ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 1, 2012

ON GENERALIZATIONS AND REINFORCEMENTS OF A HILBERT'S TYPE INEQUALITY

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ABSTRACT. In this paper, by using the Euler-Maclaurin expansion for the zeta function, we establish an inequality of a weight coefficient. Using this inequality, we derive generalizations and reinforcements of a Hilbert's type inequality.

1. Introduction. If $p, q > 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in \mathbf{N}$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}, \tag{1.2}$$

where the constant $\pi/(\sin(\pi/p))$ and pq is best possible for each inequality, respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [**1**].

In [**4, 6, 7**], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [**2**], Kuang Jichang and Debnath gave a reinforcement of inequality (1.2):

2010 AMS *Mathematics subject classification.* Primary 26D15.
Keywords and phrases. Hilbert's type inequality, weight coefficient, Hölder inequality, generalization, reinforcement.
This research was funded by Research Foundation of Chongqing University of Science and Technology (Grant No. CK2010B03) and the Natural Science Foundation of Henan Province (Grant No. 2008B110011).
Received by the editors on November 17, 2008, and in revised form on May 23, 2009.
DOI:10.1216/RMJ-2012-42-1-329

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty}[pq-G(p,n)]a_n^p\right\}^{1/p} \times \left\{\sum_{n=1}^{\infty}[pq-G(q,n)]b_n^q\right\}^{1/q} \tag{1.3}$$

where $G(r,n)=[r+(1/3r)-(4/3)]/[(2n+1)^{1/r}]>0$ $(r=p,q)$.

In this paper, by using the Euler-Maclaurin expansion for the zeta function, we establish an inequality for a weight coefficient. Using this inequality, we derive a generalization and reinforcement of inequalities (1.2) and (1.3).

2. A lemma. First, we need the following formula of the Riemann-ζ function (see [**3, 5, 8**]):

$$\begin{aligned}\zeta(\sigma) &= \sum_{k=1}^{n}\frac{1}{k^\sigma}-\frac{n^{1-\sigma}}{1-\sigma}-\frac{1}{2n^\sigma}\\ &\quad -\sum_{k=1}^{l-1}\frac{B_{2k}}{2k}\binom{-\sigma}{2k-1}\frac{1}{n^{\sigma+2k-1}}\\ &\quad -\frac{B_{2l}}{2l}\binom{-\sigma}{2l-1}\frac{\varepsilon}{n^{\sigma+2l-1}},\end{aligned} \tag{2.1}$$

where $\sigma>0$, $\sigma\neq 1$, $n,\ l\geq 1$, $n,\ l\in\mathbf{N}$, $0<\varepsilon=\varepsilon(\sigma,l,n)<1$. The numbers $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30,\ldots$ are Bernoulli numbers. In particular, $\zeta(\sigma)=\sum_{k=1}^{\infty}\frac{1}{k^\sigma}$ $(\sigma>1)$.

Since $\zeta(0)=-1/2$, then the formula of the Riemann-ζ function (2.1) is also true for $\sigma=0$.

Lemma 2.1. *If* $p,q>1$, $1/p+1/q=1$, $2-\min\{p,q\}<\lambda\leq 2$, $n\geq 1$ *and* $n\in\mathbf{N}$, *then*

$$\omega(n,\lambda,p)=\sum_{k=1}^{\infty}\frac{1}{\max\{k^\lambda,n^\lambda\}}\left(\frac{n}{k}\right)^{(2-\lambda)/p} < n^{1-\lambda}\left[\kappa(\lambda)-\frac{1}{3pn^{(p+\lambda-2)/p}}\right], \tag{2.2}$$

and

$$\omega(n,\lambda,q)=\sum_{k=1}^{\infty}\frac{1}{\max\{k^{\lambda},n^{\lambda}\}}\left(\frac{n}{k}\right)^{(2-\lambda)/q} < n^{1-\lambda}\left[\kappa(\lambda)-\frac{1}{3qn^{(q+\lambda-2)/q}}\right], \tag{2.3}$$

where $\kappa(\lambda)=(pq\lambda)/[(p+\lambda-2)(q+\lambda-2)]$. *When* $\lambda=1$, *we have following the stronger inequality*:

$$\omega(n,1,p)=\sum_{k=1}^{\infty}\frac{1}{\max\{k,n\}}\left(\frac{n}{k}\right)^{1/p} < \left[pq-\frac{12q^2+3q+5p}{12pqn^{1/q}}\right], \tag{2.4}$$

and

$$\omega(n,1,q)=\sum_{k=1}^{\infty}\frac{1}{\max\{k,n\}}\left(\frac{n}{k}\right)^{1/q} < \left[pq-\frac{12p^2+3p+5q}{12pqn^{1/p}}\right]. \tag{2.5}$$

