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THE TRACIAL RANK FOR CROSSED PRODUCTS BY FINITE GROUP ACTIONS

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ABSTRACT. We define the second tracial Rokhlin property for finite group actions. Let A be an infinite dimensional finite separable unital C^*-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Suppose that A is α-simple. If A is an AF-algebra, we prove that the tracial rank of the crossed product $A \times_\alpha G$ is zero. If A is an A$\mathbf{T}$-algebra with the (SP)-property, we prove that the tracial rank of the crossed product $A \times_\alpha G$ is no more than one.

1. Introduction. The concept of the tracial rank of C^*-algebras was introduced by Lin [**11**]. The purpose to introduce the rank was motivated by the Elliott program of classification of nuclear C^*-algebras. C^*-algebras with tracial rank no more than k for some $k \in \mathbf{N}$ are C^*-algebras that can be approximated by C^*-subalgebras in $\mathcal{I}^{(k)}$ in trace (or in "measure"). The C^*-algebras of real rank zero can be determined by K-theory and hence can be classified. For example, Lin proved that if a simple separable amenable unital C^*-algebra A has tracial rank zero and satisfies the Universal Coefficient theorem, then A is a simple AH-algebra with slow dimension growth and with real rank zero [**12, 14**].

The concept of the Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by Connes [**2**]. Then the Rokhlin property was adopted to the context of UHF-algebras by Herman and Ocneanu [**7**]. Rødam [**19**] and Kishimoto [**9**] introduced the Rokhlin property to a much more general context of C^*-algebras. More recently, Phillips, Osaka and Lin et al. studied integer group or finite group actions which satisfy certain types of Rokhlin property on some C^*-

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algebras (see [**13, 15–18**]). The Rokhlin property of actions on C^*-algebras can help us to understand the properties of induced crossed products. In [**17**], Osaka and Phillips proved that, for a separable unital C^*-algebra A, a finite group G, and an action $\alpha : G \to \mathrm{Aut}\,(A)$ with the Rokhlin property, if A has stable rank one and real rank zero, then the associated crossed product $A \times_\alpha G$ also has stable rank one and real rank zero. In [**16**] Osaka and Phillips proved that, for a stably finite simple unital C^*-algebra A, and an action $\alpha \in \mathrm{Aut}\,(A)$ with the tracial Rokhlin property, if A has stable rank one, real rank zero, and the order on projections over A is determined by traces, then the induced crossed product $A \times_\alpha \mathbf{Z}$ also has these three properties. In [**13**] Lin proved that if A is a simple unital C^*-algebra with tracial rank zero, if an action $\alpha \in \mathrm{Aut}\,(A)$ has the tracial cyclic Rokhlin property, and if there is an integer $J \geq 1$ such that $\alpha_{*0}^J|_G = \mathrm{id}_G$ for some subgroup $G \subseteq K_0(A)$ for which $\rho_A(G)$ is dense in $\rho_A(K_0(A))$, then the induced crossed product $A \times_\alpha \mathbf{Z}$ has tracial rank zero. In [**18**], Phillips proved that the crossed product of an infinite dimensional simple separable unital C^*-algebra with tracial rank zero by a finite group action with the tracial Rokhlin property again has tracial rank zero.

In this note, we attempt to introduce a certain type of Rokhlin property for finite group actions on simple or non-simple C^*-algebras. Let A be an infinite dimensional finite unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of a finite group G on A. Suppose that A is α-simple and α has the second tracial Rokhlin property. If A is an AF-algebra, we show that the tracial rank of the crossed product $A \times_\alpha G$ is zero. If A is an A$\mathbf{T}$-algebra with the (SP)-property, we show that the tracial rank of the crossed product $A \times_\alpha G$ is no more than one. It should be pointed out that the key ideas of the proof for the main results in this note come from [**18**].

2. Definitions and preliminaries. Let A be a C^*-algebra. We use $\mathrm{Aut}\,(A)$ to denote the automorphism group of A. If A is unital and $u \in A$ is a unitary, we denote by $\mathrm{Ad}\,u$ the inner automorphism defined by $\mathrm{Ad}\,u(a) = uau^*$ for all $a \in A$. If an automorphism α is not inner, we say it is outer.

