform by using the inverse wavelet transform. In addition, we characterize another subspace of $L^2(\mathbf{H}_1^2)$ such that the inversion formula of Radon transform holds in the weak sense.

Let $L = -\partial/(i\,\partial t_2)$, $f \in \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2)$. Then

$$\int_R L(f)(x,y,t)e^{i\lambda_2 t_2}\,dt_2 = \lambda_2 \mathcal{F}_3(f)(x,y,t_1,\lambda_2).$$

Therefore,

$$\widehat{L(f)}(\lambda) = \lambda_2 \widehat{f}(\lambda). \tag{4.1}$$

By (3.5) and (4.1) we have

$$(\widehat{L\mathcal{R}^{(1,1)}}(f)(\lambda)\Phi_k)(\eta) = (-1)^{|k_2|}\pi(\widehat{f}(\lambda)\Phi_k)(\eta). \tag{4.2}$$

It is easy to see that $(\widehat{L\mathcal{R}^{(1,1)} f})^2(\lambda) = \pi^2 \widehat{f}(\lambda)$, thus ${\mathcal{R}^{(1,1)}}^{-1}(f) = \pi^{-2}L\mathcal{R}^{(1,1)}L(f)$. Also, let $\mathcal{L} = (\partial/i\partial t_1)(\partial/i\partial t_2)$. Then ${\mathcal{R}^{(2,0)}}^{-1}(f) = \pi^{-4}\mathcal{L}\mathcal{R}^{(2,0)}\mathcal{L}(f)$. Therefore we have

Theorem 4.1. (1) *Let* $f \in \mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$. *Then* ${\mathcal{R}^{(2,0)}}^{-1}(f) = \pi^{-4}\mathcal{L}\mathcal{R}^{(2,0)}\mathcal{L}(f)$.

(2) *Let* $f \in \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2)$. *Then* ${\mathcal{R}^{(1,1)}}^{-1}(f) = \pi^{-2}L\mathcal{R}^{(1,1)}L(f)$.

Proof. We shall demonstrate part (1). Similar to (3.8) and (3.9), we can define

(4.3)

$$L^2_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2) = \Big\{ f \in L^2(\mathbf{H}_1^2) : \pi^{2j-4}\int_{(R^+)^2} |\lambda_1||\lambda_2|^{-2j+1}\|\widehat{f}(\lambda)\|^2_{HS}d\lambda < +\infty \text{ for all } j \in Z \Big\}$$

and

(4.4)

$$L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2) = \Big\{ f \in L^2(\mathbf{H}_1^2) : \pi^{4j-4} \int_{(R^+)^2} |\lambda_1\lambda_2|^{-2j+1} \|\widehat{f}(\lambda)\|^2_{HS} d\lambda < +\infty \text{ for all } j \in Z \Big\}.$$

It is not hard to show that $\mathcal{R}^{(2,0)}$ is a bijection from $L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$ onto itself. In fact, let $f \in L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$; by (3.7) together with the Plancherel identity (1.10) we have $\mathcal{R}^{(2,0)}(f) \in L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$. If $\mathcal{R}^{(2,0)}(f) = \mathcal{R}^{(2,0)}(g)$, then for almost everywhere $\lambda \in \Lambda$,

$$0 = \widehat{\mathcal{R}^{(2,0)}}(f-g)(\lambda) = (-1)^{|k|}\pi^2|\lambda|^{-1}(\widehat{f}(\lambda) - \widehat{g}(\lambda)),$$

which implies $f = g$. Thus $\mathcal{R}^{(2,0)}$ is a bijection from $L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$ onto itself. Because $\mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2) = L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2) \bigcap \mathcal{S}(\mathbf{H}_1^2)$, so $\mathcal{L}\mathcal{R}^{(2,0)}\mathcal{L}$ is well defined on $\mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)$, which completes the proof. □

