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EQUIVALENCE OF REAL MILNOR FIBRATIONS FOR QUASI-HOMOGENEOUS SINGULARITIES

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ABSTRACT. We show that for real quasi-homogeneous singularities $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^2, 0)$ with isolated singular point at the origin, the projection map of the Milnor fibration $S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^1$ is given by $f/\|f\|$. Moreover, for these singularities the two versions of the Milnor fibration, on the sphere and on a Milnor tube, are equivalent. In order to prove this, we show that the flow of the Euler vector field plays and important role. In addition, we present, in an easy way, a characterization of the critical points of the projection $(f/\|f\|) : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^1$.

1. Introduction. It is well known that in [**8**] Milnor studies certain fibrations associated to analytic functions in a neighborhood of an isolated critical point.

For real analytic mappings, he showed that given $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^p, 0)$, $m \geq p \geq 2$, a real polynomial map germ whose derivative Df has rank p on a punctured neighborhood U of $0 \in \mathbf{R}^m$, there exist $\varepsilon > 0$ small and $\eta > 0$, $0 < \eta \ll \varepsilon \ll 1$, such that if we set $E := B_\varepsilon^m(0) \cap f^{-1}(S_\eta^{p-1})$ (called a Milnor tube), then

$$f_{|_E} : E \longrightarrow S_\eta^{p-1}$$

is a smooth (locally trivial) fibre bundle, where $B_\varepsilon^m(0)$ is the closed ball of radius $\varepsilon > 0$ and center 0 in $\mathbf{R}^m$, and S_η^{p-1} is the boundary of the closed ball $B_\eta^p(0)$ in $\mathbf{R}^p$.

This fibration induces a fibration given by

$$(1) \qquad f_{|_{E^\circ}} : E^\circ \longrightarrow S_\eta^{p-1},$$

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where $E^\circ := B^\circ_\varepsilon(0) \cap f^{-1}(S^{p-1}_\eta)$ and $B^\circ_\varepsilon(0)$ denotes the interior of the ball $B^m_\varepsilon(0)$.

Furthermore, he used the flow of a vector field, let us say $\nu(x)$, in $B^m_\varepsilon(0) \setminus \{f = 0\}$ with the following conditions:

$$\begin{cases} \langle \nu(x), x \rangle > 0; \\ \langle \nu(x), \nabla(\|f(x)\|^2) \rangle > 0, \end{cases}$$

to construct a diffeomorphism that "pushes" E to $S^{m-1}_\varepsilon \setminus N_{K_\varepsilon}$, where N_{K_ε} denotes an open tubular neighborhood of the link K_ε in S^{m-1}_ε.

It is easy to see that, (see [**8**, page 99]) this diffeomorphism is the identity map on the boundary manifold $S^{m-1}_\varepsilon \cap f^{-1}(S^{p-1}_\eta)$. This fibration can always be extended to the complement of the link K_ε in the sphere S^{m-1}_ε, providing in this way the fibration

$$S^{m-1}_\varepsilon \setminus K_\varepsilon \longrightarrow S^{p-1}. \tag{2}$$

The problem is that, unlike in the complex case, the projection map of fibration (2), in general, may not be the obvious one, as is easily seen in Example 1.1 (or Example 3.2 for details) below.

Example 1.1 [**8**, page 99].

$$\begin{cases} P = x \\ Q = x^2 + y(x^2 + y^2). \end{cases}$$

That is, in the real case, we cannot expect that the map projection of the fibration (2) will be given by $f/\|f\|$.

Many mathematicians, see for instance [**2, 3, 6, 7, 9–13**] and their references, have been studying natural conditions on the real case under which the map

$$\frac{f}{\|f\|} : S^{m-1}_\varepsilon \setminus K_\varepsilon \longrightarrow S^{p-1} \tag{3}$$

is a smooth (locally trivial) fibration (or satisfies the strong Milnor's condition at the origin, see definition below), as well as the equivalence between fibrations (1) and (3).

In what follows we are first going to present an easy way to see a characterization of the critical points of the map $f/\|f\| : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^1$ (see Lemma 3.1), in order to understand when this map is a submersion. Then we will show that for quasi-homogeneous map germs from $(\mathbf{R}^m, 0) \to (\mathbf{R}^2, 0)$ the fibrations (1) and (3) are equivalent (see Definition 5.1).

These results show that quasi-homogeneous isolated singularities satisfy the so-called "strong Milnor condition" at the origin (see Definition 2.1 below), defined by Seade, Ruas and Verjovsky in [**11**].