Let A be a C^*-algebra and $\alpha : G \to \mathrm{Aut}\,(A)$ an action of a finite group G on A. We say that A is α-simple if no closed two-sided ideal other than 0 or A is invariant under all α_g, $g \in G$.

We denote by $\mathcal{I}^{(0)}$ the class of finite dimensional C^*-algebras, and by $\mathcal{I}^{(k)}$ the class of C^*-algebras with the form $p(C(X)\otimes F)p$, where $F\in\mathcal{I}^{(0)}$, X a finite CW complex with dimension k and $p\in C(X)\otimes F$ a projection.

Let A be a C^*-algebra, and let a be a positive element in A. We denote by $\mathrm{Her}\,(a)$ the hereditary C^*-subalgebra of A generated by a. We know that $\mathrm{Her}\,(a)=\overline{aAa}$.

Let A be a C^*-algebra, and let a,b be positive elements in A. We write $[a]\leq[b]$ if there exists a partial isometry $v\in A''$ such that, for every $c\in\mathrm{Her}(a)$, $v^*c, cv\in A$, $vv^*=p_a$, where p_a is the range projection of a in A'', and $v^*cv\in\mathrm{Her}\,(b)$. We write $[a]=[b]$ if moreover $v^*\mathrm{Her}\,(a)v=\mathrm{Her}\,(b)$. Hence, if $a\in\mathrm{Her}\,(b)$, then $[a]\leq[b]$. If p and q are two projections in A, then $[p]\leq[q]$ if and only if p is Murray-von Neumann equivalent to a subprojection of q, and $[p]=[q]$ if and only if p is Murray-von Neumann equivalent to q. If $a,b\in A$ are positive elements satisfying $ab=0$, then $[a+b]=[a]+[b]$.

Let A be a unital C^*-algebra, and let $\alpha: G\to\mathrm{Aut}\,(A)$ be an action of a finite group G on A. Let $a,b\in A$ be positive elements. Set $\mathrm{card}\,(G)=n$. We say $[a]\leq_{\alpha,G}[b]$ if for $i=1,2,\ldots,n$, there exist mutually orthogonal positive elements a_i, there exist elements $g_i,h_i\in G$, there exist mutually orthogonal positive elements $b_i\in\mathrm{Her}\,(b)$, such that $\sum_{i=1}^n a_i=a$ and $[\alpha_{g_i}(a_i)]\leq[\alpha_{h_i}(b_i)]$ in A, $i=1,2,\ldots,n$.

For any $g\in G$ and any positive element $d\in A$, we have $[\alpha_g(d)]=[u_g d u_{g^{-1}}]=[d]$ in $A\times_\alpha G$. Therefore, $[a]\leq_{\alpha,G}[b]$ in A implies that $[a]\leq[b]$ in $A\times_\alpha G$.

Let A be a C^*-algebra, and let $\mathcal{F}$ be a subset of A, $x\in A$, $\varepsilon>0$. If there exists an element $y\in\mathcal{F}$ such that $\|x-y\|<\varepsilon$, then we write $x\in_\varepsilon\mathcal{F}$.

Definition 2.1 [**10**, Definition 3.6.2]. Let A be a simple unital C^*-algebra and $k\in\mathbf{N}$. A is said to have tracial rank no more than k if, for any $\varepsilon>0$, any finite subset $\mathcal{F}\subseteq A$ and any nonzero positive element $b\in A$, there exist a nonzero projection $p\in A$ and a C^*-subalgebra $B\subseteq A$ with $1_B=p$ and $B\in\mathcal{I}^{(k)}$ such that

(1) $\|pa-ap\|<\varepsilon$ for any $a\in\mathcal{F}$.

(2) $pap\in_\varepsilon B$ for all $a\in\mathcal{F}$.

(3) The projection $1-p$ is Murray-von Neumann equivalent to a projection in $\mathrm{Her}\,(b)$, that is, $[1-p]\leq[b]$.

If A has tracial rank no more than k, we will write $\mathrm{TR}\,(A)\leq k$. If furthermore, $\mathrm{TR}\,(A)\nleq k-1$, then we say $\mathrm{TR}\,(A)=k$.