Let $h_\rho(x,y,t) = h((x/\sqrt{\rho}),(y/\sqrt{\rho}),(t/\rho 0)$. Then

$$\widehat{h_\rho}(\lambda) = \rho^4 \widehat{h}(\rho\lambda). \tag{4.5}$$

Define an operator $\widetilde{W}$ by

$$\left(\widetilde{W_h} f\right)(x,y,t) = \int_{\mathbf{H}_1^2} f(x',y',t')\overline{h}((x,y,t)(x',y',t')^{-1})dx'dy'dt'.$$

Then

$$\left(\widehat{\widetilde{W_h} f}\right)(\lambda) = \widehat{f}(\lambda)\widehat{h}(\lambda)^*. \tag{4.6}$$

By identities (4.1) and (4.5) together with (4.6), we can derive that

$$\left(\widehat{\widetilde{W}_{L(h)_\rho} f}\right)(\lambda) = \widehat{f}(\lambda)\widehat{L(h)_\rho}(\lambda)^* = \lambda_2\rho^5\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.$$

Similarly, we have

$$(\widehat{\widetilde{W}_{h_\rho} L(f)})(\lambda) = \widehat{L(f)}(\lambda)\widehat{h_\rho}(\lambda)^* = \lambda_2\rho^4\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.$$

Hence, we obtain

$$\left(\widetilde{W}_{L(h)_\rho} f\right)(x,y,t) = \rho(\widetilde{W}_{h_\rho} L(f))(x,y,t). \tag{4.7}$$

Theorem 4.2. *Let $h \in \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2) \cap AW_k$, $f \in \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2)$. Then*

$$\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}L(h)_\rho} \mathcal{R}^{(1,1)}(f)\right)(x,y,t) = \pi^2\rho^3(W_h f)(x,y,t,\rho). \tag{4.8}$$

Proof. It is easy to verify the commutative relation of L and $\mathcal{R}^{(1,1)}$, i.e., $L\mathcal{R}^{(1,1)} = \mathcal{R}^{(1,1)}L$. Thus by (4.7) we have

$$\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}L(h)_\rho} \mathcal{R}^{(1,1)}(f)\right)(x,y,t) = \rho\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}(h)_\rho} L\mathcal{R}^{(1,1)}(f)\right)(x,y,t).$$

Using identities (4.1) and (4.2) together with (4.5), we obtain

$$\begin{aligned}\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}\widehat{L(h)_\rho}} \mathcal{R}^{(1,1)}(f)\right)(\lambda) &= \rho\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}\widehat{(h)_\rho}} L\mathcal{R}^{(1,1)}(f)\right)(\lambda)\\ &= \pi^2\rho^5\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.\end{aligned}$$

On the other hand, we can deduce that

$$\widehat{W_h f}(\lambda) = \rho^2\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.$$

Thus our proposition is established. □

We now state the following

Theorem 4.3. *Let $h \in \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2) \cap AW_k$, $f \in \mathcal{S}_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2) \cap H_k$. Then for all $(x,y,t) \in \mathbf{H}_1^2$, we have*

$$\begin{aligned}f(x,y,t) = \frac{1}{\pi^2 C_h}\int_{\mathbf{H}_1^2\times R^+} &\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}L(h)_\rho} \mathcal{R}^{(1,1)}(f)\right)(x',y',t')\\ &U(x',y',t',\rho)h(x,y,t)\frac{dx'dy'dt'd\rho}{\rho^8}.\end{aligned} \tag{4.9}$$

And the inverse Radon transform $\mathcal{R}^{(1,1)^{-1}}$ *holds*:

(4.10)

$$\mathcal{R}^{(1,1)^{-1}}(f)(x,y,t) = \frac{1}{\pi^2 C_h}\int_{\mathbf{H}_1^2\times R^+}\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}L(h)_\rho}f\right)(x',y',t')\, U(x',y',t',\rho)h(x,y,t)\frac{dx'dy'dt'd\rho}{\rho^8}.$$

Generally, if $f \in L^2_{\mathcal{R}^{(1,1)}}(\mathbf{H}_1^2)\cap H_k$*, then* (4.9) *and* (4.10) *hold in the weak sense.*