It is worth saying that the problem studied in this article is approached in [**3**] in a more general setting.

2. Definitions and set-up.

Definition 2.1 [**11**]. Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^2, 0)$, $m \geq 2$, be an analytic map-germ with isolated singularity at the origin. If for all $\varepsilon > 0$ sufficiently small, the map $f/\|f\| : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^1$ is the projection of a smooth locally trivial fibre bundle, where K_ε is the link of singularity at 0 (or empty), then we say that the map satisfies the *strong Milnor condition* at the origin.

It is easy to see that the condition above holds for any holomorphic function germ with isolated singularity at origin, just considering the real and the imaginary parts as a real analytic map germ into the plane.

So, a natural question is: What kind of real analytic map-germs satisfy the strong Milnor condition at the origin?

This problem was first approached in [**6, 7**], and then further developed in [**2, 10–13**], and recently in [**3, 5**].

In [**10**], using stratification theory and Bekka's (c)-regularity (see [**4**]), the authors have proved that for quasi-homogeneous map-germs, as defined below, the map (3) is a smooth locally trivial fibration. In this paper we show that the fibrations (1) and (3) are equivalent. Moreover, the flow of the Euler vector field will provide the equivalence between them.

Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^2, 0)$, $f(x) = (P(x), Q(x))$, where the coordinate functions $P, Q : (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ are polynomials. Suppose that

$P, Q : (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ are quasi (weighted-)homogeneous of type $(w_1, \dots, w_m; b)$, i.e,

$$P(\lambda^{w_1} x_1, \dots, \lambda^{w_m} x_m) = bP(x)$$

and

$$Q(\lambda^{w_1} x_1, \dots, \lambda^{w_m} x_m) = bQ(x),$$

for some $b, w_i \in \mathbf{Z}_+^*$, $i = 1, \dots, m$ and $\lambda \in \mathbf{Z}_+^*$. In this case we will say that the functions P and Q are quasi-homogeneous of weights $\{w_1, \dots, w_m\}$ and degree b, or quasi-homogeneous of the same type and degree.

Let $\omega(x) = P(x)\nabla Q(x) - Q(x)\nabla P(x)$ and $\mathbf{e}(x) = (w_1 x_1, \dots, w_m x_m)$, where w_i are the weights given above. The vector field $\mathbf{e}(x)$ is called the Euler vector field of the functions P and Q. It is easy to see that this well-known vector field satisfies the following property:

$$\langle \nabla P(x), \mathbf{e}(x) \rangle = bP(x), \qquad \langle \nabla Q(x), \mathbf{e}(x) \rangle = bQ(x).$$

The main result is:

Theorem 2.2. *Let $f = (P, Q) : (\mathbf{R}^m, 0) \to (\mathbf{R}^2, 0)$ be a quasi-homogeneous polynomial map germ, with same type and degree, with an isolated critical point at the origin. Then, the fibrations* (1) *and* (3) *are equivalent.*

3. Critical points of the projection map. In what follows we are considering a real analytic map germ

$$f = (P, Q) : (\mathbf{R}^m, 0) \longrightarrow (\mathbf{R}^2, 0)$$

and the map

$$\frac{f}{\|f\|} : S_\varepsilon^{m-1} \setminus K_\varepsilon \longrightarrow S^1,$$

for all $\varepsilon > 0$ small enough.

Notation. For a general smooth function $g : \mathbf{R}^m \to \mathbf{R}$, for convenience we will denote the derivative $\partial g(x)/\partial x_i$ by $\partial_i g(x)$.

In the sequel we will see that the vector field $\omega(x)$ defined above plays an important role in the description of the critical points of the smooth projection map $f/\|f\|$.

The following result can be deduced from Chapter 4 in Milnor's book [**8**]. In [**6**] the author presented a different proof using the Curve Selection lemma. Below, I give a different argument.