Definition 2.2 [**16**, Definition 1.1]. Let A be an infinite dimensional finite simple separable unital C^*-algebra, and let $\alpha : G\to \mathrm{Aut}\,(A)$ be an action of a finite group G on A. We say that α has the tracial Rokhlin property if, for every $\varepsilon>0$, every finite set $\mathcal{F}\subseteq A$, every positive element $b\in A$, there are mutually orthogonal projections $\{e_g : g\in G\}$ such that

(1) $\|\alpha_g(e_h)-e_{gh}\|<\varepsilon$ for all $g,h\in G$.

(2) $\|e_ga-ae_g\|<\varepsilon$ for all $g\in G$ and all $a\in\mathcal{F}$.

(3) With $e=\sum_{g\in G}e_g$, the projection $1-e$ is Murray-von Neumann equivalent to a projection in $\mathrm{Her}\,(b)$, that is, $[1-e]\leq[b]$.

Definition 2.3. Let A be an infinite dimensional finite separable unital C^*-algebra and $\alpha : G\to\mathrm{Aut}\,(A)$ an action of a finite group G on A. We say that α has the second tracial Rokhlin property if, for every $\varepsilon>0$, every finite set $\mathcal{F}\subseteq A$, every positive elements $b,x\in A$, there are $g_0\in G$ and mutually orthogonal projections $\{e_g : g\in G\}$ such that

(1) $\|\alpha_g(e_h)-e_{gh}\|<\varepsilon$ for all $g,h\in G$.

(2) $\|e_ga-ae_g\|<\varepsilon$ for all $g\in G$ and all $a\in\mathcal{F}$.

(3) $\|e_{g_0}xe_{g_0}\|>\|x\|-\varepsilon$.

(4) With $e=\sum_{g\in G}e_g$, $[(1-e)]\leq_{\alpha,G}[b]$.

For example, suppose that G is a finite group and D is an infinite dimensional simple unital AF-algebra. Let $\beta : G\to\mathrm{Aut}\,(D)$ be an action satisfying the tracial Rokhlin property. Suppose that $\mathrm{card}\,(G)=n$, and we may write $G=\{g_1,g_2,\ldots,g_n\}$. Let $A=\overbrace{D\oplus D\oplus\cdots\oplus D}^{n}$, where $n>1$, and define $\alpha : G\to\mathrm{Aut}\,(A)$ by $\alpha_{g_i}(a_1,a_2,\ldots,a_n)=(\beta_{g_i}(a_i),\beta_{g_i}(a_2),\ldots,\beta_{g_i}(a_{i-1}),\beta_{g_i}(a_1),\beta_{g_i}(a_{i+1}),\ldots,\beta_{g_i}(a_n))$. It is obvious that A is non-simple but α-simple. Moreover, it is easy to check that α has the second tracial Rokhlin property.

Remark 2.4. In Definition 2.3, if the finite set $\mathcal{F}$ contains a full element and is large enough, then (2) always implies (3). When A is an infinite dimensional finite simple separable unital C^*-algebra and $\alpha : G \to \mathrm{Aut}\,(A)$ an action of a finite group G on A, then α has the tracial Rokhlin property implies that α has the second tracial Rokhlin property.

By perturbation of projections (see [**10**, Theorem 2.5.9]), we have Lemma 2.5, and then we get Corollary 2.6. We can also get Corollary 2.6 by semi-projectivity of finite dimensional C^*-algebras directly (see Definition 2.10, Proposition 2.18, Proposition 2.23 and Corollary 2.28 of [**1**]).

Lemma 2.5. *For any $n \in \mathbf{N}$, any $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever A is a unital C^*-algebra, and whenever $w_{i,j}$, for $1 \le i,j \le n$, are elements of A such that $\|w_{i,j}\| \le 1$ for $1 \le i,j \le n$, such that $\|w_{i,j}^* - w_{j,i}\| < \delta$ for $1 \le i,j \le n$, such that $\|w_{i_1,j_1}w_{i_2,j_2} - \delta_{j_1,i_2}w_{i_1,j_2}\| < \delta$ for $1 \le i_1, i_2, j_1, j_2 \le n$, and such that $w_{i,i}$ are mutually orthogonal projections with $\sum_{i=1}^n w_{i,i} = 1$, then there exists a $\{v_{i,j}\} \subseteq A$ for $1 \le i,j \le n$ such that $\|v_{i,j}\| \le 1$ for $1 \le i,j \le n$, such that $v_{i,j}^* = v_{j,i}$ for $1 \le i,j \le n$, such that $v_{i_1,j_1}v_{i_2,j_2} = \delta_{j_1,i_2}v_{i_1,j_2}$ for $1 \le i_1, i_2, j_1, j_2 \le n$, such that $v_{i,i}$ are mutually orthogonal projections for $1 \le i \le n$ with $v_{i,i} = w_{i,i}$, and such that $\|v_{i,j} - w_{i,j}\| < \varepsilon$ for $1 \le i,j \le n$.*