Proof. By using (2.5) and (4.8), we then obtain

$$\begin{aligned}
&\frac{1}{\pi^2 C_h}\int_{\mathbf{H}_1^2\times R^+}\left(\widetilde{W}_{L\mathcal{R}^{(1,1)}L(h)_\rho}\mathcal{R}^{(1,1)}(f)\right)(x',y',t') \\
&\qquad\qquad\times U(x',y',t',\rho)h(x,y,t)\frac{dx'dy'dt'd\rho}{\rho^8} \\
&= \frac{1}{C_h}\int_{\mathbf{H}_1^2\times R^+}(W_h f)(x',y',t')U(x',y',t',\rho)h(x,y,t)\frac{dx'dy'dt'd\rho}{\rho^5} \\
&= f(x,y,t).
\end{aligned}$$

Taking the inverse Radon transform $\mathcal{R}^{(1,1)^{-1}}$ on f, we can get formula (4.10) immediately. □

For the Radon transform $\mathcal{R}^{(2,0)}$, we have the following inversion formula.

Theorem 4.4. *Let* $h \in \mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)\cap AW_k$*,* $f \in \mathcal{S}_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2)\cap H_k$*. Then for all* $(x,y,t)\in \mathbf{H}_1^2$*, we have*

$$f(x,y,t) = \frac{1}{\pi^4 C_h}\int_{\mathbf{H}_1^2\times R^+}\left(\widetilde{W}_{\mathcal{L}\mathcal{R}^{(2,0)}\mathcal{L}(h)_\rho}\mathcal{R}^{(2,0)}(f)\right)(x',y',t')\, U(x',y',t',\rho)h(x,y,t)\frac{dx'dy'dt'd\rho}{\rho^9}. \tag{4.11}$$

And the inverse operator $\mathcal{R}^{(2,0)^{-1}}$ *is the following*:

(4.12)

$$\mathcal{R}^{(2,0)^{-1}}(f)(x,y,t) = \frac{1}{\pi^4 C_h} \int_{\mathbf{H}_1^2 \times R^+} \left(\widetilde{W}_{\mathcal{L}\mathcal{R}^{(2,0)}\mathcal{L}(h)_\rho} f\right)(x',y',t') \\ U(x',y',t',\rho) h(x,y,t) \frac{dx'dy'dt'd\rho}{\rho^9}.$$

Generally, if $f \in L^2_{\mathcal{R}^{(2,0)}}(\mathbf{H}_1^2) \cap H_k$, *then the above two formulae are also valid in the weak sense.*

For general n, let l be a positive integer satisfying $1 \le l \le n$; we can define the Radon transforms $\mathcal{R}^{(l,n-l)}$ and the partial differential operator L such that $\mathcal{R}^{(l,n-l)^{-1}} = \pi^{-2(n-l)} L \mathcal{R}^{(l,n-l)} L$ on the subspace $\mathcal{S}_{\mathcal{R}^{(l,n-l)}}(\mathbf{H}_1^n)$.

Finally, we conclude this section by exhibiting an analog of Theorems 4.3 and 4.4 for this case.

(4.13)

$$f(x,y,t) = \frac{1}{\pi^{2(n-l)} C_h} \int_{\mathbf{H}_1^n \times R^+} \left(\widetilde{W}_{L\mathcal{R}^{(l,n-l)}L(h)_\rho} \mathcal{R}^{(l,n-l)}(f)\right)(x',y',t') \\ U(x',y',t',\rho) h(x,y,t) \frac{dx'dy'dt'd\rho}{\rho^{3n+l+1}}$$

and

(4.14)

$$\mathcal{R}^{(l,n-l)^{-1}}(f)(x,y,t) = \frac{1}{\pi^{2(n-l)} C_h} \int_{\mathbf{H}_1^n \times R^+} \left(\widetilde{W}_{L\mathcal{R}^{(l,n-l)}L(h)_\rho} f\right)(x',y',t') \\ U(x',y',t',\rho) h(x,y,t) \frac{dx'dy'dt'd\rho}{\rho^{3n+l+1}}.$$

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