Lemma 3.1 [**6**]. *A point $x \in S_\varepsilon^{m-1} \setminus K_\varepsilon$ is a critical point of $f/\|f\|$ if and only if $\omega(x) = \lambda x$, for some real number λ.*

Proof. Consider

$$\frac{f(x)}{\|f(x)\|} = \left(\frac{P(x)}{\sqrt{P(x)^2+Q(x)^2}}, \frac{Q(x)}{\sqrt{P(x)^2+Q(x)^2}}\right).$$

The Jacobian matrix is

$$J\left(\frac{f(x)}{\|f(x)\|}\right) = \begin{pmatrix} \nabla\left(\frac{P(x)}{\sqrt{P(x)^2+Q(x)^2}}\right) \\ \nabla\left(\frac{Q(x)}{\sqrt{P(x)^2+Q(x)^2}}\right) \end{pmatrix}.$$

So, using the notation above, we have:

$$(*) \quad \partial_i\left(\frac{P(x)}{\sqrt{P(x)^2+Q(x)^2}}\right) = \frac{\partial_i(P(x))\sqrt{P(x)^2+Q(x)^2} - P(x)\partial_i(\sqrt{P(x)^2+Q(x)^2})}{P(x)^2+Q(x)^2}.$$

The second summand is equal to

$$(\#) \qquad \frac{2(P(x)\partial_i P(x) + Q(x)\partial_i Q(x))}{2\sqrt{P(x)^2+Q(x)^2}}.$$

Combining $(*)$ and $(\#)$, we have:

$$\partial_i\left(\frac{P(x)}{\sqrt{P(x)^2+Q(x)^2}}\right) = \frac{-Q(x)}{(P(x)^2+Q(x)^2)^{3/2}}(P(x)\partial_i Q(x) - Q(x)\partial_i P(x)),$$

for all $i = 1, \dots, n$. This means that the first row of the Jacobian matrix is

$$\left(\frac{-Q(x)}{(P(x)^2 + Q(x)^2)^{3/2}} \right) \omega(x).$$

Similarly, we have that the second row is

$$\left(\frac{P(x)}{(P(x)^2 + Q(x)^2)^{3/2}} \right) \omega(x).$$

Therefore, the Jacobian matrix has the following rows

$$\begin{pmatrix} \left(\frac{-Q(x)}{(P(x)^2+Q(x)^2)^{3/2}} \right) \omega(x) \\ \left(\frac{P(x)}{(P(x)^2+Q(x)^2)^{3/2}} \right) \omega(x) \end{pmatrix}.$$

Since

$$J\left(\frac{f(x)}{\|f(x)\|} \right) : T_x(S^{m-1} \setminus K_\varepsilon) \longrightarrow T_{f(x)} S^1 \simeq \mathbf{R},$$

then $x \in S^{m-1} \setminus K_\varepsilon$ is a critical point of the projection $f(x)/\|f(x)\|$ if, and only if, for all vectors $v \in T_x(S^{m-1} \setminus K_\varepsilon)$ we have

$$J\left(\frac{f(x)}{\|f(x)\|} \right) \cdot v = \begin{pmatrix} \left(\frac{-Q(x)}{(P(x)^2+Q(x)^2)^{3/2}} \right) \cdot \langle \omega(x), v \rangle \\ \left(\frac{P(x)}{(P(x)^2+Q(x)^2)^{3/2}} \right) \cdot \langle \omega(x), v \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now, it is pretty easy to see that the critical locus of the map $f(x)/\|f(x)\|$ is precisely the set of points x such that the vector field $\omega(x)$ points towards the vector position x. □

Let us see next that in Milnor's example (see [**8**, page 99]) there is a curve of critical points of the map $f/\|f\|$ passing through the origin.

Example 3.2.

$$\begin{cases} P = x \\ Q = x^2 + y(x^2 + y^2). \end{cases}$$

It is easy to see that $\nabla P(x,y) = (1,0)$, $\nabla Q(x,y) = (2x+2xy, x^2+3y^2)$, so $\omega(x,y) = (x^2 + yx^2 - y^3, x^3 + 3xy^2)$. The points where the vector field $\omega(x, y)$ is parallel to the vector position (x, y) satisfy the following equation: $\langle \omega(x, y), (-y, x)\rangle = 0$.

From the last equation we have $0 = \langle \omega(x, y), (-y, x)\rangle = (x^2 + y^2)^2 - yx^2$. Using polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, we get $r = \sin(\theta) \cdot \cos^2(\theta)$, which is a non-degenerate curve through the origin.

The Milnor example clearly shows that, even thought in this case the link K_ε is empty, the map projection $f/\|f\|$ can have a curve of critical points going through the origin.

4. Preliminary results. In order to prove the main theorem, we will present some lemmas that give us important information. Let us see first that in our setting, i.e., for quasi-homogeneous maps, the projection $f/\|f\|$ has maximal rank in a small neighborhood of the origin.

Lemma 4.1. *Under the conditions of Theorem* 2.2 *above, we have* $\langle \omega(x), \mathbf{e}(x)\rangle = 0$.