Proof. Equalities (2.2) and (2.3) define the weight coefficient. When $2-\min\{p,q\}<\lambda\le 2$, taking $\sigma=(2-\lambda)/p\ge 0$, $l=1$, in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right)=\sum_{k=1}^{n}\frac{1}{k^{(2-\lambda)/p}}-\frac{pn^{(p+\lambda-2)/p}}{p+\lambda-2}-\frac{1}{2n^{(2-\lambda)/p}}+\frac{2-\lambda}{12pn^{1+(2-\lambda/p)}}\varepsilon_1, \tag{2.6}$$

where $0<\varepsilon_1<1$.

Taking $\sigma=(2/p)+(\lambda/q)$, $l=1$, we obtain

$$\zeta\left(\frac{2}{p}+\frac{\lambda}{q}\right)=\sum_{k=1}^{n-1}\frac{1}{k^{(2/p)+(\lambda/q)}}+\frac{qn^{-(q+\lambda-2)/q}}{q+\lambda-2}+\frac{1}{2n^{(2/p)+(\lambda/q)}}+\frac{p\lambda+2q}{12pqn^{1+(2/p)+(\lambda/q)}}\varepsilon_2, \tag{2.7}$$

where $0 < \varepsilon_2 < 1$.

In addition,

$$\begin{aligned}\omega(n,\lambda,p) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{(2-\lambda)/p} \\ &= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{(2-\lambda)/p} - \frac{1}{n^\lambda} \\ &\quad + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{(2-\lambda)/p} \\ &= \sum_{k=1}^{n} \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{(2-\lambda)/p} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{(2-\lambda)/p} \\ &= \frac{1}{n^{[(p+1)\lambda-2]/p}} \sum_{k=1}^{n} \frac{1}{k^{(2-\lambda)/p}} - \frac{1}{n^\lambda} \\ &\quad + n^{(2-\lambda)/p} \sum_{k=n}^{\infty} \frac{1}{k^{(2/p)+(\lambda/q)}}.\end{aligned}$$

By (2.6) and (2.7)

$$\begin{aligned}\omega(n,\lambda,p) &< \frac{1}{n^{[(p+1)\lambda-2]/p}} \left[\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{(p+\lambda-2)/p}}{p+\lambda-2} + \frac{1}{2n^{(2-\lambda)/p}}\right] - \frac{1}{n^\lambda} \\ &\quad + n^{(2-\lambda)/p}\left[\frac{qn^{-(q+\lambda-2)/q}}{q+\lambda-2} + \frac{1}{2n^{(2/p)+(\lambda/q)}}\right. \\ &\qquad\qquad \left. + \frac{p\lambda+2q}{12p\,qn^{1+(2/p)+(\lambda/q)}}\right] \\ &= \frac{1}{n^{[(p+1)\lambda-2]/p}} \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{1-\lambda}}{p+\lambda-2} + \frac{1}{2n^\lambda} - \frac{1}{n^\lambda} \\ &\quad + \frac{qn^{1-\lambda}}{q+\lambda-2} + \frac{1}{2n^\lambda} + \frac{p\lambda+2q}{12p\,qn^{1+\lambda}} \\ &= \frac{1}{n^{[(p+1)\lambda-2]/p}} \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pq\lambda n^{1-\lambda}}{(p+\lambda-2)(q+\lambda-2)} \\ &\quad + \frac{p\lambda+2q}{12pqn^{1+\lambda}}\end{aligned}$$

$$= n^{1-\lambda}\left\{\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} - \frac{1}{n^{(p+\lambda-2)/p}}\left[-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12p\,qn^{(p-\lambda+2)/p}}\right]\right\}.$$