If $w_{i,j}$, for $1 \le i,j \le n$, are elements of a C^*-algebra A satisfying the conditions in Lemma 2.5, we say that $w_{i,j}$ $(1 \le i,j \le n)$ form a δ-approximate system of $n \times n$ matrix units in A.

By Lemma 2.5, we have the following result.

Corollary 2.6. *For any $n \in \mathbf{N}$, any $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $(f_{i,j})_{1 \le i,j \le n}$ is a system of matrix units for M_n, whenever B is a unital C^*-algebra, and whenever $w_{i,j}$, for $1 \le i,\, j \le n$, are elements of B which form a δ-approximate system of $n \times n$ matrix units, then there exists a unital $*$-homomorphism $\phi : M_n \to B$ such that $\phi(f_{i,i}) = w_{i,i}$ for $1 \le i \le n$ and $\|\phi(f_{i,j}) - w_{i,j}\| < \varepsilon$ for $1 \le i,j \le n$.*

By [**10**, Lemma 2.5.5, Lemma 2.5.6], we can get the following result:

Lemma 2.7. *For any $\varepsilon > 0$ and any integer n, there exists $\delta = \delta(\varepsilon, n)$ satisfying the following: Let D be a C^*-subalgebra of A, let $p_1, p_2, \ldots, p_n$ be mutually orthogonal projections in A. If $p_i \in_\delta D$ for all $1 \leq i \leq n$, then there are mutually orthogonal projections $q_1, q_2, \ldots, q_n \in D$ such that $\|p_i - q_i\| < \varepsilon$ for all $1 \leq i \leq n$.*

We say that a C^*-algebra A has the (SP)-property if every non-zero hereditary C^*-subalgebra of A has a non-zero projection.

Lemma 2.8. *Let A be an infinite dimensional finite separable unital C^*-algebra with the (SP)-property and $\alpha : G \to Aut(A)$ an action of a finite group G on A which has the second tracial Rokhlin property. Then any non-zero hereditary C^*-subalgebra of the crossed product $A \times_\alpha G$ has a non-zero projection which is Murray-von Neumann equivalent to a projection in A.*

Proof. The proof is similar to [**8**, Theorem 4.2] and we omit it. □

Lemma 2.9. *Let A be an infinite dimensional finite separable unital C^*-algebra, and let $\alpha : G \to Aut(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Suppose that A is α-simple. Then $A \times_\alpha G$ is simple.*

Proof. We claim that α_g is outer for every $g \in G \setminus \{1\}$, where 1 denotes the unit of G. For any $0 < \eta < 1/2$, any unitary u in A, apply Definition 2.3 with $\mathcal{F} = \{u\}$ and with η in place of ε. Then there exist mutually orthogonal projections $e_g \in A$ for $g \in G$ such that $\|\alpha_g(e_1) - e_g\| < \eta$ and $\|ue_1u^* - e_1\| < \eta$.

For every $g \in G \setminus \{1\}$, since e_1 and e_g are mutually orthogonal projections, we have that

$$\|\alpha_g(e_1) - ue_1u^*\| \geq \|e_1 - e_g\| - \|\alpha_g(e_1) - e_g\| - \|ue_1u^* - e_1\| > 0.$$

Hence, $\alpha_g \neq \operatorname{Ad} u$. This proves the claim.

Let x be an element in A, let $\mathcal{F}$ be a finite subset of A, and let ε be a positive number. Apply Definition 2.3; then there is a projection $e_{g_0} \in A$ such that

(1) $\|e_{g_0}\alpha_h(e_{g_0})\| < \varepsilon$ for all $h \in G \setminus \{1\}$.

(2) $\|e_{g_0}a - ae_{g_0}\| < \varepsilon$ for all $a \in \mathcal{F}$.

(3) $\|e_{g_0}xe_{g_0}\| > \|x\| - \varepsilon$.