Proof. This follows from an easy calculation since

$$\begin{aligned}\langle \omega(x), \mathbf{e}(x)\rangle &= P(x)\langle \nabla Q(x), \mathbf{e}(x)\rangle - Q(x)\langle \nabla P(x), \mathbf{e}(x)\rangle \\ &= P(x)bQ(x) - Q(x)bP(x) = 0. \qquad \square\end{aligned}$$

Remark 4.2. Since $\langle \mathbf{e}(x), x\rangle = \sum_{i=1}^{m} w_i x_i^2$, then $\langle \mathbf{e}(x), x\rangle = 0$ if and only if $x = 0$.

Lemma 4.3. *Consider* $V = \{x \in R^m : P(x) = Q(x) = 0\} = f^{-1}(0)$. *Then, for all* $x \in B_\varepsilon(0) \setminus V$ *the vector field* $\omega(x)$ *and the vector position* x *are not parallel.*

Proof. If $\omega(x)$ is parallel to x, then applying Lemma 4.1 we have $\langle x, \mathbf{e}(x)\rangle = 0$, but by Remark 4.2, this is a contradiction since $x \in B_\varepsilon(0) \setminus V$. $\square$

Lemma 4.4 [**7**]. *For $\varepsilon > 0$ small enough, the restriction α of the Euler vector field $\mathbf{e}(x)$ to $B_\varepsilon(0) \setminus V$ satisfies the following conditions*:

a) $\langle \alpha(x), x \rangle > 0$;

b) $\langle \alpha(x), \nabla(\|f(x)\|^2) \rangle > 0$, *where* $\nabla(\|f(x)\|^2) = 2 \cdot (P(x)\nabla P(x) + Q(x)\nabla Q(x))$;

c) $\langle \alpha(x), \omega(x) \rangle = 0$.

Proof. Conditions a) and c) follow from Lemma 4.1 and Remark 4.2. Let us prove condition b).

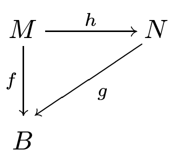
Since $\nabla(\|f(x)\|^2)(x) = 2 \cdot (P(x)\nabla P(x) + Q(x)\nabla Q(x))$, then

$$\langle \nabla(\|(P,Q)\|^2)(x), \alpha(x) \rangle = 2b \cdot (P^2(x) + Q^2(x)),$$

where b is the degree of quasi-homogeneity. Therefore, $\langle \nabla(\|(P,Q)\|^2)(x), \alpha(x) \rangle > 0$. $\square$

5. Equivalence of Milnor fibrations.

Definition 5.1. We say that two smooth locally trivial fibre bundles $f : M \to B$ and $g : N \to B$ are $\mathcal{C}^k-$equivalent (or $\mathcal{C}^k-$isomorphic), $k = 0, 1, \ldots, \infty$, if there exists a $\mathcal{C}^k-$diffeomorphism h such that the following diagram is commutative:



Proof (*Main theorem* 2.2). Given a small enough $\varepsilon > 0$ and η such that $0 < \eta \ll \varepsilon \ll 1$, by Milnor's theorem [**8**] we have that the map $f| : B_\varepsilon(0) \cap f^{-1}(S^1_\eta) \to S^1_\eta$ is a smooth locally trivial fibration. By a diffeomorphism in $\mathbf{R}^2$ we can consider $(1/\eta)f| : B_\varepsilon(0) \cap f^{-1}(S^1_\eta) \to S^1$.

For each $x \in B_\varepsilon(0) \setminus V$ consider the flow $p(s,x) = p_x(s)$ of Euler's vector field α given by Lemma 4.4. That is, $p'(s) = \alpha(p(s))$ with $p_x(s_0) = x$.

By item a) this flow is transverse to all small spheres of radius $0 < \varepsilon_1 \leq \varepsilon$, and by b) it cuts transversally all Milnor tubes $B_\varepsilon(0) \cap f^{-1}(S^1_\eta)$, for all $\eta > 0$, sufficiently small.

Now item c) says that, if the flow starts in the fiber $F_\theta = (f/\|f\|)^{-1}(\theta)$, then it will stay there all time, because $(d/ds)(f(\alpha(s))/\|f(\alpha(s))\|) = 0$. Therefore, $f(\alpha(s))/\|f(\alpha(s))\| = \theta$, for all s.