In (2.6), taking $n = 1$, by $2 - \min\{p, q\} < \lambda \le 2$, we obtain

$$\begin{aligned}\zeta\left(\frac{2-\lambda}{p}\right) &= 1 - \frac{p}{p+\lambda-2} - \frac{1}{2} + \frac{(2-\lambda)\varepsilon_1}{12p} \\ &< \frac{1}{2} - \frac{p}{p+\lambda-2} + \frac{2-\lambda}{12p} \\ &= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} \\ &< 0.\end{aligned}$$

So for $n \ge 1$, $n \in \mathbf{N}$, $2 - \min\{p, q\} < \lambda \le 2$, we have

$$\begin{aligned}&-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12pqn^{(p-\lambda+2)/p}} \\ &\quad > \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} - \frac{p\lambda+2q}{12pq} \\ &\quad = \frac{q(\lambda-2-3p)(\lambda-2-2p) - (p\lambda+2q)(p+\lambda-2)}{12pq(p+\lambda-2)} \\ &\quad = \frac{q(\lambda-2)^2 + (p\lambda+5pq+2q)(2-\lambda) - p(p\lambda+2q) + 6p^2q}{12pq(p+\lambda-2)} \\ &\quad > \frac{-p(p\lambda+2q) + 6p^2q}{12pq(p+\lambda-2)} \\ &\quad \ge \frac{-(2p+2q) + 6pq}{12q(p+\lambda-2)} \\ &\quad > \frac{1}{3(p+\lambda-2)} \\ &\quad > \frac{1}{3p}.\end{aligned}$$

Using the last result and the inequality for $\omega(n, \lambda, p)$ above, we obtain (2.2).

When $\lambda = 1$, we have

$$\begin{aligned}
-\zeta\left(\frac{2-\lambda}{p}\right) &- \frac{p\lambda+2q}{12pqn^{(p-\lambda+2)/p}} \\
&> \frac{q(\lambda-2)^2+(p\lambda+5pq+2q)(2-\lambda)-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)} \\
&= \frac{p+3q-p^2+3pq+6p^2q}{12pq(p-1)} \\
&= \frac{5p^2+10p+12q}{12pq(p-1)} \\
&= \frac{(5p^2+10p+12q)(q-1)}{12pq} \\
&= \frac{12q^2+3q+5p}{12pq}.
\end{aligned}$$

Using the last result and the inequality for $\omega(n,\lambda,p)$ above, we obtain (2.4).

In a similar way, one can prove (2.3) and (2.5). □

3. Main results.

Theorem 3.1. *If* $p, q > 1$, $(1/p)+(1/q) = 1$, $2-\min\{p,q\} < \lambda \le 2$, $a_n \ge 0$, $b_n \ge 0$, *for* $n \ge 1$, $n \in \mathbf{N}$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

(3.1)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^\lambda, n^\lambda)} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{(q+\lambda-2)/q}}\right]n^{1-\lambda}a_n^p\right\}^{1/p} \times \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{(p+\lambda-2)/p}}\right]n^{1-\lambda}b_n^q\right\}^{1/q},$$

and

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda, n^\lambda\}}\right)^p < \kappa(\lambda)^{p-1}\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{(q+\lambda-2)/q}}\right]n^{1-\lambda}a_n^p, \quad (3.2)$$

where $\kappa(\lambda) = [(p\,q\lambda)/(p+\lambda-2)(q+\lambda-2)] > 0$. *When* $\lambda = 1$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m,n)} < \left[\sum_{n=1}^{\infty}\left(pq-\frac{12p^2+3p+5q}{12pqn^{1/p}}\right)a_n^p\right]^{1/p} \times\left[\sum_{n=1}^{\infty}\left(pq-\frac{12q^2+3q+5p}{12pqn^{\frac{1}{q}}}\right)b_n^q\right]^{1/q}. \tag{3.3}$$