Then by [**3**, Lemma 3.3] and the proof of [**3**, Theorem 3.2], we have that the crossed product $A \times_\alpha G$ is simple. □

3. Main results.

Theorem 3.1. *Let A be an infinite dimensional simple or non-simple unital* AF*-algebra, and let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Suppose that A is α-simple, then $TR\,(A \times_\alpha G) = 0$.*

Proof. By Lemma 2.9, $A \times_\alpha G$ is simple. By Definition 2.1, it suffices to show the following:

For any $\varepsilon > 0$, any finite subset $\mathcal{F} = \mathcal{F}_0 \cup \{u_g | \, g \in G\}$, where $\mathcal{F}_0$ is a finite subset of the unit ball of A and $u_g \in A \times_\alpha G$ is the canonical unitary implementing the automorphism α_g, and any nonzero positive element $b \in A \times_\alpha G$, there exist a nonzero projection $e \in A \times_\alpha G$ and a finite dimensional C^*-subalgebra $B \subseteq A \times_\alpha G$ with $1_B = e$ such that

(1) $\|ea - ae\| < \varepsilon$ for any $a \in \mathcal{F}$.

(2) $eae \in_\varepsilon B$ for all $a \in \mathcal{F}$.

(3) $[1 - e] \leq [b]$ in $A \times_\alpha G$.

Since AF-algebras are real rank zero, and hence have the (SP)-property, by Lemma 2.8, there exists a non-zero projection $r \in A$ which is Murray-von Neumann equivalent to a projection in the hereditary C^*-subalgebra of the crossed product $A \times_\alpha G$ generated by b, that is, $[r] \leq [b]$ in $A \times_\alpha G$.

Set $n = \mathrm{card}\,(G)$ and set $\delta = \varepsilon/(8n)$. Choose $\eta > 0$ according to Corollary 2.6 for n given above and δ in place of ε. Moreover we may require $\eta < \min\{(\varepsilon/12n), (\varepsilon/4n(n-1))\}$.

Set $\eta_1 = \eta/3$. Apply Definition 2.3 with $\mathcal{F}_0$ given above, with η_1 in place of ε, with r in place of b. There are mutually orthogonal projections $\{\overline{e}_g : g \in G\}$ such that

(1′) $\|\alpha_g(\overline{e}_h) - \overline{e}_{gh}\| < \eta_1$ for all $g, h \in G$.

(2′) $\|\overline{e}_g a - a\overline{e}_g\| < \eta_1$ for all $g \in G$ and all $a \in \mathcal{F}_0$.

(3′) $[1 - \overline{e}] \leq_{\alpha,G} [r]$ in A, where $\overline{e} = \sum_{g\in G} \overline{e}_g$.

Since A is a unital AF-algebra, we may write $A = \overline{\cup_{n=1}^{\infty} A_n}$, where each A_n is a finite dimensional C^*-subalgebra of A.

Set $\eta_2 = \min\{(1/2n), (\eta/6)\}$. Let η_3 be a positive number according to Lemma 2.7 for n given above and η_2 in place of ε. Also require $\eta_3 < (\varepsilon/4)$. There exists an $m \in \mathbf{N}$ such that $d \in_{\eta_3} A_m$ for all $d \in \{\alpha_g(a) | a \in \mathcal{F}_0, g \in G\} \cup \{\overline{e}_g | g \in G\}$. So, for all $g \in G$ and $a \in \mathcal{F}_0$, there exists an $r_{g,a} \in A_m$ such that $\|r_{g,a} - \alpha_g(a)\| < \eta_3$.

By Lemma 2.7, there exist mutually orthogonal projections $e_g \in A_m$ for all $g \in G$ such that $\|\overline{e}_g - e_g\| < \eta_2$.

Then we have the following:

(1″) $\|\alpha_g(e_h) - e_{gh}\| \leq \|\alpha_g(e_h) - \alpha_g(\overline{e}_h)\| + \|\alpha_g(\overline{e}_h) - \overline{e}_{gh}\| + \|\overline{e}_{gh} - e_{gh}\| < 2\eta_2 + \eta_1 < \eta$, for all $g, h \in G$.

(2″) $\|e_g a - a e_g\| \leq \|e_g a - \overline{e}_g a\| + \|\overline{e}_g a - a\overline{e}_g\| + \|a\overline{e}_g - a e_g\| < 2\eta_2 + \eta_1 < \eta$, for all $g \in G$ and all $a \in \mathcal{F}_0$.