Of course, from item (a) this flow will intersect $S_\varepsilon \setminus K_\varepsilon$ in some point, say y. Then, using the uniqueness of the flow, we can define $h : B_\varepsilon(0) \cap f^{-1}(S^1_\eta) \to S_\varepsilon \setminus N_{K_\varepsilon}$ by $h(x) = y$. It follows from the basic results of classical ODE's that h is a smooth diffeomorphism and it implies that the map

$$\frac{f}{\|f\|} : S^{m-1}_\varepsilon \setminus N_{K_\varepsilon} \longrightarrow S^1 \tag{4}$$

is a smooth locally trivial fibration. It is clear that the restriction $h_| : B^\circ_\varepsilon(0) \cap f^{-1}({}^{p-1}_{\eta}) \to S_\varepsilon \setminus \overline{N}_{K_\varepsilon}$ is also a diffeomorphism, where $\overline{N}_{K_\varepsilon}$ denotes the topological closure of N_{K_ε} in the sphere.

Since the diffeomorphism h is the identity map on the boundary manifold $S_\varepsilon \cap f^{-1}(S^1_\eta)$ of N_{K_ε}, then to get the equivalence between the fibrations (1) and (3), it will be enough to show how to extend the fibration (4) to the complement of the link into N_{K_ε}.

In [**6, 7**] the author explained how to do that using Lemma 4.4. The reader can see the reference for details. Another way was done in [**3**] using tools of differential topology. Below I will explain the main idea.

Since f has an isolated singularity at the origin, then $0 \in \mathbf{R}^2$ is a regular value of the map $f_| : S^{m-1}_\varepsilon \to \mathbf{R}^2$. So, for each $\varepsilon > 0$, we can choose η, $0 < \eta \ll \varepsilon \ll 1$, such that

$$f_| : S^{m-1}_\varepsilon \cap f^{-1}(D_\eta) \longrightarrow D_\eta \tag{5}$$

is a smooth trivial fibration, where D_η denotes the closed disk of $\mathbf{R}^2$ centered at the origin with radius η and boundary S^1_η.

Now, it is clear that $N_{K_\varepsilon} = S^{m-1}_\varepsilon \cap f^{-1}(D^\circ_\eta - \{0\})$, where D°_η is the open disk of $\mathbf{R}^2$. That is, the link K_ε is embedded in the sphere S_ε with a trivial normal bundle.

The fibration (5) induces the following smooth trivial fibrations:

$$f_| : S^{m-1}_\varepsilon \cap f^{-1}(D_\eta - \{0\}) \longrightarrow D_\eta - \{0\}$$

and

$$\frac{1}{\eta}f_{|}\left(=\frac{f}{\|f\|}\right): S_{\varepsilon}^{m-1}\cap f^{-1}(S_{\eta}^{1}) \longrightarrow S^{1}. \tag{6}$$

Since the fibrations (4) and (6) coincide on $S_{\varepsilon}^{m-1}\cap f^{-1}(S_{\eta}^{1})$, then the following composition of trivial fibrations concludes the proof

$$S_{\varepsilon}^{m-1}\cap f^{-1}(D_{\eta}-\{0\}) \xrightarrow{f_{|}} D_{\eta}-\{0\} \xrightarrow{y/\|y\|} S^{1}. \qquad \square$$

Remark 5.2. The technique used above can be applied for any quasi-homogeneous map-germ $f : (\mathbf{R}^m,0) \to (\mathbf{R}^p,0)$, $m \geq p \geq 2$, with isolated singular point at the origin. The interested reader can see [**3**] (and [**5**]) for further details, as well as some interesting results on the author's point of view.

Example 5.3 (A'Campo's example, [**1**]). Let $\psi : (\mathbf{R}^{4+2m},0) \to (\mathbf{R}^2,0)$, $\psi(x,y,z,w,x_1,y_1,\dots,x_m,y_m) = (x(z^2+w^2)+z(x^2+y^2)+\sum_{i=1}^{m} x_i^2 - y_i^2, y(z^2+w^2)+w(x^2+y^2)+\sum_{i=1}^{m} 2x_iy_i)$, the weighted-homogeneous map germ of type $(3,3,3,3,2,\dots,2;6)$ with isolated singularity at origin. In [**1**], using the Lefschetz number of the monodromy of the Milnor fibration, the author showed that this fibration is not equivalent to a holomorphic one, i.e., there is no holomorphic function germ $f : (\mathbf{C}^{2+m},0) \to (\mathbf{C},0)$ such that, the Milnor fibration given by ψ is equivalent to the Milnor fibration of f. However, Theorem 2.2 shows that the fibration (1) in the "open" Milnor tube can be pushed up to get a fibration (3) on the sphere and they are the same, in the sense of Definition 5.1.

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