Proof. By the Hölder inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^\lambda,n^\lambda)} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{\max\{m^\lambda,n^\lambda\}^{1/p}}\left(\frac{m}{n}\right)^{(2-\lambda)/pq}\right] \\ &\quad\times\left[\frac{b_n}{\max\{m^\lambda,n^\lambda\}^{1/q}}\left(\frac{n}{m}\right)^{(2-\lambda)/pq}\right] \\ &\le \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m^p}{\max\{m^\lambda,n^\lambda\}}(\frac{m}{n})^{(2-\lambda)/q}\right]\right\}^{1/p} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\max\{m^\lambda,n^\lambda\}}\left(\frac{n}{m}\right)^{(2-\lambda)/p}\right]\right\}^{1/q} \\ &= \left\{\sum_{m=1}^{\infty}\omega(m,\lambda,q)a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\omega(n,\lambda,p)b_n^q\right\}^{1/q}.\end{aligned}$$

By (2.2), (2.3), (2.4) and (2.5), we obtain (3.1) and (3.2).

By the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda,n^\lambda\}} \\ &= \sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda,n^\lambda\}^{1/p}}\left(\frac{n}{m}\right)^{(2-\lambda)/pq} a_n\frac{1}{\max\{m^\lambda,n^\lambda\}^{1/q}}\left(\frac{m}{n}\right)^{(2-\lambda)/pq}\right] \\ &\le \left\{\sum_{n=1}^{\infty}\left[\frac{1}{\max\{m^\lambda,n^\lambda\}}\left(\frac{n}{m}\right)^{(2-\lambda)/q} a_n^p\right]\right\}^{1/p}\end{aligned}$$

$$\times \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n} \right)^{(2-\lambda)/p} \right] \right\}^{1/q}$$
$$= \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m} \right)^{(2-\lambda)/q} a_n^p \right] \right\}^{1/p} [\omega(m, \lambda, p)]^{1/q}$$
$$< \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m} \right)^{(2-\lambda)/q} a_n^p \right] \right\}^{1/p} [m^{1-\lambda}\kappa(\lambda)]^{1/q}.$$

So

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda, n^\lambda\}} \right)^p$$
$$< \kappa(\lambda)^{p-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m} \right)^{(2-\lambda)/q} a_n^p \right]$$
$$< \kappa(\lambda)^{p-1} \sum_{n=1}^{\infty} \omega(n, \lambda, q) a_n^p.$$

By Lemma 2.1, the proof of the theorem is completed. □

In inequality (3.3), taking $p = q = 2$, we have:

Corollary 3.2. *Let* $a_n \geq 0$, $b_n \geq 0$ *and* $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$. *Then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m, n)} < 4 \left[\sum_{n=1}^{\infty} \left(1 - \frac{1}{3\sqrt{n}} \right) a_n^2 \right]^{1/2} \left[\sum_{n=1}^{\infty} \left(1 - \frac{1}{3\sqrt{n}} \right) b_n^2 \right]^{1/2}. \tag{3.4}$$

In inequality (3.1), taking $\lambda = 2$, we obtain:

Corollary 3.3. *If* $p, q > 1$, $(1/p) + (1/q) = 1$, $a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in \mathbf{N}$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m^2, n^2)} < \left[\sum_{n=1}^{\infty} \left(2 - \frac{1}{3qn} \right) \frac{1}{n} a_n^p \right]^{1/p} \left[\sum_{n=1}^{\infty} \left(2 - \frac{1}{3pn} \right) \frac{1}{n} b_n^q \right]^{1/q}. \tag{3.5}$$

Remark. When $\lambda = 1$, we have

$$\left[\kappa(\lambda) - \frac{1}{3pn^{(p+\lambda-2)/p}}\right] n^{1-\lambda} = \frac{pq}{(p-1)(q-1)} - \frac{1}{3pn^{(p-1)/p}}$$
$$= pq - \frac{1}{3pn^{(p-1)/p}}$$
$$< pq.$$

Then, inequality (3.1) is a generalization and reinforcement of inequality (1.2). Since

$$\frac{12q^2+3q+5p}{12pq} - \left(q + \frac{1}{3q} - \frac{4}{3}\right) = \frac{5p+7q}{12pq} > 0,$$

so

$$\frac{12q^2+3q+5p}{12pqn^{1/q}} - G(q,n) > 0.$$

Thus, inequality (3.3) is a reinforcement of inequality (1.3).

Acknowledgments. The referee made several helpful comments.

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