(3″) Set $e = \sum_{g\in G} e_g$; then $\|e - \overline{e}\| \leq \sum_{g\in G} \|e_g - \overline{e}_g\| \leq n\eta_2 < 1$.

Since $\|e - \overline{e}\| < 1$, by [**10**, Lemma 2.5.1], $1-e$ is Murray-von Neumann equivalent to $1 - \overline{e}$ in A. Therefore, in $A \times_\alpha G$,

$$[1 - e] = [1 - \overline{e}] \leq [r] \leq [b]. \tag{1.1}$$

Define $w_{g,h} = u_{g,h^{-1}} e_h$ for all $g, h \in G$. We claim that $w_{g,h}$ $(g, h \in G)$ form a η-approximate system of $n \times n$ matrix units in $A \times_\alpha G$. We estimate:

$$\begin{aligned} \|w_{g,h}^* - w_{h,g}\| &= \|e_h u_{g^{-1}h} - u_{hg^{-1}} e_g\| \\ &\leq \|u_{gh^{-1}} e_h u_{g^{-1}h} - e_g\| \\ &= \|\alpha_{gh^{-1}}(e_h) - e_g\| < \eta. \end{aligned}$$

Moreover, using $e_g e_h = \delta_{g,h} e_h$, we have

$$\begin{aligned}
&\|w_{g_1,h_1} w_{g_2,h_2} - \delta_{g_2,h_1} w_{g_1,h_2}\| \\
&\qquad = \|u_{g_1 h_1^{-1}} e_{h_1} u_{g_2 h_2^{-1}} e_{h_2} - \delta_{g_2,h_1} u_{g_1 h_2^{-1}} e_{h_2}\| \\
&\qquad = \|u_{g_1 h_1^{-1}} e_{h_1} u_{g_2 h_2^{-1}} e_{h_2} - u_{g_1 h_1^{-1} g_2 h_2^{-1}} e_{h_2 g_2^{-1} h_1} e_{h_2}\| \\
&\qquad = \|u_{g_1 h_1^{-1} g_2 h_2^{-1}} (u_{g_2^{-1} h_2} e_{h_1} u_{g_2 h_2^{-1}} - e_{h_2 g_2^{-1} h_1}) e_{h_2}\| < \eta.
\end{aligned}$$

Moreover, $\sum_{g \in G} w_{g,g} = \sum_{g \in G} e_g = e$. This proves the claim.

Let $(f_{g,h})_{g,h \in G}$ be a system of matrix units for M_n. By Corollary 2.6, there exists a unital $*$-homomorphism $\psi_0 : M_n \to e(A \times_\alpha G)e$ such that

$$\|\psi_0(f_{g,h}) - w_{g,h}\| < \delta$$

for all $g, h \in G$, and $\psi_0(f_{g,g}) = e_g$ for all $g \in G$.

Define an injective unital $*$-homomorphism $\psi : M_n \otimes e_1 A_m e_1 \to e(A \times_\alpha G)e$ by

$$\psi(f_{g,h} \otimes a) = \psi_0(f_{g,1}) a \psi_0(f_{1,h})$$

for all $g, h \in G$ and $a \in e_1 A_m e_1$. Then

$$\psi(1_{M_n} \otimes e_1) = \sum_{g \in G} e_g = e, \psi(f_{1,1} \otimes a) = a$$

for all $a \in e_1 A_m e_1$ and

$$\begin{aligned}
\psi(f_{g,h} \otimes e_1) &= \psi_0(f_{g,1}) e_1 \psi_0(f_{1,h}) = \psi_0(f_{g,1}) \psi_0(f_{1,1}) \psi_0(f_{1,h}) \\
&= \psi_0(f_{g,h}) = e_g \psi_0(f_{g,h}) e_h.
\end{aligned}$$

By $(2'')$, for all $a \in \mathcal{F}_0$, we have

$$\|ae - ea\| \le \sum_{g \in G} \|ae_g - e_g a\| < n\eta < \varepsilon. \tag{1.2}$$

By $(1'')$, for all $g \in G$, we get

$$\|u_g e - e u_g\| \le \|u_g e u_{g^{-1}} - e\| = \|\sum_{h \in G} \alpha_g(e_h) - \sum_{h \in G} e_{gh}\| \le n\eta < \varepsilon. \tag{1.3}$$

Set $B = \psi(M_n \otimes e_1 A_m e_1)$; then B is a finite dimensional C^*-subalgebra of $A \times_\alpha G$. For all $g \in G$, we have

$$\begin{aligned}
\|eu_g e - \sum_{h\in G} \psi(f_{gh,h} \otimes e_1)\| & \\
&\leq \|eu_g e - u_g e\| + \|u_g e - \sum_{h\in G} \psi(f_{gh,h} \otimes e_1)\| \\
&< n\eta + \|u_g e - \sum_{h\in G} \psi(f_{gh,h} \otimes e_1)\| \\
&= n\eta + \|\sum_{h\in G} u_g e_h - \sum_{h\in G} \psi(f_{gh,h} \otimes e_1)\| \\
&= n\eta + \|\sum_{h\in G} w_{gh,h} - \sum_{h\in G} \psi_0(f_{gh,h})\| < n\eta + n\delta < \varepsilon.
\end{aligned}$$

That is, for all $g \in G$, we have

$$eu_g e \in_\varepsilon B. \tag{1.4}$$

Now let $a \in \mathcal{F}_0$, and set $b = \sum_{g\in G} f_{g,g} \otimes e_1 r_{g^{-1},a} e_1 \in M_n \otimes e_1 A_m e_1$; then

$$\|b - \sum_{g\in G} f_{g,g} \otimes e_1 \alpha_{g^{-1}}(a) e_1\| < \eta_3.$$

Using $\|e_g a e_h - a e_g e_h\| < \eta$, we get

$$\|eae - \sum_{g\in G} e_g a e_g\| \leq \sum_{g\neq h} \|e_g a e_h\| < n(n-1)\eta.$$

We also have

$$\begin{aligned}
\|\psi_0(f_{g,1})e_1 - u_g e_1\| &\leq \|\psi_0(f_{g,1}) - u_g e_1\| \\
&= \|\psi_0(f_{g,1}) - w_{g,1}\| < \delta, \\
\|\psi_0(f_{1,g}) - e_1 u_{g^{-1}}\| &\leq \|\psi_0(f_{1,g}) - u_{g^{-1}} e_g\| \\
&\quad + \|e_1 u_{g^{-1}} - u_{g^{-1}} e_g\| < \delta + \eta
\end{aligned}$$

and

$$\|e_1 \alpha_{g^{-1}}(a) e_1 - \alpha_{g^{-1}}(e_g a e_g)\| < 2\eta.$$

Then, for all $a \in \mathcal{F}_0$, we have

$$\begin{aligned}
\|eae - \psi(b)\| &= \|eae - \psi\Big(\sum_{g\in G} f_{g,g} \otimes e_1 r_{g^{-1},a} e_1\Big)\| \\
&= \|eae - \sum_{g\in G} \psi_0(f_{g,1}) e_1 r_{g^{-1},a} e_1 \psi_0(f_{1,g})\| \\
&< \Big\|eae - \sum_{g\in G} u_g e_1 \alpha_{g^{-1}}(a) e_1 u_{g^{-1}}\Big\| + 2n\delta + n\eta + \eta_3 \\
&< \|eae - \sum_{g\in G} u_g \alpha_{g^{-1}}(e_g a e_g) u_{g^{-1}}\| + 2n\delta + 3n\eta + \eta_3 \\
&= \Big\|eae - \sum_{g\in G} e_g a e_g\Big\| + 2n\delta + 3n\eta + \eta_3 \\
&< n(n-1)\eta + 2n\delta + 3n\eta + \eta_3 \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

That is, for all $a \in \mathcal{F}_0$,

$$eae \in_\varepsilon B. \tag{1.5}$$

From (1.1)–(1.5), we have $TR(A \times_\alpha G) = 0$. □

Theorem 3.2. *Let A be an infinite dimensional unital $A\mathbf{T}$-algebra with the* (SP)*-property, and let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Suppose that A is α-simple. Then $TR(A \times_\alpha G) \leq 1$.*

Proof. By Lemma 2.9, $A \times_\alpha G$ is a simple C^*-algebra. By Definition 2.1, it suffices to show the following:

For any $\varepsilon > 0$, any finite subset $\mathcal{F} = \mathcal{F}_0 \cup \{u_g |\, g \in G\}$, where $\mathcal{F}_0$ is a finite subset of the unit ball of A and $u_g \in A \times_\alpha G$ is the canonical unitary implementing the automorphism α_g, and any nonzero positive element $b \in A \times_\alpha G$, there exist a nonzero projection $e \in A \times_\alpha G$ and C^*-subalgebra $B \subseteq A \times_\alpha G$ with $1_B = e$ and $B \in \mathcal{I}^{(1)}$, such that

(1) $\|ea - ae\| < \varepsilon$ for any $a \in \mathcal{F}$.

(2) $eae \in_\varepsilon B$ for all $a \in \mathcal{F}$.

(3) $[1-e] \leq [b]$.

As in the proof of Theorem 3.1, there exists a non-zero projection $r \in A$ such that $[r] \leq [b]$ in $A \times_\alpha G$.

Set $n = \mathrm{card}\,(G)$ and set $\delta = \varepsilon/(8n)$. Choose $\eta > 0$ according to Corollary 2.6 for n given above and δ in place of ε. Moreover we may require $\eta < \min\{(\varepsilon/12n), (\varepsilon/4n(n-1))\}$.

Since A is a unital A**T**-algebra, we may write

$$A = \overline{\cup_{n=1}^{\infty} \oplus_{j=1}^{k_n} C(X_{n,j}, M_{s(n,j)})},$$

where $k_n, s(n,j) \in \mathbf{N}$ and $X_{n,j}$ are closed subsets of the circle $\mathbf{T}$ for all $n \in \mathbf{N}$ and $j = 1, 2, \ldots, k_n$.

By the proof of Theorem 3.1, there exist an $m \in \mathbf{N}$ and mutually orthogonal projections $e_g \in \oplus_{j=1}^{k_m} C(X_{m,j}, M_{s(m,j)})$ for all $g \in G$ satisfying the following:

(1′) $\|\alpha_g(e_h) - e_{gh}\| < \eta$, for all $g, h \in G$.

(2′) $\|e_g a - a e_g\| < \eta$, for all $g \in G$ and all $a \in \mathcal{F}_0$.

(3′) Set $e = \sum_{g \in G} e_g$; then $[1-e] \leq_{\alpha,G} [r]$ in A.

As in the proof of Theorem 3.1, in $A \times_\alpha G$,

$$[1-e] \leq [b], \quad \|ae - ea\| < \varepsilon, \quad \|u_g e - e u_g\| < \varepsilon. \tag{1.1}$$

Define $w_{g,h} = u_{g,h^{-1}} e_h$ for $g, h \in G$. Using the same estimation as in the proof of Theorem 3.1, we get that $w_{g,h}\,(g, h \in G)$ form an η-approximate system of $n \times n$ matrix units in $A \times_\alpha G$.

Let $(f_{g,h})_{g,h \in G}$ be a system of matrix units for M_n. By Corollary 2.6, there exists a unital $*$-homomorphism $\psi_0 : M_n \to e(A \times_\alpha G)e$ such that

$$\|\psi_0(f_{g,h}) - w_{g,h}\| < \delta$$

for all $g, h \in G$, and $\psi_0(f_{g,g}) = e_g$ for all $g \in G$.

Define an injective unital $*$-homomorphism $\psi : M_n \otimes e_1(\oplus_{j=1}^{k_m} C(X_{m,j}, M_{s(m,j)}))e_1 \to e(A \times_\alpha G)e$ by

$$\psi(f_{g,h} \otimes a) = \psi_0(f_{g,1}) a \psi_0(f_{1,h})$$

for all $g, h \in G$ and $a \in e_1(\oplus_{j=1}^{k_m} C(X_{m,j}, M_{s(m,j)}))e_1$.

Set $B = \psi(M_n \otimes e_1(\oplus_{j=1}^{k_m} C(X_{m,j}, M_{s(m,j)}))e_1)$; then $B \in \mathcal{I}^{(1)}$.

Then, by the proof of Theorem 3.1, we estimate that

$$eu_g e \in_\varepsilon B, \tag{1.2}$$

for all $g \in G$, and

$$eae \in_\varepsilon B. \tag{1.3}$$

for all $a \in \mathcal{F}_0$.

From (1.1)–(1.3) we have $\mathrm{TR}\,(A \times_\alpha G) \leq 1$. □

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